

Moments of $S(t, f)$ Associated with Holomorphic Hecke Cusp Forms

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Abstract. Let $S(t, f) := \pi^{-1} \arg L(1/2 + it, f)$, where f is a holomorphic Hecke cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k . We establish an asymptotic formula for the moments of $S(t, f)$.

1. Introduction

It is well known that the function $S(t) = \frac{1}{\pi} \arg \zeta(1/2 + it)$ is related to the number of nontrivial zeros ρ of the Riemann zeta-function $\zeta(s)$ with $0 < \mathrm{Im}(\rho) \leq t$. There are many research papers that have been devoted to studying the behavior of $S(t)$ (see, for example, [1, 2, 11, 13, 14, 18]). In particular, Selberg [15, 17] showed that for $n \in \mathbb{N}$,

$$\frac{1}{T} \int_T^{2T} S(t)^{2n} dt = \frac{(2n)!}{n!(2\pi)^{2n}} (\log \log T)^n + O((\log \log T)^{n-1/2}).$$

Using the same method one can derive a similar result for $S(t, \chi) = \pi^{-1} \arg L(1/2 + it, \chi)$, where χ is a primitive Dirichlet character of modulus q . In the subsequent paper [16] Selberg proved an analog result in the conductor aspect. More precisely, for a prime q , he showed

$$(1.1) \quad \frac{1}{q} \sum_{\chi \pmod{q}}^* S(t, \chi)^{2n} = \frac{(2n)!}{n!(2\pi)^{2n}} (\log \log q)^n + O_{t,n}((\log \log q)^{n-1/2}),$$

where the summation runs over the primitive characters $\chi \pmod{q}$.

In [6], Hejhal and Luo considered a GL_2 analog of (1.1). They proved an asymptotic formula for the spectral moments of $S(t, f_j) = \pi^{-1} \arg L(1/2 + it, f_j)$ assuming the Generalized Riemann Hypothesis (GRH), i.e., $S(t, f_j)^n$ is averaged over the Hecke–Maass cusp forms f_j for $\mathrm{SL}_2(\mathbb{Z})$ with a smooth test function, for each fixed positive t . Recently, a GL_3 analogous result was obtained in [12] assuming the GRH. In this paper, we establish a GL_2 analogous result of Hejhal and Luo in the weight aspect *without assuming the GRH*.

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To state our main result, let H_k denote the set of holomorphic Hecke cusp forms f of weight k for $\mathrm{SL}_2(\mathbb{Z})$, where $f(z)$ has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}$$

with $\lambda_f(1) = 1$. For $f \in H_k$, the L -function associated to f is given by

$$L(s, f) := \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

and this has Euler product

$$\begin{aligned} L(s, f) &= \prod_p (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1} \\ &= \prod_p (1 - \alpha_f(p) p^{-s})^{-1} (1 - \beta_f(p) p^{-s})^{-1}. \end{aligned}$$

The Ramanujan–Petersson conjecture (proved by Deligne [4]) asserts that

$$(1.2) \quad |\alpha_f(p)| = |\beta_f(p)| = 1, \quad \text{and thus} \quad |\lambda_f(p)| \leq 2.$$

The complete L -function

$$\Lambda(s, f) := \pi^{-s} \Gamma\left(\frac{s + (k-1)/2}{2}\right) \Gamma\left(\frac{s + (k+1)/2}{2}\right) L(s, f)$$

admits an entire continuation to $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, f) = i^k \Lambda(1-s, f).$$

Next we define the analog of $S(t)$ for f by

$$S(t, f) := \frac{1}{\pi} \arg L(1/2 + it, f),$$

where the argument $\arg L(1/2 + it, f)$ is obtained by continuous variation along the line $\mathrm{Im}(s) = t$ from $\sigma = +\infty$ to $\sigma = 1/2$. Our main result is the following theorem.

Theorem 1.1. *Let $t > 0$ and $n \in \mathbb{N}$ be given. For sufficiently large even integer k we have*

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{S(t, f)^n}{L(1, \mathrm{sym}^2 f)} = C_n (\log \log k)^{n/2} + O_{t,n}((\log \log k)^{(n-1)/2}),$$

where

$$C_n = \begin{cases} \frac{n!}{(n/2)!(2\pi)^n} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Remark 1.2. Theorem 1.1 and Corollary 2.2 indicate that the values of $|S(t, f)|$ on average have order of magnitude $\sqrt{\log \log k}$.

2. Preliminaries

The following proposition is the well-known Petersson trace formula, which can be found in [8].

Proposition 2.1 (Petersson trace formula).

$$\sum_{f \in H_k} \omega_f^{-1} \lambda_f(m) \lambda_f(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{mn}}{c} \right),$$

where $\omega_f = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \|f\|^2$, $\delta_{m,n}$ equals 1 if $m = n$ and 0 otherwise, $S(m, n; c)$ is the Kloosterman sum, and $J_{k-1}(x)$ is the J -Bessel function.

From the integral representation (see [5, 8.411 10])

$$J_{\nu}(x) = \frac{1}{\Gamma(\nu + 1/2)\Gamma(1/2)} \left(\frac{x}{2}\right)^{\nu} \int_{-1}^1 e^{ixt} (1 - t^2)^{\nu-1/2} dt$$

and the Stirling's formula, we deduce that

$$(2.1) \quad J_{k-1}(x) \ll \left(\frac{ex}{2k}\right)^{k-1}.$$

Using the bound (2.1) and the relation (see [10])

$$\omega_f = \frac{k-1}{2\pi^2} L(1, \text{sym}^2 f),$$

one can deduce the following corollary.

Corollary 2.2. *For any $m, n \geq 1$ with $8\pi\sqrt{mn} \leq k$ we have*

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{\lambda_f(n) \lambda_f(m)}{L(1, \text{sym}^2 f)} = \delta_{m,n} + O(k^{-A})$$

for any $A > 0$.

We will need the following zero-density estimate which was established by Hough [7]. In fact, [7, Theorem 1.1] states a result without the weight $1/L(1, \text{sym}^2 f)$. However, it is easy to derive a weighted version as Proposition 2.3 below by using [7, Proposition 5.1].

