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A Poisson Problem of Transmission-type for the Stokes and Generalized Brinkman Systems in Complementary Lipschitz Domains in $\mathbb{R}^3$

Andrei-Florin Albişoru

Abstract. The purpose of this paper is to give a well-posedness result for a boundary value problem of transmission-type for the Stokes and generalized Brinkman systems in two complementary Lipschitz domains in $\mathbb{R}^3$. In the first part of the paper, we have introduced the classical and weighted L^2-based Sobolev spaces on Lipschitz domains in $\mathbb{R}^3$. Afterwards, the trace and conormal derivative operators are defined in the case of both Stokes and generalized Brinkman systems. Also, a summary of the main properties of the layer potential operators for the Stokes system, is provided. In the second part of the work, we exploit the well-posedness of another transmission problem concerning the Stokes system on two complementary Lipschitz domains in $\mathbb{R}^3$ which is based on the Potential Theory for the Stokes system. Then, certain properties of Fredholm operators will allow us to show our main well-posedness result in L^2-based Sobolev spaces.

1. Introduction

Recall that the Brinkman system with constant coefficients is the following system of PDEs:

$$\mathcal{B}_\alpha(\boldsymbol{w}, p) := (\Delta - \alpha\mathbb{I})\boldsymbol{w} - \nabla p = \boldsymbol{\eta}, \quad \operatorname{div} \boldsymbol{w} = 0, \tag{1.1}$$

where $\alpha > 0$ is a constant. Note that when $\alpha = 0$ in (1.1), we obtain the well-known Stokes system:

$$\mathcal{B}_0(\boldsymbol{w}, p) := \Delta\boldsymbol{w} - \nabla p = \boldsymbol{\eta}, \quad \operatorname{div} \boldsymbol{w} = 0.$$

Both systems presented above are linear and elliptic in the sense of Agmon-Douglis-Nirenberg (see, e.g., [9,12]) and they play a main role in fluid mechanics and porous media (see, e.g., [16, 22] and the references therein).

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Layer potential methods have been exploited in the study of elliptic boundary value problems and among many valuable references concerning the application of L^p-theory, we mention [3, 9, 16, 21, 23]. Fabes et al. [4] have developed a layer potential method in order to show the solvability of the Dirichlet problem for the Stokes system on Lipschitz domains in $\mathbb{R}^n$, for $n \geq 3$, with L^2-boundary data. Mitrea and Wright [21] have exploited layer potential methods in order to analyze the Dirichlet and Neumann problems for the Stokes system in arbitrary Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. Kohr et al. [13] have studied Robin type problems for the Brinkman and Darcy-Forchheimer-Brinkman systems, respectively, on Lipschitz domains in Euclidean setting. Also, they tackled mixed Dirichlet-Robin problems for the Brinkman and Darcy-Forchheimer-Brinkman systems, respectively, on bounded, creased Lipschitz domains in an Euclidean setting. Medkova [18] has obtained existence and uniqueness results for L^2-solutions of the transmission problem, the Robin-transmission problem and the Dirichlet-transmission problem for the Brinkman system in Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, by using integral equation method. Choe and Kim [2], have exploited the linear theory in [4] and have obtained complete solvability results of the Dirichlet problem for the Navier-Stokes system on a bounded Lipschitz domain in $\mathbb{R}^3$ with connected boundary. Groşan et al. [7] have proved the well-posedness of the Dirichlet problem associated to the generalization of the Darcy-Forchheimer-Brinkman system in Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$. Kohr et al. [11] have studied transmission problems for the Stokes and Brinkman systems and for the Stokes and Darcy-Forchheimer-Brinkman systems in $\mathbb{R}^3$, using the layer potential methods in the linear case and the combination of well-posedness results from the linear case and fixed point theorems. Kohr et al. [15] have tackled transmission-type boundary value problems for the Stokes and generalized Brinkman systems, Navier-Stokes and Darcy-Forchheimer-Brinkman systems, respectively, in the setting of complementary Lipschitz domains on compact Riemannian manifolds.

The aim of this paper is to give an existence and uniqueness result in L^2-weighted Sobolev spaces, for a Poisson problem of transmission-type for the generalized Brinkman and Stokes system in two complementary Lipschitz domains in $\mathbb{R}^3$, the first being a bounded Lipschitz domain D, with connected boundary denoted by ∂D, and the latter is the complementary (or exterior) Lipschitz domain $\mathbb{R}^3 \setminus \overline{D}$. We use a layer potential method combined with the means of Fredholm operator theory to obtain the desired result. After laying the theoretical foundation required for the formulation of our boundary value problem in Section 3, we review the main properties of the layer potential operators for the Stokes system in Section 4. We state and prove the uniqueness of a solution for the Poisson problem of transmission-type associated to the Stokes system in both complementary Lipschitz domains in $\mathbb{R}^3$ (see Lemma 4.1). Then we establish the existence of a solution for the same transmission problem by using a layer potential analysis (see Theorems 4.3 and 4.4). Finally, we use these auxiliary results to prove the well-posedness of our boundary value problem for the Stokes and generalized Brinkman systems in complementary Lipschitz domains D and $\mathbb{R}^3 \setminus \overline{D}$ (see Theorem 4.5).

2. Preliminary results

We follow a similar approach as in the work of Kohr et al. [11]. In the latter we shall denote $D_+ := D \subset \mathbb{R}^3$, the bounded Lipschitz domain, and by $D_- := \mathbb{R}^3 \setminus \overline{D}$, the exterior (unbounded) Lipschitz domain. Recall that, a Lipschitz domain is an open, connected set, whose boundary is locally the graph of a Lipschitz function (see, e.g., [13, Definition 2.1]).

Everywhere in this paper, we adopt the repeated index summation convention.

2.1. Standard Sobolev spaces

Recall that, for $1 \leq p < \infty$, $L^p(\mathbb{R}^3)$ is the Lebesgue space of (equivalence classes of) measurable functions p-th power integrable on $\mathbb{R}^3$, and denote by $L^\infty(\mathbb{R}^3)$ the space of (equivalence classes of) essentially bounded measurable functions in $\mathbb{R}^3$. To keep the presentation as smooth as possible, we shall introduce the standard Sobolev (or Bessel potential) spaces, by the means of Fourier transform. In this section, by D_0, we denote either D_+, D_- or $\mathbb{R}^3$.

Note that, the Fourier transform $\mathcal{F}$, and its inverse $\mathcal{F}^{-1}$, are defined on $L^1(\mathbb{R}^3)$ functions by

$$(\mathcal{F}u)(\xi) := \int_{\mathbb{R}^3} \exp^{-2\pi i x\cdot\xi} u(x)\,\mathrm{d}x, \quad (\mathcal{F}^{-1}u)(x) := \int_{\mathbb{R}^3} \exp^{2\pi i x\cdot\xi} (\mathcal{F}u)(\xi)\,\mathrm{d}\xi,$$

and one generalizes these formulas to the space of temperate distributions.

Also, denote by $\mathcal{D}(D_0)$ the space of test functions (i.e., $C_0^\infty(D_0)$ endowed with the inductive limit topology) and by $\mathcal{D}'(D_0)$ the space of distributions, i.e., the dual of $\mathcal{D}(D_0)$. Also, the space $\mathcal{D}(\overline{D}_0)$ is given by $\mathcal{D}(D_0) = \{u|_{D_0} : u \in \mathcal{D}(\mathbb{R}^3)\}$.

For $s \in \mathbb{R}$, we introduce the scalar L^2-based Sobolev (Bessel potential) spaces:

$$\begin{aligned}
&H^s(\mathbb{R}^3) := \{\mathcal{F}^{-1}(1+|\xi|^2)^{-s/2}\mathcal{F}u : u \in L^2(\mathbb{R}^3)\},\\
&H^s(D_0) := \{u \in \mathcal{D}'(D_0) : \exists U \in H^s(\mathbb{R}^n) \text{ such that } U|_{D_0} = u\},\\
&\widetilde{H}^s(D_0) \equiv \text{the closure of } \mathcal{D}(D_0) \text{ in } H^s(\mathbb{R}^3),
\end{aligned}$$

and the vector-valued L^2-based Sobolev (Bessel potential) spaces:

$$\begin{aligned}
H^s(\mathbb{R}^3)^3 &:= \{\boldsymbol{u} = (u_1, u_2, u_3) : u_i \in H^s(\mathbb{R}^3), i = 1,2,3\},\\
H^s(D_0)^3 &:= \{\boldsymbol{u} = (u_1, u_2, u_3) : u_i \in H^s(D_0), i = 1,2,3\},\\
\widetilde{H}^s(D_0)^3 &:= \{\boldsymbol{u} = (u_1, u_2, u_3) : u_i \in \widetilde{H}^s(D_0), i = 1,2,3\}.
\end{aligned}$$

We have the following norm on $H^s(\mathbb{R}^3)$:

$$\|u\|_{H^s(\mathbb{R}^3)} := \|(1+|\xi|^2)^{-s/2}\mathcal{F}u\|_{L^2(\mathbb{R}^3)}.$$

Remark that, if $f = \mathcal{F}^{-1}(1+|\xi|^2)^{-s/2}\mathcal{F}u$, then

$$\|f\|_{H^s(\mathbb{R}^3)} = \|u\|_{L^2(\mathbb{R}^3)}.$$

We also have the following norm on $H^s(D_0)$, given by

$$\|u\|_{H^s(D_0)} = \inf\{\|U\|_{H^s(\mathbb{R}^3)} : U|_{D_0} = u\}.$$

The norms on the vector-valued spaces are introduced similarly.

Note that another characterization for $\widetilde{H}^s(D_0)$ is given by (see, e.g., [10, Remark 2.7])

$$\widetilde{H}^s(D_0) = \{U \in H^s(\mathbb{R}^3) : \operatorname{supp} U \subseteq \overline{D}_0\}.$$

For any $s \in \mathbb{R}$, $\mathcal{D}(\overline{D}_0)$ is dense in the space $H^s(D_0)$, and the following duality relations hold (see, e.g., [10, Proposition 2.9], [5, (1.9)]):

$$(H^s(D_0))' = \widetilde{H}^{-s}(D_0), \quad H^{-s}(D_0) = (\widetilde{H}^s(D_0))',$$

where the upper script $'$ refers to the topological dual.

Recall that $H^0(\partial D) := L^2(\partial D)$ is the space (of equivalence classes) of all measurable functions, square-integrable on the boundary.