Proposition 2.3. *Let*

$$N_f(\sigma, T) := \#\{\rho = \beta + i\gamma \mid L(\rho, f) = 0, \sigma < \beta, |\gamma| < T\}.$$

Let $\frac{1}{2} + \frac{2}{\log k} < \sigma < 1$. For some sufficiently small $\delta_1, \theta_1 > 0$ we have uniformly in $\frac{10}{\log k} < T < k^{\delta_1}$,

$$\sum_{f \in H_k} \frac{N_f(\sigma, T)}{L(1, \text{sym}^2 f)} \ll T k^{1-\theta_1(\sigma-1/2)} \log k.$$

3. Proof of the main theorem

We will follow the framework of Selberg [16, 17] and Hejhal–Luo [6]. The argument here is more complicated than in [6] since we did not assume the GRH.

For a positive parameter x (to be determined later), let

$$M(t, f) := \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x^3} \frac{C_f(p)}{p^{1/2+it}} \quad \text{and} \quad R(t, f) := S(t, f) - M(t, f),$$

where $C_f(p) := \alpha_f(p) + \beta_f(p) = \lambda_f(p)$.

Proposition 3.1. *Let $t > 0$ be given. For even $k \in \mathbb{N}$ sufficiently large and $x = k^{\delta/3}$ with sufficiently small $\delta > 0$, we have*

$$(3.1) \quad \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{M(t, f)^n}{L(1, \operatorname{sym}^2 f)} = C_n (\log \log k)^{n/2} + O_{t,n}((\log \log k)^{n/2-1}),$$

$$(3.2) \quad \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{|R(t, f)|^{2n}}{L(1, \operatorname{sym}^2 f)} = O_{t,n}(1).$$

The proof of Proposition 3.1 shall be given in Section 5. Now we deduce the main theorem from this result.

Proof of Theorem 1.1. By the binomial theorem, we have

$$(3.3) \quad S(t, f)^n = M(t, f)^n + O_n \left(\sum_{\ell=1}^n |M(t, f)|^{n-\ell} |R(t, f)|^\ell \right).$$

For $1 \leq \ell < n$, we apply the generalized Hölder's inequality with exponents

$$p = 2, \quad q = \frac{2n}{n-\ell} \quad \text{and} \quad r = \frac{2n}{\ell}$$

and Proposition 3.1 to deduce that

$$\begin{aligned} & \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \operatorname{sym}^2 f)} |M(t, f)|^{n-\ell} |R(t, f)|^\ell \\ & \ll \left(\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \operatorname{sym}^2 f)} \right)^{1/p} \left(\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \operatorname{sym}^2 f)} |M(t, f)|^{2n} \right)^{1/q} \\ & \quad \times \left(\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \operatorname{sym}^2 f)} |R(t, f)|^{2n} \right)^{1/r} \\ & \ll_{t,n} (\log \log k)^{(n-\ell)/2} \ll_{t,n} (\log \log k)^{(n-1)/2}. \end{aligned}$$

The assertion follows from (3.3), Proposition 3.1 and this bound. \square

4. An approximation of $S(t, f)$

In this section we will prove several technical lemmas and derive an approximation of $S(t, f)$.

We denote by $\rho = \beta + i\gamma$ a typical zero of $L(s, f)$ inside the critical strip, i.e., $0 < \beta < 1$. For $\operatorname{Re}(s) > 1$, we have

$$(4.1) \quad -\frac{L'}{L}(s, f) = \sum_{n=1}^{\infty} \frac{\Lambda(n)C_f(n)}{n^s},$$

where $\Lambda(n)$ denotes the von Mangoldt function, and

$$C_f(n) = \begin{cases} \alpha_f(p)^m + \beta_f(p)^m & \text{if } n = p^m \text{ for a prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.1. *Let $x > 1$. For $s \neq \rho$, and $s \neq -2m - \frac{k \pm 1}{2}$ ($m = 0, 1, 2, \dots$), we have the following identity*

$$\begin{aligned} \frac{L'}{L}(s, f) = & - \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^s} - \frac{1}{\log^2 x} \sum_{\rho} \frac{x^{\rho-s}(1-x^{\rho-s})^2}{(\rho-s)^3} \\ & - \frac{1}{\log^2 x} \sum_{m=0}^{\infty} \frac{x^{-2m-\frac{k \pm 1}{2}-s}(1-x^{-2m-\frac{k \pm 1}{2}-s})^2}{(-2m-\frac{k \pm 1}{2}-s)^3}, \end{aligned}$$

where

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{if } n \leq x, \\ \Lambda(n) \frac{\log^2(x^3/n) - 2\log^2(x^2/n)}{2\log^2 x} & \text{if } x \leq n \leq x^2, \\ \Lambda(n) \frac{\log^2(x^3/n)}{2\log^2 x} & \text{if } x^2 \leq n \leq x^3, \\ 0 & \text{if } n \geq x^3. \end{cases}$$

Proof. First we recall the discontinuous integral

$$(4.2) \quad \frac{1}{2\pi i} \int_{(\alpha)} \frac{y^s}{s^3} ds = \begin{cases} \frac{\log^2 y}{2} & \text{if } y \geq 1, \\ 0 & \text{if } 0 < y \leq 1 \end{cases}$$

for $\alpha > 0$. It follows from (4.1) and (4.2) that

$$-\log^2 x \sum_{n=1}^{\infty} \frac{C_f(n)\Lambda_x(n)}{n^s} = \frac{1}{2\pi i} \int_{(\alpha)} \frac{x^u(1-x^u)^2}{u^3} \frac{L'}{L}(s+u, f) du,$$

where $\alpha = \max\{2, 1 + \operatorname{Re}(s)\}$. By moving the line of integration all way to the left, we pick up the residues at $u = 0$, $u = \rho - s$ and $u = -2m - \frac{k \pm 1}{2} - s$ ($m = 0, 1, 2, \dots$) and

deduce that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(2)} \frac{x^u(1-x^u)^2}{u^3} \frac{L'}{L}(s+u, f) du \\ &= \frac{L'}{L}(s, f) \log^2 x + \sum_{\rho} \frac{x^{\rho-s}(1-x^{\rho-s})^2}{(\rho-s)^3} + \sum_{m=0}^{\infty} \frac{x^{-2m-\frac{k+1}{2}-s}(1-x^{-2m-\frac{k+1}{2}-s})^2}{(-2m-\frac{k+1}{2}-s)^3}. \end{aligned}$$

Thus the lemma follows immediately. \square

Lemma 4.2. For $s = \sigma + it$, $s' = \sigma' + it'$ such that $1/2 \leq \sigma, \sigma' \leq 10$, $s \neq \rho$, and $s' \neq \rho$, we have

$$\operatorname{Im} \left(\frac{L'}{L}(s, f) - \frac{L'}{L}(s', f) \right) = \operatorname{Im} \sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{s'-\rho} \right) + O(1),$$

and

$$(4.3) \quad \operatorname{Re} \frac{L'}{L}(s, f) = \sum_{\rho} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} + O(\log(|t| + k)).$$

Proof. By Hadamard's factorization of the entire function $\Lambda(s, f)$ we have

$$\frac{L'}{L}(s, f) = b_f + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s + (k-1)/2}{2} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s + (k+1)/2}{2} \right) + \log \pi$$

for some $b_f \in \mathbb{C}$ with $\operatorname{Re}(b_f) = -\operatorname{Re} \sum_{\rho} \frac{1}{\rho}$ (see [9, Proposition 5.7]). Now Lemma 4.2 follow from

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right). \quad \square$$

Let $x \geq 4$. We define

$$(4.4) \quad \sigma_x = \sigma_{x,f} = \sigma_{x,f,t} := \frac{1}{2} + 2 \max_{\rho} \left\{ \left| \beta - \frac{1}{2} \right|, \frac{5}{\log x} \right\},$$

where $\rho = \beta + i\gamma$ runs through the zeros of $L(s, f)$ for which

$$(4.5) \quad |t - \gamma| \leq \frac{x^{3|\beta-1/2|}}{\log x}.$$

We shall display the dependence of σ_x as needed.

Lemma 4.3. Let $x \geq 4$. For $\sigma \geq \sigma_x$, we have

$$\begin{aligned} \frac{L'}{L}(\sigma + it, f) &= - \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma+it}} + O \left(x^{1/4-\sigma/2} \left| \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x+it}} \right| \right) \\ &\quad + O(x^{1/4-\sigma/2} \log(|t| + k)), \end{aligned}$$

and

$$\sum_{\rho} \frac{\sigma_x - 1/2}{(\sigma_x - \beta)^2 + (t - \gamma)^2} = O \left(\sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x+it}} \right) + O(\log(|t| + k)).$$

Proof. By (4.3), we have

$$(4.6) \quad \operatorname{Re} \frac{L'}{L}(\sigma_x + it, f) = \sum_{\rho} \frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} + O(\log(|t| + k)).$$

On the other hand, if $\beta + i\gamma$ is a zero of $L(s, f)$, then $(1 - \beta) + i\gamma$ is also a zero of $L(s, f)$. Thus we have

$$(4.7) \quad \begin{aligned} & \sum_{\rho} \frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} \\ &= \frac{1}{2} \sum_{\rho} \left(\frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} + \frac{\sigma_x - (1 - \beta)}{(\sigma_x - 1 - \beta)^2 + (t - \gamma)^2} \right) \\ &= \left(\sigma_x - \frac{1}{2} \right) \sum_{\rho} \frac{(\sigma_x - 1/2)^2 - (\beta - 1/2)^2 + (t - \gamma)^2}{((\sigma_x - \beta)^2 + (t - \gamma)^2)((\sigma_x - 1 + \beta)^2 + (t - \gamma)^2)}. \end{aligned}$$

Case (i): If $|\beta - 1/2| \leq \frac{\sigma_x - 1/2}{2}$, then

$$\begin{aligned} (\sigma_x - 1/2)^2 - (\beta - 1/2)^2 &\geq \frac{1}{2}((\sigma_x - 1/2)^2 + (\beta - 1/2)^2) \\ &= \frac{1}{4}((\sigma_x - \beta)^2 + (\sigma_x - 1 + \beta)^2). \end{aligned}$$

Thus

$$(\sigma_x - 1/2)^2 - (\beta - 1/2)^2 + (t - \gamma)^2 \geq \frac{1}{4}((\sigma_x - 1 + \beta)^2 + (t - \gamma)^2).$$

Case (ii): If $|\beta - 1/2| > \frac{\sigma_x - 1/2}{2}$, then by (4.4) and (4.5) we have

$$|t - \gamma| > \frac{x^{3|\beta - 1/2|}}{\log x} > 3|\beta - 1/2|.$$

Thus

$$\begin{aligned} & (\sigma_x - 1/2)^2 - (\beta - 1/2)^2 + (t - \gamma)^2 \\ &= ((\sigma_x - 1/2)^2 + (\beta - 1/2)^2) + (t - \gamma)^2 - 2(\beta - 1/2)^2 \\ &\geq \frac{1}{2}((\sigma_x - \beta)^2 + (\sigma_x - 1 + \beta)^2) + \frac{7}{9}(t - \gamma)^2 \\ &\geq \frac{1}{4}((\sigma_x - 1 + \beta)^2 + (t - \gamma)^2). \end{aligned}$$

From Cases (i), (ii) and (4.7), we have

$$\sum_{\rho} \frac{\sigma_x - \beta}{(\sigma_x - \beta)^2 + (t - \gamma)^2} \geq \frac{1}{4}(\sigma_x - 1/2) \sum_{\rho} \frac{1}{(\sigma_x - \beta)^2 + (t - \gamma)^2}.$$

Using this bound and (4.6), we obtain

$$(4.8) \quad \sum_{\rho} \frac{1}{(\sigma_x - \beta)^2 + (t - \gamma)^2} \leq \frac{4}{\sigma_x - 1/2} \left| \frac{L'}{L}(\sigma_x + it, f) \right| + O\left(\frac{\log(|t| + k)}{\sigma_x - 1/2}\right).$$

On the other hand, we have (by Lemma 4.1)

$$(4.9) \quad \begin{aligned} \frac{L'}{L}(\sigma + it, f) = & - \sum_{n \leq x^3} \frac{\Lambda_x(n) C_f(n)}{n^{\sigma+it}} + \frac{w(x, \sigma, t)}{\log^2 x} \sum_{\rho} \frac{x^{\beta-\sigma} (1 + x^{\beta-\sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} \\ & + O\left(\frac{x^{-\sigma}}{\log^2 x}\right) \end{aligned}$$

with $|w(x, \sigma, t)| \leq 1$.