For $s \in (0,1)$, we can introduce the space $H^s(\partial D)$ as natural trace spaces, as presented in the book of Hsiao and Wendland [9, Chapter 4]. Also. we define the space $H^{-s}(\partial D)$ as follows:

$$H^{-s}(\partial D) = (H^s(\partial D))'$$

for $s \in (0,1)$.

All the above spaces have structure of Hilbert spaces (for more details, see e.g., [1, 9, 11]).

2.2. Weighted Sobolev spaces and related results

In this paper, we will work with the Stokes system in an exterior Lipschitz domain in the context of our transmission-type problems. This situation requires the introduction of the weighted Sobolev spaces, as seen in the work of Hanouzet [8]. This approach is useful in order to compensate for the behavior of the fundamental solution of the Stokes at infinity, in $\mathbb{R}^3$.

To this end, we consider the weight function $\rho(\mathbf{x}) := (1+|\mathbf{x}|^2)^{1/2}$ for all $\mathbf{x} \in \mathbb{R}^3$.

We say that a function $u\colon D_- \to \mathbb{R}$ belongs to the weighted space $L^2(\rho^{-1}; D_-)$ if and only if $\rho^{-1}u \in L^2(D_-)$.

We can introduce now the weighted Sobolev spaces

$$\begin{aligned}&\mathcal{H}^1(D_-) := \{u \in \mathcal{D}'(D_-) : \rho^{-1}u \in L^2(D_-), \nabla u \in L^2(D_-)^3\},\\ &\widetilde{\mathcal{H}}^1(D_-) \equiv \text{the closure of } \mathcal{D}(D_-) \text{ in } \mathcal{H}^1(D_-),\end{aligned}$$

and similarly, we introduce the vector-valued spaces

$$\begin{aligned}&\mathcal{H}^1(D_-)^3 := \{\boldsymbol{u} = (u_1, u_2, u_3) : u_i \in \mathcal{H}^1(D_-), i = 1,2,3\},\\ &\widetilde{\mathcal{H}}^1(D_-)^3 := \{\boldsymbol{u} = (u_1, u_2, u_3) : u_i \in \widetilde{\mathcal{H}}^1(D_-), i = 1,2,3\}.\end{aligned}$$

We have the following norm on $\mathcal{H}^1(D_-)$:

$$\|u\|_{\mathcal{H}^1(D_-)} := \left[\|\rho^{-1}u\|^2_{L^2(D_-)} + \|\nabla u\|^2_{L^2(D_-)^3}\right]^{1/2}.$$

By duality, we introduce the negative order weighted Sobolev spaces

$$\mathcal{H}^{-1}(D_-) = (\widetilde{\mathcal{H}}^1(D_-))', \quad \widetilde{H}^{-1}(D_-) = (\mathcal{H}^1(D_-))'.$$

We conclude this section with an important definition regarding the Leray condition at infinity and a very useful corollary. Both will play a main role in the analysis of the transmission problem studied in the last two sections (see, e.g., [11, Definition 2.3] and the references therein).

Definition 2.1. A function w tends to a constant w_∞ at ∞, in the sense of Leray if

$$\lim_{r\to\infty} \int_{\mathcal{S}^2} |w(ry) - w_\infty| \,\mathrm{d}\sigma_y = 0,$$

where $\mathcal{S}^2$ denotes the unit sphere in $\mathbb{R}^3$.

Corollary 2.2. *If $w \in \mathcal{H}^1(D_-)$, then w tends to zero at ∞ in the sense of Leray.*

For the proof of this corollary, see, e.g., [11, Corollary 2.4] and the references therein.

2.3. Trace and conormal derivative operators

In order to formulate the transmission conditions that appear in our boundary value problems, we need to introduce the trace operator and the conormal derivative operator.

Note that, the trace operator is just the extension from the case of smooth functions to Sobolev spaces of the restriction operator to the boundary (see, e.g., [3], [9, Theorem 4.2.1], [20, Theorem 2.3, Lemma 2.6]).

Lemma 2.3 (Gagliardo Trace Lemma: standard Sobolev space)**.** *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary ∂D and denote by $D_- := \mathbb{R}^3 \setminus \overline{D}$ the complementary Lipschitz domain. Then, there exist linear, continuous trace operators $\gamma_\pm \colon H^1(D_\pm) \to H^{1/2}(\partial D)$ such that*

$$\gamma_\pm u = u|_{\partial D}, \quad \forall\, u \in C^\infty(\overline{D}_\pm).$$

Moreover, these operators are surjective, having (non-unique) linear and continuous right inverse operators $\mathcal{Z}_\pm \colon H^{1/2}(\partial D) \to H^1(D_\pm)$. Hence $\gamma_\pm \circ \mathcal{Z}_\pm = \mathbb{I}$.

A similar lemma holds in the case of an exterior trace operator defined on the weighted Sobolev space $\mathcal{H}^1(D_-)$. Due to the fact that the embedding $H^1(D_-) \hookrightarrow \mathcal{H}^1(D_-)$ is continuous, we can state the following lemma (see, e.g., [11, Lemma 2.2]).

Lemma 2.4 (Gagliardo Trace Lemma: weighted Sobolev space)**.** *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary ∂D and denote by $D_- := \mathbb{R}^3 \setminus \overline{D}$ the complementary Lipschitz domain. Then, there exists a linear, continuous trace operator $\gamma_- \colon \mathcal{H}^1(D_-) \to H^{1/2}(\partial D)$ such that*

$$\gamma_- u = u|_{\partial D}, \quad \forall\, u \in C^\infty(\overline{D}_\pm).$$

Moreover, this operator is surjective, having a (non-unique) linear and continuous right inverse operator $\mathcal{Z}_- \colon H^{1/2}(\partial D) \to \mathcal{H}^1(D_-)$. Hence $\gamma_- \circ \mathcal{Z}_- = \mathbb{I}$.

Note that, in this paper, we maintain the same notation for the trace operators $\gamma_\pm$ in the case of vector-valued functions.

In the latter, by $\langle \cdot\,, \cdot \rangle_A$ we denote the duality pairing of two dual Sobolev spaces defined on A, where A is either an open set or a surface in $\mathbb{R}^3$.

Denote by $\boldsymbol{\nu}$ the outward unit normal to our bounded Lipschitz domain $D \subset \mathbb{R}^3$, which exists a.e. on ∂D and has the components ν_l, $l = 1, 2, 3$.

Through the following lemma, we can define the conormal derivative operator for the Stokes system in the setting of Sobolev spaces (see, e.g., [3, Lemma 3.2], [20, Defintion 3.1, Theorem 3.2] and [21, Theorem 10.4.1]).

Lemma 2.5. *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $D_- := \mathbb{R}^3 \setminus \overline{D}$. Consider the following space*

$$\begin{aligned}\boldsymbol{H}^1(D_\pm, \mathcal{B}_0) := \{(\boldsymbol{w}_\pm, p_\pm, \boldsymbol{\eta}_\pm) \in H^1(D_\pm)^3 \times L^2(D_\pm) \times \widetilde{H}^{-1}(D_\pm)^3 :\\ \mathcal{B}_0(\boldsymbol{w}_\pm, p_\pm) = \boldsymbol{\eta}_\pm|_{D_\pm} \text{ and } \operatorname{div} \boldsymbol{w}_\pm = 0 \text{ in } D_\pm\}.\end{aligned}$$

Define the conormal derivative operators for the Stokes system in $D_\pm$,

$$\partial^\pm_{0,\nu} \colon \boldsymbol{H}^1(D_\pm, \mathcal{B}_0) \to H^{-1/2}(\partial D)^3,$$

by the following relation

$$\pm\langle\partial_{0,\nu}^{\pm}(\boldsymbol{w}_{\pm},p_{\pm},\boldsymbol{\eta}_{\pm}),\boldsymbol{\phi}\rangle_{\partial D}:=2\langle\mathbf{E}(\boldsymbol{w}_{\pm}),\mathbf{E}(\mathcal{Z}_{\pm}\boldsymbol{\phi})\rangle_{D_{\pm}}-\langle p_{\pm},\operatorname{div}(\mathcal{Z}_{\pm}\boldsymbol{\phi})\rangle_{D_{\pm}}\\ +\langle\boldsymbol{\eta}_{\pm},\mathcal{Z}_{\pm}\boldsymbol{\phi}\rangle_{D_{\pm}},\quad\forall\,\boldsymbol{\phi}\in H^{1/2}(\partial D)^3,$$

where $\mathbf{E}(\boldsymbol{w})$ *is the symmetric part of* $\nabla\boldsymbol{w}$*, and* $\mathcal{Z}_{\pm}$ *are right inverses of the trace operators* $\gamma_{\pm}\colon H^1(D_{\pm})^3\to H^{1/2}(\partial D)^3$*. The operators* $\partial_{0,\nu}^{\pm}$ *are linear, bounded and do not depend on the choice of the right inverses* $\mathcal{Z}_{\pm}$ *of the trace operators* $\gamma_{\pm}$*.*

Moreover, the following Green formulas hold:

$$\pm\langle\partial_{0,\nu}^{\pm}(\boldsymbol{w}_{\pm},p_{\pm},\boldsymbol{\eta}_{\pm}),\gamma_{\pm}\boldsymbol{\psi}_{\pm}\rangle_{\partial D}=2\langle\mathbf{E}(\boldsymbol{w}_{\pm}),\mathbf{E}(\boldsymbol{\psi}_{\pm})\rangle_{D_{\pm}}-\langle p_{\pm},\operatorname{div}\boldsymbol{\psi}_{\pm}\rangle_{D_{\pm}}\\ +\langle\boldsymbol{\eta}_{\pm},\boldsymbol{\psi}_{\pm}\rangle_{D_{\pm}}\tag{2.1}$$

for all $(\boldsymbol{w}_{\pm},p_{\pm},\boldsymbol{\eta}_{\pm})\in\boldsymbol{H}^1(D_{\pm},\mathcal{B}_0)$ *and for any* $\boldsymbol{\psi}_{\pm}\in H^1(D_{\pm})^3$*.*

We have a similar version of the Lemma 2.5 in the case of weighted spaces (cf. [11, Lemma 2.9]).