Next we claim that

$$(4.10) \quad \frac{x^{\beta-\sigma} (1 + x^{\beta-\sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} \leq 2 \log x \frac{x^{1/4-\sigma/2}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}.$$

If $\beta \leq \frac{\sigma_x + 1/2}{2}$, then

$$\begin{aligned} \frac{x^{\beta-\sigma} (1 + x^{\beta-\sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} & \leq \frac{4x^{1/4-\sigma/2}}{(\sigma_x - \beta)((\sigma_x - \beta)^2 + (t - \gamma)^2)} \\ & \leq \frac{8}{\sigma_x - 1/2} \frac{x^{1/4-\sigma/2}}{(\sigma_x - \beta)^2 + (t - \gamma)^2} \\ & \leq \frac{4}{5} \log x \frac{x^{1/4-\sigma/2}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}. \end{aligned}$$

If $\beta > \frac{\sigma_x + 1/2}{2}$, then by the definition of σ_x in (4.4) and (4.5),

$$|t - \gamma| > \frac{x^{3|\beta-1/2|}}{\log x} > 3|\beta - 1/2| \geq 3|\beta - \sigma_x|.$$

Thus $(t - \gamma)^2 > \frac{8}{9}((\beta - \sigma_x)^2 + (t - \gamma)^2)$. Hence

$$\begin{aligned} \frac{x^{\beta-\sigma} (1 + x^{\beta-\sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} & \leq \frac{x^{\beta-\sigma} (1 + x^{\beta-1/2})^2}{|t - \gamma|(t - \gamma)^2} \leq \frac{\log x}{x^{3|\beta-1/2|}} \cdot \frac{9}{8} \frac{x^{\beta-\sigma} (1 + x^{\beta-1/2})^2}{(\beta - \sigma_x)^2 + (t - \gamma)^2} \\ & = \frac{9}{8} (\log x) (1 + x^{-(\beta-1/2)})^2 \frac{x^{1/2-\sigma}}{(\beta - \sigma_x)^2 + (t - \gamma)^2} \\ & < \frac{9}{8} (1 + e^{-5})^2 (\log x) \frac{x^{1/2-\sigma}}{(\beta - \sigma_x)^2 + (t - \gamma)^2} \\ & < 2(\log x) \frac{x^{1/2-\sigma}}{(\beta - \sigma_x)^2 + (t - \gamma)^2}. \end{aligned}$$

So in both cases, we have (4.10). Using (4.8) and (4.10), we get

$$\begin{aligned} & \sum_{\rho} \frac{x^{\beta-\sigma} (1 + x^{\beta-\sigma})^2}{((\sigma - \beta)^2 + (t - \gamma)^2)^{3/2}} \\ & \leq 8 \log x \frac{x^{1/4-\sigma/2}}{\sigma_x - 1/2} \left| \frac{L'}{L}(\sigma_x + it, f) \right| + O\left(\frac{(\log x) x^{1/4-\sigma/2} \log(|t| + k)}{\sigma_x - 1/2}\right) \\ & \leq \frac{4}{5} (\log x)^2 x^{1/4-\sigma/2} \left| \frac{L'}{L}(\sigma_x + it, f) \right| + O((\log x)^2 x^{1/4-\sigma/2} \log(|t| + k)). \end{aligned}$$

Inserting this bound into (4.9), we have

$$(4.11) \quad \begin{aligned} \frac{L'}{L}(\sigma + it, f) = & - \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}} + \frac{4}{5}w'(x, \sigma, t)x^{1/4-\sigma/2}\frac{L'}{L}(\sigma_x + it, f) \\ & + O(x^{1/4-\sigma/2}\log(|t| + k)) \end{aligned}$$

with $|w'(x, \sigma, t)| \leq 1$. By taking $\sigma = \sigma_x$,

$$\begin{aligned} \left(1 - \frac{4}{5}w'(x, \sigma_x, t)x^{1/4-\sigma_x/2}\right)\frac{L'}{L}(\sigma_x + it, f) = & O\left(\left|\sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right) \\ & + O(x^{1/4-\sigma/2}\log(|t| + k)). \end{aligned}$$

Since $|1 - \frac{4}{5}w'(x, \sigma_x, t)| \geq 1 - \frac{4}{5} = \frac{1}{5}$,

$$(4.12) \quad \frac{L'}{L}(\sigma_x + it, f) = O\left(\left|\sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right) + O(x^{1/4-\sigma/2}\log(|t| + k)).$$

The results follow from substituting (4.12) into (4.11) and (4.8). \square

The following theorem provides an approximation of $S(t, f)$.

Theorem 4.4. *For $t \neq 0$, and $x \geq 4$, we have*

$$\begin{aligned} S(t, f) = & \frac{1}{\pi} \operatorname{Im} \sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it} \log n} + O\left((\sigma_x - 1/2) \left|\sum_{n \leq x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right) \\ & + O((\sigma_x - 1/2)\log(|t| + k)), \end{aligned}$$

where σ_x is defined in (4.4).

Proof. We begin with

$$\begin{aligned} \pi S(t, f) = & - \int_{1/2}^{\infty} \operatorname{Im} \frac{L'}{L}(\sigma + it, f) d\sigma \\ = & - \int_{\sigma_x}^{\infty} \operatorname{Im} \frac{L'}{L}(\sigma + it, f) d\sigma - (\sigma_x - 1/2) \operatorname{Im} \frac{L'}{L}(\sigma_x + it, f) \\ & + \int_{1/2}^{\sigma_x} \operatorname{Im} \left(\frac{L'}{L}(\sigma_x + it, f) - \frac{L'}{L}(\sigma + it, f) \right) d\sigma \\ =: & J_1 + J_2 + J_3. \end{aligned}$$

By Lemma 4.3, we have

$$\begin{aligned}
 J_1 &= - \int_{\sigma_x}^{\infty} \operatorname{Im} \frac{L'}{L}(\sigma + it, f) d\sigma \\
 &= \int_{\sigma_x}^{\infty} \operatorname{Im} \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma+it}} d\sigma + O \left(\left| \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x+it}} \right| \int_{\sigma_x}^{\infty} x^{1/4-\sigma/2} d\sigma \right) \\
 &\quad + O \left(\log(|t| + k) \int_{\sigma_x}^{\infty} x^{1/4-\sigma/2} d\sigma \right) \\
 &= \operatorname{Im} \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x+it} \log n} + O \left(\frac{1}{\log x} \left| \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x+it}} \right| \right) + O \left(\frac{\log(|t| + k)}{\log x} \right).
 \end{aligned}$$

Taking $\sigma = \sigma_x$ in Lemma 4.3 we see

$$\begin{aligned}
 |J_2| &\leq (\sigma_x - 1/2) \left| \frac{L'}{L}(\sigma_x + it, f) \right| \\
 &\ll (\sigma_x - 1/2) \left| \sum_{n \leq x^2} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x+it}} \right| + (\sigma_x - 1/2) \log(|t| + k).
 \end{aligned}$$

To bound J_3 , we first apply Lemma 4.2 to see that

$$\begin{aligned}
 &\operatorname{Im} \left(\frac{L'}{L}(\sigma_x + it, f) - \frac{L'}{L}(\sigma + it, f) \right) \\
 &= \sum_{\rho} \frac{(\sigma_x - \sigma)(\sigma + \sigma_x - 2\beta)(t - \gamma)}{\{(\sigma_x - \beta)^2 + (t - \gamma)^2\} \{(\sigma - \beta)^2 + (t - \gamma)^2\}} + O(1).
 \end{aligned}$$

Hence

$$\begin{aligned}
 |J_3| &\leq \sum_{\rho} \int_{1/2}^{\sigma_x} \frac{(\sigma_x - \sigma) |\sigma + \sigma_x - 2\beta| |t - \gamma|}{\{(\sigma_x - \beta)^2 + (t - \gamma)^2\} \{(\sigma - \beta)^2 + (t - \gamma)^2\}} d\sigma + O(\sigma_x - 1/2) \\
 &\leq \sum_{\rho} \frac{\sigma_x - 1/2}{(\sigma_x - \beta)^2 + (t - \gamma)^2} \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta| |t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma + O(\sigma_x - 1/2).
 \end{aligned}$$

Case (i): If $|\beta - 1/2| \leq \frac{1}{2}(\sigma_x - 1/2)$, then for $1/2 \leq \sigma \leq \sigma_x$,

$$\begin{aligned}
 |\sigma + \sigma_x - 2\beta| &= |(\sigma - 1/2) + (\sigma_x - 1/2) - 2(\beta - 1/2)| \\
 &\leq |\sigma - 1/2| + |\sigma_x - 1/2| + 2|\beta - 1/2| \leq 3(\sigma_x - 1/2).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta| |t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma &\leq 3(\sigma_x - 1/2) \int_{-\infty}^{\infty} \frac{|t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma \\
 &< 10(\sigma_x - 1/2).
 \end{aligned}$$

Case (ii): If $|\beta - 1/2| > \frac{1}{2}(\sigma_x - 1/2)$, then by (4.4) and (4.5),

$$|t - \gamma| > \frac{x^{3|\beta-1/2|}}{\log x} > 3|\beta - 1/2|$$

and for $1/2 \leq \sigma \leq \sigma_x$,

$$|\sigma + \sigma_x - 2\beta| \leq (\sigma - 1/2) + (\sigma_x - 1/2) + 2|\beta - 1/2| \leq 6|\beta - 1/2|.$$

Thus

$$\begin{aligned} \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2} d\sigma &< \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta|}{|t - \gamma|} d\sigma \\ &< \int_{1/2}^{\sigma_x} \frac{6|\beta - 1/2|}{3|\beta - 1/2|} d\sigma = 2(\sigma_x - 1/2). \end{aligned}$$

It follows from Cases (i) and (ii) that

$$\begin{aligned} |J_3| &\leq 10(\sigma_x - 1/2) \sum_{\rho} \frac{\sigma_x - 1/2}{(\sigma_x - \beta)^2 + (t - \gamma)^2} + O(\sigma_x - 1/2) \\ &= O\left((\sigma_x - 1/2) \left| \sum_{n \leq x^3} \frac{\Lambda_x(n) C_f(n)}{n^{\sigma_x + it}} \right| \right) + O((\sigma_x - 1/2) \log(|t| + k)), \end{aligned}$$

where the last step is obtained by Lemma 4.3. The theorem follows from these bounds for J_1 , J_2 and J_3 . \square

We end this section by giving a consequence of Theorem 4.4, which is an analog to those obtained by Littlewood [11], Selberg [16], and Hejhal–Luo [6].

Theorem 4.5. *Assuming the GRH, we have*

$$S(t, f) \ll \frac{\log(|t| + k)}{\log \log(|t| + k)}.$$

Proof. We have $\sigma_x = \frac{1}{2} + \frac{10}{\log x}$ by virtue of the GRH. By (1.2),

$$|C_f(p^m) \Lambda_x(p^m)| \leq 2 \log p.$$

Thus

$$\left| \operatorname{Im} \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x + it} \log n} \right| \ll \sum_{p \leq x^3} p^{-1/2} \ll \frac{x^{3/2}}{\log x}$$

and

$$(\sigma_x - 1/2) \left| \sum_{n \leq x^3} \frac{C_f(n) \Lambda_x(n)}{n^{\sigma_x + it}} \right| \ll \frac{1}{\log x} \sum_{p \leq x^3} p^{-1/2} \log p \ll \frac{x^{3/2}}{\log x}.$$

The theorem follows by taking $x = \{\log(|t| + k)\}^{2/3}$ in Theorem 4.4 and above estimates. \square

5. Proof of Proposition 3.1

5.1. The main term (3.1)

First recall that $C_f(p) = \alpha(p) + \beta(p) = \lambda_f(p)$ and

$$M(t, f) = \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x^3} \frac{C_f(p)}{p^{1/2+it}} = \frac{-i}{2\pi} \left(\sum_{p \leq x^3} \frac{\lambda_f(p)}{p^{1/2+it}} - \sum_{p \leq x^3} \frac{\lambda_f(p)}{p^{1/2-it}} \right).$$