Lemma 2.6. *Let* $D_+:=D\subset\mathbb{R}^3$ *be a bounded Lipschitz domain with connected boundary* ∂D *and let* $D_-:=\mathbb{R}^3\setminus\overline{D}$*. Consider the following space*

$$\boldsymbol{\mathcal{H}}^1(D_-,\mathcal{B}_0):=\{(\boldsymbol{w}_-,p_-,\boldsymbol{\eta}_-)\in\mathcal{H}^1(D_-)^3\times L^2(D_-)\times\widetilde{\mathcal{H}}^{-1}(D_-)^3:\\ \mathcal{B}_0(\boldsymbol{w}_-,p_-)=\boldsymbol{\eta}_-|_{D_-}\ \text{and}\ \operatorname{div}\boldsymbol{w}_-=0\ \text{in}\ D_-\}.$$

Define the conormal derivative operator in D_-*,*

$$\partial_{0,\nu}^{-}\colon\boldsymbol{\mathcal{H}}^1(D_-,\mathcal{B}_0)\to H^{-1/2}(\partial D)^3,$$

by the following relation

$$\langle\partial_{0,\nu}^{-}(\boldsymbol{w}_-,p_-,\boldsymbol{\eta}_-),\boldsymbol{\phi}\rangle_{\partial D}:=-2\langle\mathbf{E}(\boldsymbol{w}_-),\mathbf{E}(\mathcal{Z}_-\boldsymbol{\phi})\rangle_{D_-}+\langle p_-,\operatorname{div}(\mathcal{Z}_-\boldsymbol{\phi})\rangle_{D_-}\\ -\langle\boldsymbol{\eta}_-,\mathcal{Z}_-\boldsymbol{\phi}\rangle_{D_-},\quad\forall\,\boldsymbol{\phi}\in H^{1/2}(\partial D)^3,$$

where $\mathcal{Z}_-$ *is a right inverse of the trace operator* $\gamma_-\colon\mathcal{H}^1(D_-)^3\to H^{1/2}(\partial D)^3$*. The operator* $\partial_{0,\nu}^{-}$ *is linear, bounded and does not depend on the choice of the right inverse* $\mathcal{Z}_-$ *of the trace operator* γ_-*.*

Moreover, the following Green formula holds:

$$\langle\partial_{0,\nu}^{-}(\boldsymbol{w}_-,p_-,\boldsymbol{\eta}_-),\gamma_-\boldsymbol{\psi}\rangle_{\partial D}=-2\langle\mathbf{E}(\boldsymbol{w}_-),\mathbf{E}(\boldsymbol{\psi})\rangle_{D_-}+\langle p_-,\operatorname{div}\boldsymbol{\psi}\rangle_{D_-}-\langle\boldsymbol{\eta}_-,\boldsymbol{\psi}\rangle_{D_-}\tag{2.2}$$

for all $(\boldsymbol{w}_-,p_-,\boldsymbol{\eta}_-)\in\boldsymbol{\mathcal{H}}^1(D_-,\mathcal{B}_0)$ *and for any* $\boldsymbol{\psi}\in H^1(D_-)^3$*.*

2.4. The generalized Brinkman system and related results

Let $D_+ := D \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. In this setting we consider a generalized form of the Brinkman system,

$$\mathcal{B}_{\mathcal{P}}(\boldsymbol{w}, p) := \Delta \boldsymbol{w} - \mathcal{P}\boldsymbol{w} - \nabla p = \boldsymbol{\eta} \text{ in } D_+, \quad \operatorname{div} \boldsymbol{w} = 0 \text{ in } D_+,$$

where $\mathcal{P} \in L^\infty(D_+)^{3\times 3}$, such that

$$\langle \mathcal{P}\boldsymbol{v}, \boldsymbol{v}\rangle_{D_+} \geq c_{\mathcal{P}} \|\boldsymbol{v}\|^2_{L^2(D_+)^3}, \quad \forall\, \boldsymbol{v} \in L^2(D_+)^3, \tag{2.3}$$

where $c_{\mathcal{P}} > 0$ is a constant.

This system of partial differential equations is understood in a distributional sense. Indeed, let $(\boldsymbol{w}, p) \in H^1(D_+)^3 \times L^2(D_+)$. In this case, we have the following

$$\langle \mathcal{B}_{\mathcal{P}}(\boldsymbol{w}, p), \boldsymbol{\psi}\rangle_{D_+} = \langle \boldsymbol{\eta}, \boldsymbol{\psi}\rangle_{D_+}, \quad \langle \operatorname{div} \boldsymbol{w}, g\rangle_{D_+} = 0$$

for all $(\boldsymbol{w}, g) \in \mathcal{D}(D_+)^3 \times \mathcal{D}(D_+)$, where

$$\begin{aligned}\langle \mathcal{B}_{\mathcal{P}}(\boldsymbol{w}, p), \boldsymbol{\psi}\rangle_{D_+} &:= \langle \Delta \boldsymbol{w} - \mathcal{P}\boldsymbol{w} - \nabla p, \boldsymbol{\psi}\rangle_{D_+} \\ &= -\langle \nabla \boldsymbol{w}, \nabla \boldsymbol{\psi}\rangle_{D_+} - \langle \mathcal{P}\boldsymbol{w}, \boldsymbol{\psi}\rangle_{D_+} + \langle p, \operatorname{div} \boldsymbol{\psi}\rangle_{D_+}.\end{aligned}$$

Note that, due to the continuous embedding

$$L^2(D_+) \hookrightarrow H^{-1}(D_+),$$

we deduce that the operator

$$\mathcal{B}_{\mathcal{P}} \colon H^1(D_+)^3 \times L^2(D_+) \to H^{-1}(D_+)^3 = (\widetilde{H}^1(D_+)^3)'$$

is linear and bounded.

Remark 2.7. If $\mathcal{P} \equiv 0$, one finds the Stokes system. If $\mathcal{P} \equiv \alpha\mathbb{I}$, where $\alpha > 0$ is a constant, one finds the classical Brinkman system.

Now, just as in the case of the classical Brinkman system and Stokes systems, we introduce the lemma that lets us define the conormal derivative operator for our generalized version of the Brinkman system. The proof in this general case, follows similar arguments (cf. [7, Lemma 2.2]).

Lemma 2.8. *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with boundary ∂D. Let $\mathcal{P} \in L^\infty(D_+)^{3\times 3}$. Consider the following space*

$$\begin{aligned}\boldsymbol{H}^1(D_+, \mathcal{B}_{\mathcal{P}}) := \{(\mathbf{w}, p, \boldsymbol{\eta}) \in H^1(D_+)^3 \times L^2(D_+) \times \widetilde{H}^{-1}(D_+)^3 :\\ \mathcal{B}_{\mathcal{P}}(\boldsymbol{w}, p) := \Delta \boldsymbol{w} - \mathcal{P}\boldsymbol{w} - \nabla p = \boldsymbol{\eta}|_{D_+} \textit{ and } \operatorname{div} \boldsymbol{w} = 0 \textit{ in } D_+\}.\end{aligned}$$

Define the conormal derivative operator for the generalized Brinkman system,

$$\partial^+_{\mathcal{P},\nu} \colon \boldsymbol{H}^1(D_+,\mathcal{B}_{\mathcal{P}}) \to H^{-1/2}(\partial D)^3,$$

by the following relation

$$\begin{aligned}\langle \partial^+_{\mathcal{P},\nu}(\boldsymbol{w},p,\boldsymbol{\eta}),\boldsymbol{\phi}\rangle_{\partial D} := 2\langle \mathbf{E}(\boldsymbol{w}),\mathbf{E}(\mathcal{Z}_+\boldsymbol{\phi})\rangle_{D_+} + \langle \mathcal{P}\boldsymbol{w},\mathcal{Z}_+\boldsymbol{\phi}\rangle_{D_+}\\ - \langle p,\operatorname{div}(\mathcal{Z}_+\boldsymbol{\phi})\rangle_{D_+} + \langle \boldsymbol{\eta},\mathcal{Z}_+\boldsymbol{\phi}\rangle_{D_+}, \quad \forall\, \boldsymbol{\phi}\in H^{1/2}(\partial D)^3,\end{aligned}$$

where $\mathcal{Z}_+$ is a right inverse of the trace operator $\gamma_+\colon H^1(D_+)^3 \to H^{1/2}(\partial D)^3$. The operator $\partial^+_{\mathcal{P},\nu}$ is linear, bounded and does not depend on the choice of the right inverse $\mathcal{Z}_+$ of the trace operator γ_+.

Moreover, the following Green formula holds:

$$\langle \partial^+_{\mathcal{P},\nu}(\boldsymbol{w},p,\boldsymbol{\eta}),\gamma_+\boldsymbol{\psi}\rangle_{\partial D} = 2\langle \mathbf{E}(\boldsymbol{w}),\mathbf{E}(\boldsymbol{\psi})\rangle_{D_+} + \langle \mathcal{P}\boldsymbol{w},\boldsymbol{\psi}\rangle_{D_+} - \langle p,\operatorname{div}\boldsymbol{\psi}\rangle_{D_+} + \langle \boldsymbol{\eta},\boldsymbol{\psi}\rangle_{D_+} \tag{2.4}$$

for all $(\boldsymbol{w},p,\boldsymbol{\eta})\in \boldsymbol{H}^1(D_+,\mathcal{B}_{\mathcal{P}})$ and for any $\boldsymbol{\psi}\in H^1(D_+)^3$.

Remark 2.9. If $\mathcal{P}\equiv \alpha\mathbb{I}$, where $\alpha>0$ is a constant, one obtains the corresponding lemma for the conormal derivative associated to the classical Brinkman system. We could also obtain a similar result as in Lemma 2.8 for the classical Brinkman system in the exterior (unbounded) Lipschitz domain $D_-\subseteq\mathbb{R}^3$.

3. Layer potential operators for the Stokes system

We review in this section the main properties of the layer potential operators for the Stokes system. First of all, we denote the fundamental solution of the Stokes system by $(\mathbf{G}(\,\cdot\,,\cdot\,),\mathbf{P}(\,\cdot\,,\cdot\,))\in\mathcal{D}'(\mathbb{R}^3\times\mathbb{R}^3)^{3\times 3}\times\mathcal{D}'(\mathbb{R}^3\times\mathbb{R}^3)^3$. This fundamental solution satisfies the following equations

$$\Delta_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\mathbf{P}(\mathbf{x},\mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I}, \quad \operatorname{div}\mathbf{G}(\mathbf{x},\mathbf{y}) = 0, \tag{3.1}$$

where $\delta_{\mathbf{y}}$ is the Dirac distribution with singularity at $\mathbf{y}$. Note that the differential operators $\Delta_{\mathbf{x}}$ and $\nabla_{\mathbf{x}}$ act with respect to $\mathbf{x}$. Sometimes, we have the following notations $\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{G}(\mathbf{x}-\mathbf{y})$ and $\mathbf{P}(\mathbf{x},\mathbf{y}) = \mathbf{P}(\mathbf{x}-\mathbf{y})$.