Set $x = k^{\delta/3}$ for a suitably small $\delta > 0$ (to be specified later). Thus we have

$$(5.1) \quad M(t, f)^n = \frac{(-i)^n}{(2\pi)^n} \left(\sum_{p \leq k^\delta} \frac{\lambda_f(p)}{p^{1/2+it}} - \sum_{p \leq k^\delta} \frac{\lambda_f(p)}{p^{1/2-it}} \right)^n.$$

A general term in the expansion of (5.1) has the form

$$(5.2) \quad \frac{\lambda_f(p_1)^{m(p_1)+n(p_1)}}{p_1^{m(p_1)(1/2+it)} (-1)^{n(p_1)} p_1^{n(p_1)(1/2-it)}} \times \cdots \times \frac{\lambda_f(p_r)^{m(p_r)+n(p_r)}}{p_r^{m(p_r)(1/2+it)} (-1)^{n(p_r)} p_r^{n(p_r)(1/2-it)}},$$

where $p_1 < p_2 < \cdots < p_r < k^\delta$, $m(p_j) + n(p_j) \geq 1$, and $\sum_{j=1}^r (m(p_j) + n(p_j)) = n$.

To study the general term (5.2), we first recall the following Hecke relation

$$(5.3) \quad \lambda_f(m) \lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right).$$

Now we discuss the contribution from the general term (5.2) in the following cases.

Case (I): In the general term (5.2), $m(p_{j_0}) \not\equiv n(p_{j_0}) \pmod{2}$ for some j_0 . By the Hecke relation (5.3) and Corollary 2.2, we have the contribution from these terms are negligible by taking $0 < \delta < 2/n$.

Thus for any $A > 0$ and odd $n \geq 1$, we have

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{M(t, f)^n}{L(1, \operatorname{sym}^2 f)} \ll k^{-A}.$$

Case (II): In the general term (5.2), $m(p_j) \equiv n(p_j) \pmod{2}$ for all j , and $m(p_{j_0}) + n(p_{j_0}) \geq 4$ for some j_0 . In this case, n is even and $n > 2r$. By the bound (1.2) and $\sum_{p \leq x} p^{-1} \ll \log \log x$, we deduce that the contribution from these terms to (3.1) is at most $O((\log \log k)^{n/2-1})$.

It remains to discuss the following: In the general term (5.2), $m(p_j) \equiv n(p_j) \pmod{2}$ and $m(p_j) + n(p_j) = 2$ for all j . Clearly n must be even, say $n = 2m$ and so $r = m$.

Case (III): $m(p_{j_0}) = 2$ or $n(p_{j_0}) = 2$ for some j_0 . We deduce that the contribution from these terms to (3.1) is at most $O((\log \log k)^{n/2-1})$ by using (1.2) and the convergence of $\sum_p p^{-1-it}$ for $t \neq 0$.

Case (IV): $m(p_j) = n(p_j) = 1$ for all j . The contribution from these terms to (3.1) is

$$(5.4) \quad \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \binom{2m}{m} (m!) (m!) \frac{(-1)^m (-i)^{2m}}{(2\pi)^{2m}} \\ \times \sum_{p_1 < \dots < p_m < k^\delta} \frac{\lambda_f(p_1)^2 \cdots \lambda_f(p_m)^2}{p_1 \cdots p_m}.$$

By the Hecke relations, we have $\lambda_f(p_1)^2 \cdots \lambda_f(p_m)^2 = \lambda_f(p_1 \cdots p_m) \lambda_f(p_1 \cdots p_m)$.

Applying Corollary 2.2, we see that (5.4) equals

$$\begin{aligned} & \frac{(2m)!}{(2\pi)^{2m}} \sum_{p_1 < \dots < p_m < k^\delta} \frac{1}{p_1 \cdots p_m} \cdot (1 + O(k^{-A})) \\ &= \frac{(2m)!}{m!(2\pi)^{2m}} \sum_{\substack{p_1, \dots, p_m < k^\delta \\ p_j \text{ distinct}}} \frac{1}{p_1 \cdots p_m} \cdot (1 + O(k^{-A})) \\ &= C_{2m} (\log \log k + O(1))^m \cdot (1 + O(k^{-A})) \\ &= C_{2m} (\log \log k)^m + O_m((\log \log k)^{m-1}). \end{aligned}$$

Now the asymptotic formula (3.1) follows from Cases (I)–(IV).

5.2. The remainder term (3.2)

Lemma 5.1. *Let $t > 0$ be given. For even $k \in \mathbb{N}$ sufficiently large and $x = k^{\delta/3}$ with $0 < \delta < 3\theta_1/(8n+3)$, we have*

$$\sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)^{4n} x^{4n(\sigma_{x,f} - 1/2)}}{L(1, \text{sym}^2 f)} \ll_{t,n,\delta} \frac{k}{(\log k)^{4n}},$$

where $\sigma_{x,f}$ is defined in (4.4) and θ_1 is as in Proposition 2.3.

Proof. By the definition of $\sigma_{x,f}$,

$$\begin{aligned} & (\sigma_{x,f} - 1/2)^{4n} x^{4n(\sigma_{x,f} - 1/2)} \\ & \leq \left(\frac{10}{\log x} \right)^{4n} x^{40n/\log x + 2^{4n+1}} \sum_{\substack{\beta > \frac{1}{2} + \frac{5}{\log x} \\ |t-\gamma| \leq \frac{x^{3(\beta-1/2)}}{\log x}}} (\beta - 1/2)^{4n} x^{8n(\beta-1/2)}. \end{aligned}$$

On the other hand,

$$\sum_{\substack{\beta > \frac{1}{2} + \frac{5}{\log x} \\ |t-\gamma| \leq \frac{x^{3(\beta-1/2)}}{\log x}}} (\beta - 1/2)^{4n} x^{8n(\beta-1/2)}$$

$$\begin{aligned}
&\leq \sum_{j=5}^{\frac{1}{2}[\log x]} \left(\frac{j+1}{\log x}\right)^{4n} x^{8n\frac{j+1}{\log x}} \sum_{\substack{\frac{1}{2} + \frac{j}{\log x} < \beta \leq \frac{1}{2} + \frac{j+1}{\log x} \\ |t-\gamma| \leq \frac{x^{3(\beta-1/2)}}{\log x}}} 1 \\
&\leq \frac{1}{(\log x)^{4n}} \sum_{j=5}^{\frac{1}{2}[\log x]} (j+1)^{4n} e^{8n(j+1)} N_f \left(\frac{1}{2} + \frac{j}{\log x}, |t| + \frac{e^{3(j+1)}}{\log x} \right).
\end{aligned}$$

By Proposition 2.3,

$$\begin{aligned}
&\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \sum_{\substack{\beta > \frac{1}{2} + \frac{5}{\log x} \\ |t-\gamma| \leq \frac{x^{3(\beta-1/2)}}{\log x}}} (\beta - 1/2)^{4n} x^{8n(\beta-1/2)} \\
&\ll_t \frac{1}{(\log x)^{4n}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{8n(j+1)} \frac{e^{3(j+1)}}{\log x} k^{1-\theta_1 \frac{j}{\log x}} \log k \\
&\ll_t \frac{k \log k}{(\log x)^{4n+1}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{8n(j+1)+3(j+1)-\theta_1 \frac{\log k}{\log x} j} \\
&\ll_{t,n,\delta} \frac{k}{(\log k)^{4n}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{(8n+3-\frac{3\theta_1}{\delta})j} \\
&\ll_{t,n,\delta} \frac{k}{(\log k)^{4n}}
\end{aligned}$$

provided that $0 < \delta < 3\theta_1/(8n+3)$. In addition,

$$\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left(\frac{10}{\log x} \right)^{4n} x^{40n/\log x} \ll \frac{k}{(\log x)^{4n}} \ll \frac{k}{(\log k)^{4n}}.$$

The lemma follows from above estimates. \square

Lemma 5.2.

$$\begin{aligned}
R(t, f) &= O \left(\left| \text{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda(p) - \Lambda_x(p))}{p^{1/2+it} \log p} \right| \right) + O \left(\left| \text{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+2it} \log p} \right| \right) \\
&\quad + O \left((\sigma_x - 1/2) x^{\sigma_x-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \right) \\
&\quad + O((\sigma_x - 1/2) \log(|t| + k)) + O(1).
\end{aligned}$$

Proof. By Theorem 4.4

$$R(t, f) = S(t, f) - M(t, f)$$

$$\begin{aligned}
&= \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda_x(p)p^{1/2-\sigma_x} - \Lambda(p))}{p^{1/2+it} \log p} + \frac{1}{\pi} \operatorname{Im} \sum_{m=2}^{\infty} \sum_{p^m \leq x^3} \frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)} \log p^m} \\
&\quad + O\left((\sigma_x - 1/2) \left| \sum_{m=1}^{\infty} \sum_{p^m \leq x^3} \frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)}} \right| \right) + O((\sigma_x - 1/2) \log(|t| + k)).
\end{aligned}$$

Using the bound (1.2), we deduce that

$$\sum_{m=3}^{\infty} \sum_{p^m \leq x^3} \frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)}} = O(1) \quad \text{and} \quad \sum_{m=3}^{\infty} \sum_{p^m \leq x^3} \frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)} \log p} = O(1).$$

Note that

$$\begin{aligned}
(\sigma_x - 1/2) \left| \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{2(\sigma_x+it)}} \right| &\ll (\sigma_x - 1/2) \sum_{p \leq x^{3/2}} \frac{\log p}{p} \\
&\ll (\sigma_x - 1/2) \log x \ll (\sigma_x - 1/2) \log k.
\end{aligned}$$

Thus

$$\begin{aligned}
R(t, f) &= \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda_x(p) - \Lambda(p))}{p^{1/2+it} \log p} - \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)(1 - p^{1/2-\sigma_x})}{p^{1/2+it} \log p} \\
&\quad + \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+it} \log p} + \frac{1}{\pi} \operatorname{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)(p^{1-2\sigma_x} - 1)}{p^{1+it} \log p} \\
&\quad + O\left((\sigma_x - 1/2) \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{\sigma_x+it}} \right| \right) + O((\sigma_x - 1/2) \log(|t| + k)) + O(1).
\end{aligned}$$

Note that for $1/2 \leq a \leq \sigma_x$,

$$\begin{aligned}
\left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{a+it}} \right| &= x^{a-1/2} \left| \int_a^{\infty} x^{1/2-\sigma} \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} d\sigma \right| \\
&\leq x^{\sigma_x-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)(1 - p^{1/2-\sigma_x})}{p^{1/2+it} \log p} \right| \\
&= \left| \int_{1/2}^{\sigma_x} \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{a+it}} da \right|
\end{aligned}$$

$$\leq (\sigma_x - 1/2)x^{\sigma_x-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma$$

and

$$(\sigma_x - 1/2) \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{\sigma_x+it}} \right| \leq (\sigma_x - 1/2)x^{\sigma_x-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma.$$

Moreover,

$$\begin{aligned} \left| \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)(p^{1-2\sigma_x} - 1)}{p^{1+it} \log p} \right| &\ll \sum_{p \leq x^{3/2}} \frac{1}{p} (1 - p^{1-2\sigma_x}) \ll \sum_{p \leq x^{3/2}} (\sigma_x - 1/2) \frac{\log p}{p} \\ &\ll (\sigma_x - 1/2) \log k. \end{aligned}$$

The results follows from above estimates. \square

Now we are ready to prove (3.2). Recall that $\sigma_x = \sigma_{x,f}$ depending on f . By Lemma 5.2,

$$\begin{aligned} (5.5) \quad &\sum_{f \in H_k} \frac{|R(t, f)|^{2n}}{L(1, \text{sym}^2 f)} \\ &\ll \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left| \text{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda(p) - \Lambda_x(p))}{p^{1/2+it} \log p} \right|^{2n} \\ &+ \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left| \text{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+2it} \log p} \right|^{2n} \\ &+ \sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)^{2n} x^{2n(\sigma_{x,f}-1/2)}}{L(1, \text{sym}^2 f)} \left(\int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \right)^{2n} \\ &+ \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} (\sigma_{x,f} - 1/2)^{2n} (\log(|t| + k))^{2n}. \end{aligned}$$

Since

$$|\Lambda(p) - \Lambda_x(p)| = O\left(\frac{\log^3 p}{\log^2 x}\right) \quad \text{and} \quad C_f(p^2) = \lambda_f(p^2) - 1,$$

the first two terms is of $O(k)$ by the same argument as in Section 5.1. The last term is of $O(k)$ by Lemma 5.1.