The components of $\mathbf{G}(\,\cdot\,,\cdot\,)$, $\mathbf{P}(\,\cdot\,,\cdot\,)$ have the following expression (see, e.g., [16, pp. 38–39])

$$\mathbf{G}_{jk}(\mathbf{x}) = \frac{1}{8\pi}\left\{\frac{\delta_{jk}}{|\mathbf{x}|} + \frac{x_jx_k}{|\mathbf{x}|^3}\right\}, \quad \mathbf{P}_j(\mathbf{x}) = \frac{1}{4\pi}\frac{x_j}{|\mathbf{x}|^3},$$

where δ_{jk} denotes the Kronecker symbol.

The stress and pressure tensors $\mathbf{S}(\mathbf{S}_{jkl}), \mathbf{R}(\mathbf{R}_{jk})$ have the components (see, e.g., [16, p. 39])

$$\mathbf{S}_{jkl}(\mathbf{x}) = -\frac{3}{4\pi}\frac{x_j x_k x_l}{|\mathbf{x}|^5}, \quad \mathbf{R}_{jk}(\mathbf{x}) = \frac{1}{2\pi}\left\{-\frac{\delta_{jk}}{|\mathbf{x}|^3} + 3\frac{x_j x_k}{|\mathbf{x}|^5}\right\}.$$

Moreover, for $\mathbf{x} \neq \mathbf{y}$, we have

$$\Delta_{\mathbf{x}}\mathbf{S}_{jkl}(\mathbf{y},\mathbf{x}) - \frac{\partial \mathbf{R}_{jl}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0, \quad \frac{\partial \mathbf{S}_{jkl}(\mathbf{y},\mathbf{x})}{\partial x_k} = 0.$$

Now, for $\mathfrak{F} \in [\mathcal{D}'(\mathbb{R}^3)]^3$, the Newtonian velocity and pressure potentials, for the Stokes system, are given by

$$(\mathcal{N}_{\mathbb{R}^3}\mathfrak{F})(\mathbf{x}) := -\langle \mathbf{G}(\mathbf{x},\cdot), \mathfrak{F}\rangle_{\mathbb{R}^3}, \quad (\mathcal{Q}_{\mathbb{R}^3}\mathfrak{F})(\mathbf{x}) := -\langle \mathbf{P}(\mathbf{x},\cdot), \mathfrak{F}\rangle_{\mathbb{R}^3}. \tag{3.2}$$

We can also consider the Newtonian potentials corresponding to $D_\pm$ as

$$\mathcal{N}_{D_\pm}\mathfrak{F} := r_{D_\pm}(\mathcal{N}_{\mathbb{R}^3}\mathfrak{F}), \quad \mathcal{Q}_{D_\pm}\mathfrak{F} := r_{D_\pm}(\mathcal{Q}_{\mathbb{R}^3}\mathfrak{F}),$$

where $r_{D_\pm}$ are the operators of restriction of vector-valued or scalar-valued distributions in $\mathbb{R}^3$ to $D_\pm$.

We have the following lemma that gives the continuity properties of the Newtonian layer potentials on our Sobolev spaces (see, e.g., [11, Lemma A.3]).

Lemma 3.1. *The Newtonian velocity and pressure potential operators for the Stokes system given by* (3.2) *are linear and continuous,*

$$\mathcal{N}_{\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to \mathcal{H}^1(\mathbb{R}^3)^3, \qquad \mathcal{Q}_{\mathbb{R}^3} : \mathcal{H}^{-1}(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3),$$

$$\mathcal{N}_{D_\pm} : \widetilde{\mathcal{H}}^{-1}(D_\pm)^3 \to \mathcal{H}^1(D_\pm)^3, \qquad \mathcal{Q}_{D_\pm} : \widetilde{\mathcal{H}}^{-1}(D_\pm)^3 \to L^2(D_\pm), \tag{3.3}$$

$$\mathcal{N}_{D_+} : \widetilde{H}^{-1}(D_+)^3 \to H^1(D_+)^3, \qquad \mathcal{Q}_{D_+} : \widetilde{H}^{-1}(D_+)^3 \to L^2(D_+). \tag{3.4}$$

In view of (3.1), the Newtonian potentials satisfy the equations (in the sense of distributions)

$$\Delta(\mathcal{N}_{\mathbb{R}^3}\mathfrak{F}) - \nabla(\mathcal{Q}_{\mathbb{R}^3}\mathfrak{F}) = \mathfrak{F}, \quad \operatorname{div}\mathcal{N}_{\mathbb{R}^3}\mathfrak{F} = 0 \quad \text{in } \mathbb{R}^3.$$

Now, for $\boldsymbol{\varphi} \in H^{-1/2}(\partial D)^3$, we introduce the single-layer potential $\mathbf{V}_{\partial D}\boldsymbol{\varphi}$ and its associated pressure potential $\mathcal{Q}^s_{\partial D}\boldsymbol{\varphi}$ defined by

$$(\mathbf{V}_{\partial D}\boldsymbol{\varphi}) := \langle \mathbf{G}(\mathbf{x},\cdot), \boldsymbol{\varphi}\rangle_{\partial D}, \quad (\mathbf{Q}^s_{\partial D}\boldsymbol{\varphi}) := \langle \mathbf{P}(\mathbf{x},\cdot), \boldsymbol{\varphi}\rangle_{\partial D}, \quad \forall\, \mathbf{x} \in \mathbb{R}^3 \setminus \partial D.$$

Let $\boldsymbol{\phi} \in H^{1/2}(\partial D)^3$. We define the double-layer potential $\mathbf{W}_{\partial D}\boldsymbol{\phi}$ and its associated pressure potential $\mathcal{Q}^d_{\partial D}\boldsymbol{\phi}$ for the Stokes system by

$$(\mathbf{W}_{\partial D}\boldsymbol{\phi})_k(\mathbf{x}) := \int_{\partial D} \mathbf{S}_{jkl}(\mathbf{y},\mathbf{x}), \nu_l(\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}, \quad \forall\, \mathbf{x} \in \mathbb{R}^3 \setminus \partial D,$$

$$(\mathcal{Q}^d_{\partial D}\boldsymbol{\phi})_k(\mathbf{x}) := \int_{\partial D} \mathbf{R}_{jl}(\mathbf{x},\mathbf{y}), \nu_l(\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}, \quad \forall\, \mathbf{x} \in \mathbb{R}^3 \setminus \partial D.$$

We also introduce, the principal value of $\mathbf{W}_{\partial D}\boldsymbol{\phi}$, denoted by $\mathbf{K}_{\partial D}\boldsymbol{\phi}$ and given by

$$(\mathbf{K}_{\partial D}\boldsymbol{\phi})_k(\mathbf{x}) := \text{p.v.}\int_{\partial D} \mathbf{S}_{jkl}(\mathbf{y},\mathbf{x})\nu_l(\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}$$
$$= \lim_{\varepsilon\to 0}\int_{\partial D\setminus(\partial D\cap B(x,\varepsilon))} \mathbf{S}_{jkl}(\mathbf{y},\mathbf{x})\nu_l(\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}\sigma_{\mathbf{y}}$$

for $x \in \partial D$ where this limit makes sense.

We have the following lemma that describes the properties of the layer potential operators associated to the Stokes system (see, e.g., [11, Lemma A.4]).

Lemma 3.2. *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary ∂D. Let $D_- := \mathbb{R}^3 \setminus \overline{D}$.*

(i) *The following operators are linear and bounded:*

$$(\mathbf{V}_{\partial D})|_{D_+}\colon H^{-1/2}(\partial D)^3 \to H^1(D_+)^3,\quad (\mathcal{Q}^s_{\partial D})|_{D_+}\colon H^{-1/2}(\partial D)^3 \to L^2(D_+), \tag{3.5}$$
$$(\mathbf{W}_{\partial D})|_{D_+}\colon H^{1/2}(\partial D)^3 \to H^1(D_+)^3,\quad (\mathcal{Q}^d_{\partial D})|_{D_+}\colon H^{1/2}(\partial D)^3 \to L^2(D_+), \tag{3.6}$$
$$(\mathbf{V}_{\partial D})|_{D_-}\colon H^{-1/2}(\partial D)^3 \to \mathcal{H}^1(D_-)^3,\quad (\mathcal{Q}^s_{\partial D})|_{D_-}\colon H^{-1/2}(\partial D)^3 \to L^2(D_-), \tag{3.7}$$
$$(\mathbf{W}_{\partial D})|_{D_-}\colon H^{1/2}(\partial D)^3 \to \mathcal{H}^1(D_-)^3,\quad (\mathcal{Q}^d_{\partial D})|_{D_-}\colon H^{1/2}(\partial D)^3 \to L^2(D_-), \tag{3.8}$$
$$(\mathbf{W}_{\partial D})|_{D_-}\colon H^{1/2}(\partial D)^3 \to H^1(D_-)^3.$$

(ii) *For $\boldsymbol{\phi} \in H^{1/2}(\partial D)^3$ and $\boldsymbol{\varphi} \in H^{-1/2}(\partial D)^3$, the following jump relations hold a.e. on ∂D:*

$$\gamma_+(\mathbf{V}_{\partial D}\boldsymbol{\varphi}) = \gamma_-(\mathbf{V}_{\partial D}\boldsymbol{\varphi}) =: \mathcal{V}_{\partial D}\boldsymbol{\varphi}, \tag{3.9}$$
$$\frac{1}{2}\boldsymbol{\phi} + \gamma_+(\mathbf{W}_{\partial D}\boldsymbol{\phi}) = -\frac{1}{2}\boldsymbol{\phi} + \gamma_-(\mathbf{W}_{\partial D}\boldsymbol{\phi}) =: \mathbf{K}_{\partial D}\boldsymbol{\phi}, \tag{3.10}$$
$$-\frac{1}{2}\boldsymbol{\varphi} + \partial^+_{0,\nu}(\mathbf{V}_{\partial D}\boldsymbol{\varphi}, \mathcal{Q}^s_{\partial D}\boldsymbol{\varphi}) = \frac{1}{2}\boldsymbol{\varphi} + \partial^-_{0,\nu}(\mathbf{V}_{\partial D}\boldsymbol{\varphi}, \mathcal{Q}^s_{\partial D}\boldsymbol{\varphi}) =: \mathbf{K}^*_{\partial D}\boldsymbol{\varphi}, \tag{3.11}$$
$$\partial^+_{0,\nu}(\mathbf{W}_{\partial D}\boldsymbol{\phi}, \mathcal{Q}^d_{\partial D}\boldsymbol{\phi}) = \partial^-_{0,\nu}(\mathbf{W}_{\partial D}\boldsymbol{\phi}, \mathcal{Q}^d_{\partial D}\boldsymbol{\phi}) =: \mathbf{D}_{\partial D}\boldsymbol{\phi},$$

where $\mathbf{K}^*_{\partial D}$ is the transpose of $\mathbf{K}_{\partial D}$. In addition, the following Stokes layer potential operators given by

$$\mathcal{V}_{\partial D}\colon H^{-1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3,\qquad \mathbf{K}_{\partial D}\colon H^{1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3,$$
$$\mathbf{K}^*_{\partial D}\colon H^{-1/2}(\partial D)^3 \to H^{-1/2}(\partial D)^3,\qquad \mathbf{D}_{\partial D}\colon H^{1/2}(\partial D)^3 \to H^{-1/2}(\partial D)^3$$

are linear and bounded.