For the third term, it follows from Cauchy's inequality that

$$\begin{aligned} & \sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)^{2n} x^{2n(\sigma_{x,f} - 1/2)}}{L(1, \text{sym}^2 f)} \left(\int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \right)^{2n} \\ & \leq \left(\sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)^{4n} x^{4n(\sigma_{x,f} - 1/2)}}{L(1, \text{sym}^2 f)} \right)^{1/2} \\ & \quad \times \left(\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left(\int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \right)^{4n} \right)^{1/2}. \end{aligned}$$

By Hölder's inequality with the exponents $4n/(4n-1)$ and $4n$,

$$\begin{aligned} & \left(\int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \right)^{4n} \\ & \leq \left(\int_{1/2}^{\infty} x^{1/2-\sigma} d\sigma \right)^{4n-1} \left(\int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right|^{4n} d\sigma \right) \\ & = \frac{1}{(\log x)^{4n-1}} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right|^{4n} d\sigma. \end{aligned}$$

Using the same argument as in Section 5.1, we have

$$\begin{aligned} \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right|^{4n} & \ll \left(\sum_{p \leq x^3} \frac{|\Lambda_x(p) \log(xp) p^{1/2-\sigma}|^2}{p} \right)^{2n} \\ & \ll \left(\sum_{p \leq x^3} \frac{(\log p)^2}{p} (\log x)^2 \right)^{2n} \ll (\log x)^{8n}, \end{aligned}$$

where the last inequality is obtained by $\sum_{p \leq y} \frac{(\log p)^2}{p} \ll (\log y)^2$. Thus

$$\begin{aligned} & \sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left(\int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p) \Lambda_x(p) \log(xp)}{p^{\sigma+it}} \right| d\sigma \right)^{4n} \\ & \ll \frac{k}{(\log x)^{4n-1}} \int_{1/2}^{\infty} x^{1/2-\sigma} (\log x)^{8n} d\sigma \ll k (\log x)^{4n}. \end{aligned}$$

Using this bound and Lemma 5.1, we have the third term in (5.5) is of $O(k)$.

Finally (3.2) follows from the above discussion.

6. A weighted central limit theorem

As pointed out in Remark 1.2, for a weight k Hecke cusp forms f , $S(t, f)$ has order of magnitude $\sqrt{\log \log k}$. Thus it makes sense to consider the following probability measure μ_k on \mathbb{R} , defined by

$$\mu_k(E) := \left(\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \mathbf{1}_E \left(\frac{S(t, f)}{\sqrt{\log \log k}} \right) \right) / \left(\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \right)$$

where $\mathbf{1}_E$ is the characteristic function on a Borel measurable set E in \mathbb{R} . As a consequence of Theorem 1.1 we obtain the following weighted central limit theorem, which should be compared with that in [6].

Theorem 6.1. *As $k \rightarrow \infty$, the probability measure μ_k converges to the Gaussian distribution of mean 0 and variance $(2\pi^2)^{-1}$; that is,*

$$\lim_{k \rightarrow \infty} \mu_k([a, b]) = \int_a^b \sqrt{\pi} \exp(-\pi^2 \xi^2) d\xi$$

for any $a < b$.

Proof. The n th moment of μ_k is

$$\int_{\mathbb{R}} \xi^n d\mu_k(\xi) = \left(\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \left(\frac{S(t, f)}{\sqrt{\log \log k}} \right)^n \right) / \left(\sum_{f \in H_k} \frac{1}{L(1, \text{sym}^2 f)} \right).$$

By Theorem 1.1 and Corollary 2.2, we deduce that for all n ,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} \xi^n d\mu_k(\xi) = C_n = \int_{\mathbb{R}} \xi^n \sqrt{\pi} \exp(-\pi^2 \xi^2) d\xi.$$

Now the result follows from the theory of moments in probability theory (see, for example, [3, Theorem 30.2]). \square

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References

- [1] R. J. Backlund, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, C. R. Acad. Sci. Paris **158** (1914), 1979–1981.

- [2] ———, *Über die Nullstellen der Riemannschen Zetafunktion*, Acta Math. **41** (1916), no. 1, 345–375.
- [3] P. Billingsley, *Probability and Measure*, Third edition, Wiley Series in Probability and Mathematical Statistics, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1995.
- [4] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. No. **43** (1974), 273–307.
- [5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian, Sixth edition, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, Academic Press, San Diego, CA, 2000.
- [6] D. A. Hejhal and W. Luo, *On a spectral analog of Selberg’s result on $S(T)$* , Internat. Math. Res. Notices **1997**, no. 3, 135–151.
- [7] B. Hough, *Zero-density estimate for modular form L -functions in weight aspect*, Acta Arith. **154** (2012), no. 2, 187–216.
- [8] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics **17**, American Mathematical Society, Providence, RI, 1997.
- [9] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications **53**, American Mathematical Society, Providence, RI, 2004.
- [10] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L -functions*, Inst. Hautes Études Sci. Publ. Math. No. **91** (2000), 55–131.
- [11] J. E. Littlewood, *On the zeros of the Riemann zeta-function*, Math. Proc. Cambridge Philos. Soc. (2) **22** (1924), no. 3, 295–318.
- [12] S.-C. Liu and S. Liu, *A GL_3 analog of Selberg’s result on $S(t)$* , Ramanujan J. **56** (2021), no. 1, 163–181.
- [13] H. von Mangoldt, *Zu Riemanns Abhandlung “Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse”*, J. Reine Angew. Math. **114** (1895), 255–305.
- [14] ———, *Zur Verteilung der Nullstellen der Riemannschen Funktion $\gamma(t)$* , Math. Ann. **60** (1905), no. 1, 1–19.
- [15] A. Selberg, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$* , Avh. Norske Vid.-Akad. Oslo I **1944** (1944), no. 1, 27 pp.

- [16] ———, *Contributions to the theory of Dirichlet's L -functions*, Skr. Norske Vid.-Akad. Oslo I **1946** (1946), no. 3, 62 pp.
- [17] ———, *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid. **48** (1946), no. 5, 89–155.
- [18] E. C. Titchmarsh, *The zeros of the Riemann zeta-function*, Proc. Royal Soc. A **151** (1935), 234–255.

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