4. The Poisson problem of transmission-type for the generalized Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$

Recall that $D_+ := D \subset \mathbb{R}^3$ is a bounded Lipschitz domain and denote by $D_- := \mathbb{R}^3 \setminus \overline{D}$ the exterior Lipschitz domain.

In the sequel, we need the following spaces

$$
\begin{aligned}
H^1_{\operatorname{div}}(D_+)^3 &:= \{\boldsymbol{w} \in H^1(D_+)^3 : \operatorname{div} \boldsymbol{w} = 0 \text{ in } D_+\},\\
\mathcal{H}^1_{\operatorname{div}}(D_-)^3 &:= \{\boldsymbol{w} \in \mathcal{H}^1(D_-)^3 : \operatorname{div} \boldsymbol{w} = 0 \text{ in } D_-\},\\
\mathfrak{X} &:= (H^1_{\operatorname{div}}(D_+)^3 \times L^2(D_+)) \times (\mathcal{H}^1_{\operatorname{div}}(D_-)^3 \times L^2(D_-)),\\
\mathfrak{Y} &:= \widetilde{H}^{-1}(D_+)^3 \times \widetilde{\mathcal{H}}^{-1}(D_-)^3 \times H^{1/2}(\partial D)^3 \times H^{-1/2}(\partial D)^3,\\
\mathfrak{Y}_\infty &:= \widetilde{H}^{-1}(D_+)^3 \times \widetilde{\mathcal{H}}^{-1}(D_-)^3 \times H^{1/2}(\partial D)^3 \times H^{-1/2}(\partial D)^3 \times \mathbb{R}^3.
\end{aligned}
$$

Note that these spaces $H^1_{\operatorname{div}}(D_+)^3$ and $\mathcal{H}^1_{\operatorname{div}}(D_-)^3$ are endowed with the norms of $H^1(D_+)^3$ and $\mathcal{H}^1(D_-)^3$, respectively. For the spaces $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Y}_\infty$, the norm of an element is defined as the sum of the norms of its components in their corresponding spaces.

4.1. The Poisson problem of transmission-type for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$

Before we state and prove the main theorem of this paper we shall give some important results that play a crucial role in the remainder of the paper.

We consider now, the Poisson problem of transmission-type for the Stokes system in both domains D_+ and D_-. The problem is stated as follows:

$$
\begin{cases}
\Delta \boldsymbol{w}_+ - \nabla p_+ = \boldsymbol{\eta}_+|_{D_+} & \text{in } D_+,\\
\Delta \boldsymbol{w}_- - \nabla p_- = \boldsymbol{\eta}_-|_{D_-} & \text{in } D_-,\\
\gamma_+\boldsymbol{w}_+ - \gamma_-\boldsymbol{w}_- = \boldsymbol{\mu}_0 & \text{on } \partial D,\\
\partial^+_{0,\nu}(\boldsymbol{w}_+, p_+, \boldsymbol{\eta}_+) - \partial^-_{0,\nu}(\boldsymbol{w}_-, p_-, \boldsymbol{\eta}_-) + \mathbf{L}\gamma_+\boldsymbol{w}_+ = \boldsymbol{\zeta}_0 & \text{on } \partial D,
\end{cases}
\tag{4.1}
$$

where $\mathbf{L} \in L^\infty(\partial D)^{3\times 3}$ is a symmetric matrix valued function, which satisfies the following positivity condition

$$
\langle \mathbf{L}\boldsymbol{v}, \boldsymbol{v}\rangle_{\partial D} \geq 0, \quad \forall\, \boldsymbol{v} \in L^2(\partial D)^3. \tag{4.2}
$$

We shall proceed on with the following lemma that shows us that our transmission problem (4.1) has at most one solution such that

$$
(\boldsymbol{w}_+, p_+, \boldsymbol{w}_- - \boldsymbol{w}_\infty, p_-) \in \mathfrak{X}, \tag{4.3}
$$

where $\boldsymbol{w}_\infty \in \mathbb{R}^3$ is a given constant.

Lemma 4.1. *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary ∂D. Let $D_- := \mathbb{R}^3 \setminus \overline{D}$ be the corresponding complementary set. Let $\mathbf{L} \in L^\infty(\partial D)^{3\times 3}$ be a symmetric matrix valued function such that the positivity condition* (4.2) *holds. Then for the given data $(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0, \boldsymbol{w}_\infty) \in \mathfrak{Y}_\infty$, the transmission problem* (4.1) *has at most one solution $(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-)$ such that* (4.3) *holds.*

Proof. Let $(\boldsymbol{w}_+^0, p_+^0, \boldsymbol{w}_-^0, p_-^0) \in \mathfrak{X}$ satisfy the homogeneous version of our problem (4.1). Then, the vector fields $\boldsymbol{w}_\pm^0$ admit the following layer potential representation (cf. e.g., [21, Proposition 10.6.1], see also [19, Relations (4.4) and (4.5), Lemma 4.2])

$$\boldsymbol{w}_+^0 = \mathbf{V}_{\partial D}(\partial_{0,\nu}^+(\boldsymbol{w}_+^0, p_+^0, 0)) - \mathbf{W}_{\partial D}(\gamma_+ \boldsymbol{w}_+^0), \tag{4.4}$$

$$\boldsymbol{w}_-^0 = -\mathbf{V}_{\partial D}(\partial_{0,\nu}^-(\boldsymbol{w}_-^0, p_-^0, 0)) + \mathbf{W}_{\partial D}(\gamma_- \boldsymbol{w}_-^0). \tag{4.5}$$

Apply γ_+ to relation (4.4) and γ_- to relation (4.5) while taking into account relations (3.9) and (3.10), and obtain

$$\left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\partial D}\right)(\gamma_+ \boldsymbol{w}_+^0) = \mathcal{V}_{\partial D}(\partial_{0,\nu}^+(\boldsymbol{w}_+^0, p_+^0, 0)), \tag{4.6}$$

$$\left(-\frac{1}{2}\mathbb{I} + \mathbf{K}_{\partial D}\right)(\gamma_- \boldsymbol{w}_-^0) = \mathcal{V}_{\partial D}(\partial_{0,\nu}^-(\boldsymbol{w}_-^0, p_-^0, 0)). \tag{4.7}$$

By subtracting (4.7) from (4.6) and taking into account the transmission conditions of the homogeneous version of (4.1) we obtain the equation

$$(\mathbb{I} + \mathcal{V}_{\partial D}\mathbf{L})(\gamma_+ \boldsymbol{w}_+^0) = 0. \tag{4.8}$$

Next, we show that the operator

$$\mathbb{I} + \mathcal{V}_{\partial D}\mathbf{L} \colon H^{1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3 \tag{4.9}$$

is an isomorphism. To this end, notice that the mappings

$$\mathbf{L} \colon L^2(\partial D)^3 \to L^2(\partial D)^3, \quad \mathcal{V}_{\partial D} \colon H^{-1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3$$

are linear and continuous operators, while the embedding

$$L^2(\partial D)^3 \hookrightarrow H^{-1/2}(\partial D)^3$$

is continuous and compact. Therefore, the operator

$$\mathcal{V}_{\partial D}\mathbf{L} \colon H^{1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3$$

is linear and continuous, even compact and we get that the operator (4.9) is a Fredholm operator of index 0.

Due to the fact that $\mathbf{L}\colon L^2(\partial D)^3 \to L^2(\partial D)^3$ and $\mathcal{V}_{\partial D}\colon H^{-1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3$ are self-adjoint operators, we get that the operator

$$\mathbb{I} + \mathbf{L}\mathcal{V}_{\partial D}\colon H^{-1/2}(\partial D)^3 \to H^{-1/2}(\partial D)^3 \tag{4.10}$$

is the adjoint of (4.9) and hence it is Fredholm of index 0 as well (see, e.g., [9, Theorem 5.3.7]).

In order to show that the operator (4.9) is an isomorphism, we would need to prove that (4.9) is one-to-one, or equivalently, that (4.10) is one-to-one.

To this end, we consider

$$\boldsymbol{\varphi}_0 \in \operatorname{Ker}\{\mathbb{I} + \mathbf{L}\mathcal{V}_{\partial D}\colon H^{-1/2}(\partial D)^3 \to H^{-1/2}(\partial D)^3\},$$

which is equivalent to the fact that

$$-\mathbf{L}\mathcal{V}_{\partial D}\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0. \tag{4.11}$$

We introduce now, the following fields $\boldsymbol{w}_0 := \mathbf{V}_{\partial D}\boldsymbol{\varphi}_0$ and $p_0 = \mathcal{Q}^s_{\partial D}\boldsymbol{\varphi}_0$. Obviously, the pair $(\boldsymbol{w}_0, p_0)$ satisfies the Stokes system in $D_\pm$ and using relations (3.9) and (3.11) the following relations hold:

$$\gamma_+\boldsymbol{w}_0 = \gamma_-\boldsymbol{w}_0, \quad \partial^+_{0,\nu}(\boldsymbol{w}_0, p_0, 0) - \partial^+_{0,\nu}(\boldsymbol{w}_0, p_0, 0) = \boldsymbol{\varphi}_0. \tag{4.12}$$

If we apply the Green formulas (2.1), we obtain

$$\langle \partial^+_{0,\nu}(\boldsymbol{w}_0, p_0, 0), \gamma_+\boldsymbol{w}_0\rangle_{\partial D} = 2\langle \mathbf{E}_{ij}(\boldsymbol{w}_0), \mathbf{E}_{ij}(\boldsymbol{w}_0)\rangle_{D_+} \tag{4.13}$$

and

$$-\langle \partial^-_{0,\nu}(\boldsymbol{w}_0, p_0, 0), \gamma_-\boldsymbol{w}_0\rangle_{\partial D} = 2\langle \mathbf{E}_{ij}(\boldsymbol{w}_0), \mathbf{E}_{ij}(\boldsymbol{w}_0)\rangle_{D_-}. \tag{4.14}$$

Adding the relations (4.13), (4.14) and taking into account relations (4.11) and (4.12), we deduce the identity

$$-\langle \mathbf{L}\mathcal{V}_{\partial D}\boldsymbol{\varphi}_0, \mathcal{V}_{\partial D}\boldsymbol{\varphi}_0\rangle_{\partial D} = 2\langle \mathbf{E}_{ij}(\boldsymbol{w}_0), \mathbf{E}_{ij}(\boldsymbol{w}_0)\rangle_{D_+} + 2\langle \mathbf{E}_{ij}(\boldsymbol{w}_0), \mathbf{E}_{ij}(\boldsymbol{w}_0)\rangle_{D_-}. \tag{4.15}$$

In view of the positivity condition (4.2), we deduce that each side of (4.15) is null and hence

$$\mathbf{E}_{ij}(\boldsymbol{w}_0) = 0 \quad \text{in } D_\pm.$$

It follows that, there exists (see, e.g., [17, Lemma 3.1]) skew-symmetric matrices $B_\pm$ (i.e., $B^T_\pm = -B_\pm$), and constants $a_\pm \in \mathbb{R}^3$ such that

$$\boldsymbol{w}_0 = a_\pm + B_\pm \times \boldsymbol{x} \quad \text{in } D_\pm.$$

However, $\boldsymbol{w}_0(x) \to 0$ as $|x| \to \infty$, and hence,

$$a_- = 0, \quad B_- = 0.$$

Consequently, $\boldsymbol{w}_0 = 0$ in D_- and thus $\gamma_- \boldsymbol{w}_0 = 0$ on ∂D.

Since $\boldsymbol{w}_0 := \mathbf{V}_{\partial D} \boldsymbol{\varphi}_0$, we have

$$\gamma_+ \boldsymbol{w}_0 = \gamma_- \boldsymbol{w}_0 = 0. \tag{4.16}$$

Moreover, $p_0 = 0$ in D_-, by the fact that $p(x) \to 0$ as $|x| \to \infty$.

Note that, due to (4.16), the pair $(\boldsymbol{w}_0, p_0)$ is a solution of the interior Dirichlet problem for the Stokes system. Since this problem has at most one solution up to an additive constant pressure in $H^1(D_+)^3 \times L^2(D_+)$, we obtain (see, e.g., [21, Theorem 10.6.2])

$$\boldsymbol{w}_0 = 0, \quad p_0 = c \quad \text{in } D_+,$$

where $c \in \mathbb{R}$ is a constant.

We have obtained that

$$\boldsymbol{w}_0 = 0 \ \text{ in } D_\pm, \quad p_0 = c \in \mathbb{R} \ \text{ in } D_+, \quad p_0 = 0 \ \text{ in } D_-. \tag{4.17}$$

In view of the second transmission condition and the relation (4.17), we deduce that

$$0 = \partial_{0,\nu}^+(\boldsymbol{w}_0, p_0, 0) = -c\boldsymbol{\nu},$$

and hence $p_0 = 0$. Now, it follows that

$$\boldsymbol{w}_0 = \mathbf{V}_{\partial D} \varphi_0 = 0, \quad p_0 = \mathcal{Q}^s_{\partial D} \varphi_0 = 0 \quad \text{in } D_\pm.$$

Since $\gamma_\pm \boldsymbol{w}_0 = \mathcal{V}_{\partial D} \boldsymbol{\varphi}_{\mathbf{0}}$, we deduce that

$$\mathcal{V}_{\partial D} \boldsymbol{\varphi}_{\mathbf{0}} = 0. \tag{4.18}$$

Moreover, by (4.11) and (4.18) it follows that $\boldsymbol{\varphi}_{\mathbf{0}} = 0$. This shows that the Fredholm operator of index zero (4.10), and thus an isomorphism.

Consequently, the operator (4.9) is an isomorphism as well (see, e.g., [9, Theorem 5.3.7]). Then, the equation (4.8) has a unique solution, i.e., $\gamma_+ \boldsymbol{w}_+^0 = 0$ and taking into account the layer potential representation (4.4), we obtain

$$\boldsymbol{w}_+^0 = \mathbf{V}_{\partial D}(\partial_{0,\nu}^+(\boldsymbol{w}_+^0, p_+^0, 0)) = 0 \quad \text{in } D_+. \tag{4.19}$$

By applying the trace operator γ_+ to the relation (4.19), we obtain

$$\mathcal{V}_{\partial D}(\partial_{0,\nu}^+(\boldsymbol{w}_+^0, p_+^0, 0)) = 0 \quad \text{on } \partial D.$$

It follows that (see, e.g., [14, Lemma 3.1]),

$$\partial^{+}_{0,\nu}(\boldsymbol{w}^0_+, p^0_+, 0) \in \operatorname{Ker}\{\mathcal{V}_{\partial D}\colon H^{-1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3\} = \mathbb{R}\nu. \tag{4.20}$$

Using (4.19), (4.20) and the fact that $\mathcal{V}_{\partial D}\nu = 0$ in $\mathbb{R}^3$ (see, e.g., [21, Lemma 5.3.1]), we obtain

$$\boldsymbol{w}^0_+ = 0 \quad \text{in } D_+. \tag{4.21}$$

The transmission conditions give $\gamma_- \boldsymbol{w}^0_- = 0$, $\partial^{-}_{0,\nu}(\boldsymbol{w}^0_-, p^0_-, 0) = 0$. So,

$$\boldsymbol{w}^0_- = 0 \quad \text{in } D_-, \tag{4.22}$$

by relation (4.5).

Using Stokes' equations and the relations (4.21), (4.22), we deduce that

$$p^0_\pm = c_\pm \in \mathbb{R} \quad \text{in } D_\pm.$$

Recall that $p^0_- \in L^2(D_-)$. Hence $c_0 = 0$ and we have

$$p^0_- = 0 \quad \text{in } D_-. \tag{4.23}$$

Using the second transmission condition of the homogeneous version of (4.1) and the relations (4.21), (4.22), (4.23), we get

$$0 = \partial^{+}_{0,\nu}(\boldsymbol{w}^0_+, p^0_+, 0) = -c_+\boldsymbol{\nu}.$$

It follows that $c_+ = 0$.

Consequently, we have shown that

$$\boldsymbol{w}^0_+ = 0,\ \ p^0_+ = 0 \ \text{ in } D_+, \quad \boldsymbol{w}^0_- = 0,\ \ p^0_- = 0 \ \text{ in } D_-,$$

i.e., the problem (4.1) has at most one solution. This completes our proof. □

In the proof of Lemma 4.1 we have proved the following corollary.

Corollary 4.2. *The operators* (4.9) *and* (4.10) *are isomorphisms.*

At this moment, we are ready to state and prove the existence result of our transmission problem (4.1) in the case $\boldsymbol{w}_\infty = 0$ (see also [11, Theorem 4.2]).

Theorem 4.3. *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary ∂D. Let $D_- := \mathbb{R}^3 \setminus \overline{D}$ be the corresponding complementary set. Let $\mathbf{L} \in L^\infty(\partial D)^{3\times 3}$ be a symmetric matrix valued function which satisfies the positivity condition* (4.2)*. Then for* *the given data* $(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0) \in \mathfrak{Y}$, *the transmission problem* (4.1) *has a unique solution* $(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) \in \mathfrak{X}$. *In addition, there is a linear and continuous 'solution' operator*

$$\mathcal{T} : \mathfrak{Y} \to \mathfrak{X} \tag{4.24}$$

that maps $(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0) \in \mathfrak{Y}$ *to the corresponding solution* $(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) \in \mathfrak{X}$ *of the transmission problem* (4.1). *Hence, there is a constant* $C \equiv C(D_+, D_-, \mathbf{L}) > 0$ *such that*

$$\|(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-)\|_{\mathfrak{X}} \leq C\|(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0)\|_{\mathfrak{Y}}.$$

Moreover, $\boldsymbol{w}_-$ *vanishes at infinity in the sense of Leray, i.e.,*

$$\lim_{r\to\infty} \int_{S^2} \boldsymbol{w}_-(r\mathbf{y})\, \mathrm{d}\sigma_{\mathbf{y}} = 0. \tag{4.25}$$

Proof. We note that, this result is a consequence of Theorem 4.2 in [11]. Next, we give an alternative proof for our statement.

We construct a solution of the problem (4.1) as a combination of Newtonian, single-layer and double-layer potentials, as follows:

$$\begin{aligned} \boldsymbol{w}_\pm &= \mathcal{N}_{D_\pm}\boldsymbol{\eta}_\pm + \mathbf{V}_{\partial D}\boldsymbol{\varphi} + \mathbf{W}_{\partial D}\boldsymbol{\phi} \quad \text{in } D_\pm, \\ p_\pm &= \mathcal{Q}_{D_\pm}\boldsymbol{\eta}_\pm + \mathcal{Q}^s_{\partial D}\boldsymbol{\varphi} + \mathcal{Q}^d_{\partial D}\boldsymbol{\phi} \quad \text{in } D_\pm \end{aligned}$$

with the unknown densities $\boldsymbol{\varphi} \in H^{-1/2}(\partial D)^3$ and $\boldsymbol{\phi} \in H^{1/2}(\partial D)^3$.

Note that, due to the mapping properties (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8), we deduce that

$$(\boldsymbol{w}_+, p_+) \in H^1(D_+)^3 \times L^2(D_+), \quad (\boldsymbol{w}_-, p_-) \in \mathcal{H}^1(D_-)^3 \times L^2(D_-). \tag{4.26}$$

Using the transmission condition $(4.1)_3$, and the second statement from Lemma 3.2, we obtain after computation that the density $\boldsymbol{\phi} \in H^{1/2}(\partial D)^3$ is given by

$$\boldsymbol{\phi} = (\gamma_+(\mathcal{N}_{D_+}\boldsymbol{\eta}_+) - \gamma_-(\mathcal{N}_{D_-}\boldsymbol{\eta}_-)) - \boldsymbol{\mu}_0. \tag{4.27}$$

Exploit now the transmission condition $(4.1)_4$, and yet again the second statement form Lemma 3.2, we get, after computations, the following equation

$$(\mathbb{I} + \mathbf{L}\mathcal{V}_{\partial D})\boldsymbol{\varphi} = \boldsymbol{\zeta} \in H^{-1/2}(\partial D)^3, \tag{4.28}$$

where $\boldsymbol{\zeta} \in H^{-1/2}(\partial D)^3$ is given by

$$\begin{aligned} \boldsymbol{\zeta} := \boldsymbol{\zeta}_0 &- (\partial^+_{0,\nu}(\mathcal{N}_{D_+}\boldsymbol{\eta}_+, \mathcal{Q}_{D_+}\boldsymbol{\eta}_+, 0) - \partial^-_{0,\nu}(\mathcal{N}_{D_-}\boldsymbol{\eta}_-, \mathcal{Q}_{D_-}\boldsymbol{\eta}_-, 0)) \\ &+ \mathbf{L}(\gamma_+\mathcal{N}_{D_+}\boldsymbol{\eta}_+) + \mathbf{L}\left(-\frac{1}{2}\mathbb{I} + \mathbf{K}_{\partial D}\right)\boldsymbol{\phi}. \end{aligned} \tag{4.29}$$

Recall that the operator

$$\mathbb{I} + \mathbf{L}\mathcal{V}_{\partial D}\colon H^{-1/2}(\partial D)^3 \to H^{-1/2}(\partial D)^3$$

is an isomorphism due to Corollary 4.2. In conclusion, the solution of equation (4.28) is unique, and given by

$$\boldsymbol{\varphi} = (\mathbb{I} + \mathbf{L}\mathcal{V}_{\partial D})^{-1}\boldsymbol{\zeta} \in H^{-1/2}(\partial D)^3, \tag{4.30}$$

where $\boldsymbol{\zeta}$ is given by relation (4.29).

Consequently, the densities $\boldsymbol{\varphi}$ and $\boldsymbol{\phi}$ given by relations (4.27) and (4.30), together with the layer potential representations (4.26) determine a solution of the problem (4.1) in the space $\mathfrak{X}$, which is unique, as we have already proved in Lemma 4.1.

Since $(\boldsymbol{w}_-, p_-)$ satisfy the Stokes system in the exterior domain, we deduce in view of Corollary 2.2, that $\boldsymbol{w}_-$ vanishes at ∞ in the sense of Leray, i.e., (4.25) holds. Moreover, the well-posedness of the problem (4.1) implies immediately that the solution operator (4.24) is linear and continuous.

This concludes the proof. □

Now, we state the result in the case $\boldsymbol{w}_\infty \neq 0$ (see also [11, Theorem 4.4]).

Theorem 4.4. *Let $D_+ := D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary ∂D. Let $D_- := \mathbb{R}^3 \setminus \overline{D}$ be the exterior Lipschitz domain. Let $\mathbf{L} \in L^\infty(\partial D)^{3\times 3}$ be a symmetric matrix valued function such that the positivity condition* (4.2) *holds. Then for the given data $(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0, \boldsymbol{w}_\infty) \in \mathcal{Y}_\infty$, the transmission problem* (4.1) *has a unique solution $(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-)$ satisfying the condition* (4.3). *Moreover, there is a constant $C \equiv C(D_+, D_-, \mathbf{L}) > 0$ such that*

$$\|(\boldsymbol{w}_+, p_+, \boldsymbol{w}_- - \boldsymbol{w}_\infty, p_-)\|_{\mathfrak{X}} \le C\|(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0, \boldsymbol{w}_\infty)\|_{\mathfrak{Y}_\infty},$$

and $\boldsymbol{w}_- - \boldsymbol{w}_\infty$ vanishes at infinity in the sense of Leray.

Proof. We consider the new unknowns $\boldsymbol{v}_+ := \boldsymbol{w}_+$ and $\boldsymbol{v}_- := \boldsymbol{w}_- - \boldsymbol{w}_\infty$ and we write the problem (4.1) in the following equivalent form

$$\begin{cases} \Delta \boldsymbol{v}_+ - \nabla p_+ = \boldsymbol{\eta}_+|_{D_+} & \text{in } D_+,\\ \Delta \boldsymbol{v}_- - \nabla p_- = \boldsymbol{\eta}_-|_{D_-} & \text{in } D_-,\\ \gamma_+\boldsymbol{v}_+ - \gamma_-\boldsymbol{v}_- = \boldsymbol{\mu}_0 + \boldsymbol{w}_\infty & \text{on } \partial D,\\ \partial^+_{0,\nu}(\boldsymbol{v}_+, p_+, \boldsymbol{\eta}_+) - \partial^-_{0,\nu}(\boldsymbol{v}_-, p_-, \boldsymbol{\eta}_-) + \mathbf{L}\gamma_+\boldsymbol{v}_+ = \boldsymbol{\zeta}_0 & \text{on } \partial D \end{cases} \tag{4.31}$$

for $(\boldsymbol{v}_+, p_+, \boldsymbol{v}_-, p_-) \in \mathfrak{X}$, already considered at Theorem 4.3. Due to the fact that $\boldsymbol{w}_\infty$ appears in the right-hand side of (4.31), it will also be involved in the right-hand side of the estimate.

It remains to apply Theorem 4.3 in order to deduce the desired result. □

4.2. The Poisson problem of transmission-type for the generalized Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$

We are ready to give the main result of this paper. The considered problem is the following

$$
\begin{cases}
\Delta \boldsymbol{w}_+ - \mathcal{P}\boldsymbol{w}_+ - \nabla p_+ = \boldsymbol{\eta}_+|_{D_+} \quad \text{in } D_+,\\
\Delta \boldsymbol{w}_- - \nabla p_- = \boldsymbol{\eta}_-|_{D_-} \quad \text{in } D_-,\\
\gamma_+\boldsymbol{w}_+ - \gamma_-\boldsymbol{w}_- = \boldsymbol{\mu}_0 \quad \text{on } \partial D,\\
\partial^+_{\mathcal{P},\nu}(\boldsymbol{w}_+, p_+, \boldsymbol{\eta}_+) - \partial^-_{0,\nu}(\boldsymbol{w}_-, p_-, \boldsymbol{\eta}_-) + \mathbf{L}\gamma_+\boldsymbol{w}_+ = \boldsymbol{\zeta}_0 \quad \text{on } \partial D.
\end{cases}
\tag{4.32}
$$

We have the following theorem in the case $\boldsymbol{w}_\infty = 0$.

Theorem 4.5. *Let $D_+ := D \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary ∂D and let $D_- := \mathbb{R}^3 \setminus \overline{D}$ the exterior Lipschitz domain. Let $\mathcal{P} \in L^\infty(D_+)^{3\times 3}$ such that condition* (2.3) *holds. Let $\mathbf{L} \in L^\infty(\partial D)^{3\times 3}$ be a symmetric matrix valued function that satisfies condition* (4.2). *Then for $(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0) \in \mathfrak{Y}$ given, the Poisson problem of transmission-type for Stokes and generalized Brinkman systems* (4.32) *has a unique solution $((\boldsymbol{w}_+, p_+), (\boldsymbol{w}_-, p_-)) \in \mathfrak{X}$. Moreover, there is a constant $C \equiv C(D_+, D_-, \mathcal{P}, \mathbf{L}) > 0$ such that*

$$
\|(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-)\|_{\mathfrak{X}} \le C\|(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0)\|_{\mathfrak{Y}},
$$

and $\boldsymbol{w}_-$ vanishes at infinity in the sense of Leray.

Proof. Step 1: Let $\mathcal{P} \equiv 0$. In this case, Theorem 4.3 asserts that the problem (4.1) is well-posed. In addition, the corresponding 'solution' operator

$$
\mathcal{T}\colon \mathfrak{Y} \to \mathfrak{X}
$$

is well-defined, linear, continuous. This result follows from Lemma 4.1 and Theorem 4.2 of [11].

Step 2*: Uniqueness in the case $\mathcal{P} \neq 0$.* Consider the homogenenous problem associated to our transmission problem (4.32), namely

$$
\begin{cases}
\Delta \boldsymbol{w}^0_+ - \mathcal{P}\boldsymbol{w}^0_+ - \nabla p^0_+ = 0 \quad \text{in } D_+,\\
\Delta \boldsymbol{w}^0_- - \nabla p^0_- = 0 \quad \text{in } D_-,\\
\gamma_+\boldsymbol{w}^0_+ - \gamma_-\boldsymbol{w}^0_- = 0 \quad \text{on } \partial D,\\
\partial^+_{\mathcal{P},\nu}(\boldsymbol{w}^0_+, p^0_+, 0) - \partial^-_{0,\nu}(\boldsymbol{w}^0_-, p^0_-, 0) + \mathbf{L}\gamma_+\boldsymbol{w}^0_+ = 0 \quad \text{on } \partial D,
\end{cases}
\tag{4.33}
$$

where we have denoted by $((\boldsymbol{w}^0_+, p^0_+), (\boldsymbol{w}^0_-, p^0_-)) \in \mathcal{X}$ the difference between two possible solutions of the problem (4.32).

By applying the Green formulas (2.4) and (2.2), we obtain the relations

$$\langle \partial^{+}_{\mathcal{P},\nu}(\boldsymbol{w}^0_+, p^0_+, 0), \gamma_+ \boldsymbol{w}^0_+\rangle_{\partial D} = 2\langle \mathbf{E}(\boldsymbol{w}^0_+), \mathbf{E}(\boldsymbol{w}^0_+)\rangle_{D_+} + \langle \mathcal{P}\boldsymbol{w}^0_+, \boldsymbol{w}^0_+\rangle_{D_+} \tag{4.34}$$

and

$$\langle \partial^{-}_{0,\nu}(\boldsymbol{w}^0_-, p^0_-, 0), \gamma_- \boldsymbol{w}^0_-\rangle_{\partial D} = -2\langle \mathbf{E}(\boldsymbol{w}^0_-), \mathbf{E}(\boldsymbol{w}^0_-)\rangle_{D_-}. \tag{4.35}$$

On the other hand, subtracting the relations (4.34) and (4.35), we get

$$\begin{aligned}&\langle \partial^{+}_{\mathcal{P},\nu}(\boldsymbol{w}^0_+, p^0_+, 0), \gamma_+ \boldsymbol{w}^0_+\rangle_{\partial D} - \langle \partial^{-}_{0,\nu}(\boldsymbol{w}^0_-, p^0_-, 0), \gamma_- \boldsymbol{w}^0_-\rangle_{\partial D}\\ &= 2\langle \mathbf{E}(\boldsymbol{w}^0_+), \mathbf{E}(\boldsymbol{w}^0_+)\rangle_{D_+} + 2\langle \mathbf{E}(\boldsymbol{w}^0_-), \mathbf{E}(\boldsymbol{w}^0_-)\rangle_{D_-} + \langle \mathcal{P}\boldsymbol{w}^0_+, \boldsymbol{w}^0_+\rangle_{D_+}.\end{aligned}$$

In addition, by using the transmission conditions, we deduce that

$$\langle \partial^{+}_{\mathcal{P},\nu}(\boldsymbol{w}^0_+, p^0_+, 0), \gamma_+ \boldsymbol{w}^0_+\rangle_{\partial D} - \langle \partial^{-}_{0,\nu}(\boldsymbol{w}^0_-, p^0_-, 0), \gamma_- \boldsymbol{w}^0_-\rangle_{\partial D} = -\langle \mathbf{L}\gamma_+ \boldsymbol{w}^0_+, \gamma_+ \boldsymbol{w}^0_+\rangle_{\partial D},$$

and hence

$$-\langle \mathbf{L}\gamma_+ \boldsymbol{w}^0_+, \gamma_+ \boldsymbol{w}^0_+\rangle_{\partial D} = 2\langle \mathbf{E}(\boldsymbol{w}^0_+), \mathbf{E}(\boldsymbol{w}^0_+)\rangle_{D_+} + 2\langle \mathbf{E}(\boldsymbol{w}^0_-), \mathbf{E}(\boldsymbol{w}^0_-)\rangle_{D_-} + \langle \mathcal{P}\boldsymbol{w}^0_+, \boldsymbol{w}^0_+\rangle_{D_+}.$$

In view of the positivity condition imposed on $\mathbf{L}$, we obtain

$$\begin{gathered}2\langle \mathbf{E}(\boldsymbol{w}^0_+), \mathbf{E}(\boldsymbol{w}^0_+)\rangle_{D_+} + 2\langle \mathbf{E}(\boldsymbol{w}^0_-), \mathbf{E}(\boldsymbol{w}^0_-)\rangle_{D_-} + \langle \mathcal{P}\boldsymbol{w}^0_+, \boldsymbol{w}^0_+\rangle_{D_+} = 0,\\ \langle \mathbf{L}\gamma_+ \boldsymbol{w}^0_+, \gamma_+ \boldsymbol{w}^0_+\rangle_{\partial D} = 0.\end{gathered}$$

In such a case, from the positivity condition satisfied by $\mathcal{P}$, we deduce that

$$\boldsymbol{w}^0_+ = 0 \text{ in } D_+, \quad \mathbf{E}(\boldsymbol{w}^0_\pm) = 0 \text{ in } D_\pm. \tag{4.36}$$

Since $\nabla p^0_+ = \boldsymbol{\Delta}\boldsymbol{w}^0_+ - \mathcal{P}\boldsymbol{w}^0_+ = 0$, the function p^0_+ is constant, i.e.,

$$p^0_+ = c \in \mathbb{R} \quad \text{in } D_+. \tag{4.37}$$

Using again the first transmission condition we obtain

$$\gamma_+ \boldsymbol{w}^0_+ = \gamma_- \boldsymbol{w}^0_- = 0.$$

Then we deduce that $(\boldsymbol{w}^0_-, p^0_-) \in \mathcal{H}^1_{\mathrm{div}}(D_-)^3 \times L^2(D_-)$ is the solution of the exterior Dirichlet problem associated to the Stokes system with homogeneous Dirichlet boundary condition. But this boundary value problem has a unique solution (see, e.g., [6, Theorem 3.4]). Hence

$$\boldsymbol{w}^0_- = 0, \quad p^0_- = 0 \quad \text{in } D_-.$$

From the second transmission condition, we deduce that

$$\partial^{+}_{\mathcal{P},\nu}(\boldsymbol{w}^0_+, p^0_+, 0) = 0. \tag{4.38}$$

A straightforward computation based on (4.36) and (4.37), yields the fact that

$$\partial^{+}_{\mathcal{P},\nu}(\boldsymbol{w}^0_+, p^0_+, 0) = -c\boldsymbol{\nu}. \tag{4.39}$$

Then, from (4.38) and (4.39), we obtain $c = 0$. Consequently, we have proved that

$$\boldsymbol{w}^0_+ = 0, \quad p^0_+ = 0 \quad \text{in } D_+,$$

and hence, that our transmission problem (4.32), has at most one solution.

Step 3: Existence in the case $\mathcal{P} \neq 0$. In the latter, we use similar arguments to those in the proof of [15, Theorem 4.4].

We rewrite our transmission problem in the following form

$$\begin{cases} \Delta \boldsymbol{w}_+ - \nabla p_+ = \boldsymbol{\eta}_+|_{D_+} + \mathring{E}(\mathcal{P}\boldsymbol{w}_+) & \text{in } D_+, \\ \Delta \boldsymbol{w}_- - \nabla p_- = \boldsymbol{\eta}_-|_{D_-} & \text{in } D_-, \\ \gamma_+\boldsymbol{w}_+ - \gamma_-\boldsymbol{w}_- = \boldsymbol{\mu}_0 & \text{on } \partial D, \\ \partial^{+}_{\mathcal{P},\nu}(\boldsymbol{w}_+, p_+, \boldsymbol{\eta}_+) - \partial^{-}_{0,\nu}(\boldsymbol{w}_-, p_-, \boldsymbol{\eta}_-) + \mathbf{L}\gamma_+\boldsymbol{w}_+ = \boldsymbol{\zeta}_0 & \text{on } \partial D, \end{cases} \tag{4.40}$$

where $\mathring{E}$ is the extension by 0 operator outside D_+.

In order to show the role of the extension operator $\mathring{E}$ in (4.40), we note that the condition $\mathcal{P} \in L^\infty(D_+)^{3\times 3}$ implies that

$$\mathcal{P}\boldsymbol{w}_+ \in L^2(D_+)^3 \hookrightarrow \widetilde{H}^{-1}(D_+)^3 \tag{4.41}$$

in the sense that

$$\mathring{E}(\mathcal{P}\boldsymbol{w}_+) \in \widetilde{H}^{-1}(D_+)^3.$$

Moreover, the embedding (4.41) is compact.

In view of the well-posedness result at Step 1, the problem (4.40) can be written equivalently in terms of the solution operator $\mathcal{T}$ as

$$(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) = \mathcal{T}(\boldsymbol{\eta}_+ + \mathring{E}(\mathcal{P}\boldsymbol{w}_+), \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0). \tag{4.42}$$

Moreover, in view of the linearity of $\mathcal{T}$, we have that

$$\mathcal{T}(\boldsymbol{\eta}_+ + \mathring{E}(\mathcal{P}\boldsymbol{w}_+), \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0) = \mathcal{T}(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0) + \mathcal{T}(\mathring{E}(\mathcal{P}\boldsymbol{w}_+), 0, 0, 0).$$

Hence, (4.42) becomes

$$(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) - \mathcal{T}(\mathring{E}(\mathcal{P}\boldsymbol{w}_+), 0, 0, 0) = \mathcal{T}(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0). \tag{4.43}$$

In view of the compact embedding (4.41), the linear operator

$$\mathcal{T}_{\mathcal{P}}\colon \mathfrak{X} \to \mathfrak{X}$$

given by

$$\mathcal{T}_{\mathcal{P}}(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) := \mathcal{T}(\mathring{E}(\mathcal{P}\boldsymbol{w}_+), 0, 0, 0)$$

is compact.

We can introduce now the operator

$$\mathcal{A}_{\mathcal{P}} := \mathbb{I} - \mathcal{T}_{\mathcal{P}}\colon \mathfrak{X} \to \mathfrak{X},$$

which is a Fredholm operator of index 0. Moreover, the equation (4.43) becomes

$$\mathcal{A}_{\mathcal{P}}(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) = \mathcal{T}(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0). \tag{4.44}$$

It remains to show that the operator $\mathcal{A}_{\mathcal{P}}$ is injective. To prove the injectivity of $\mathcal{A}_{\mathcal{P}}$, take into account that the following equation

$$\mathcal{A}_{\mathcal{P}}(\boldsymbol{w}_+^0, p_+^0, \boldsymbol{w}_-^0, p_-^0) = 0$$

is equivalent to the homogeneous problem (4.33).

However, at the second step we proved that this problem has only the trivial solution in $\mathcal{X}$. Then $\mathcal{A}_{\mathcal{P}}$ is injective as asserted.

Therefore, we conclude that the operator $\mathcal{A}_{\mathcal{P}}$ is an isomorphism, and hence, taking into account the equation (4.44), we have guaranteed the existence of a solution for our transmission problem (4.32) in the space $\mathfrak{X}$, and for the given data $(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0) \in \mathfrak{Y}$.

Moreover, the solution is delivered through the operator $\mathcal{A}_{\mathcal{P}}^{-1} \circ \mathcal{T}$ in the following manner

$$(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-) = (\mathcal{A}_{\mathcal{P}}^{-1} \circ \mathcal{T})(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0). \tag{4.45}$$

This completes the proof of the existence result.

Finally, it remains to use (4.45) and the continuity of the operators $\mathcal{T}$ and $\mathcal{A}_{\mathcal{P}}^{-1}$ to conclude that there exists a constant $C \equiv C(D_+, D_-, \mathcal{P}, \mathbf{L}) > 0$ such that

$$\|(\boldsymbol{w}_+, p_+, \boldsymbol{w}_-, p_-)\|_{\mathfrak{X}} \le C\|(\boldsymbol{\eta}_+, \boldsymbol{\eta}_-, \boldsymbol{\mu}_0, \boldsymbol{\zeta}_0)\|_{\mathfrak{Y}}$$

and Corollary 2.2 together with the fact that $\boldsymbol{w}_- \in \mathcal{H}^1(D_-)^3$ implies that $\boldsymbol{w}_-$ vanishes at infinity in the sense of Leray. □

Remark 4.6. In a similar manner one can state the more general assertion, namely, the case when $\boldsymbol{w}_\infty \in \mathbb{R}^3$ is chosen arbitrary. Since the proof follows similar arguments to those in the proof of Theorem 4.5, we omit the details for the sake of brevity (see, e.g., [11], in the special case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a constant).

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Andrei-Florin Albişoru
Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, 400084, Romania
E-mail address: `florin.albisoru@math.ubbcluj.ro`