TAIWANESE JOURNAL OF MATHEMATICS
Vol. 23, No. 3, pp. 557–561, June 2019
DOI: 10.11650/tjm/180904

Short Proof and Generalization of a Menon-type Identity by Li, Hu and Kim

László Tóth

Abstract. We present a simple proof and a generalization of a Menon-type identity by Li, Hu and Kim, involving Dirichlet characters and additive characters.

1. Motivation and main result

Menon's classical identity states that for every $n \in \mathbb{N}$,

$$\sum_{\substack{a=1 \\ (a,n)=1}}^{n} (a-1,n) = \varphi(n)\tau(n), \tag{1.1}$$

where $(a-1,n)$ stands for the greatest common divisor of $a-1$ and n, $\varphi(n)$ is Euler's totient function and $\tau(n) = \sum_{d \mid n} 1$ is the divisor function. Identity (1.1) was generalized by several authors in various directions. Zhao and Cao [7] proved that

$$\sum_{a=1}^{n} (a-1,n)\chi(a) = \varphi(n)\tau(n/d), \tag{1.2}$$

where χ is a Dirichlet character $(\bmod\, n)$ with conductor d ($n \in \mathbb{N}$, $d \mid n$). If χ is the principal character $(\bmod\, n)$, that is $d = 1$, then (1.2) reduces to Menon's identity (1.1). Generalizations of (1.2) involving even functions $(\bmod\, n)$ were deduced by the author [6], using a different approach.

Li, Hu and Kim [4] proved the following generalization of identity (1.2):

Theorem 1.1. [4, Theorem 1.1] *Let $n \in \mathbb{N}$ and let χ be a Dirichlet character* $(\bmod\, n)$ *with conductor d ($d \mid n$). Let $b \mapsto \lambda_\ell(b) := \exp(2\pi i w_\ell b/n)$ be additive characters of the group $\mathbb{Z}_n$, with $w_\ell \in \mathbb{Z}$ ($1 \le \ell \le k$). Then*

$$\sum_{a,b_1,\ldots,b_k=1}^{n} (a-1,b_1,\ldots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k) = \varphi(n)\sigma_k((n/d,w_1,\ldots,w_k)), \tag{1.3}$$

where $\sigma_k(n) = \sum_{d \mid n} d^k$.

Received August 3, 2018; Accepted September 9, 2018.
Communicated by Yu-Ru Liu.
2010 *Mathematics Subject Classification.* 11A07, 11A25.
Key words and phrases. Menon's identity, Dirichlet character, additive character, arithmetic function, Euler's totient function, congruence.

Note that in (1.2) and (1.3) the sums are, in fact, over $1 \le a \le n$ with $(a,n)=1$, since $\chi(a)=0$ for $(a,n)>1$. In the case $w_1=\dots=w_k=0$, identity (1.3) was deduced by the same authors in paper [3]. For the proof, Li, Hu and Kim computed first the given sum in the case $n=p^t$, a prime power, and then they showed that the sum is multiplicative in n.

It is the goal of this paper to present a simple proof of Theorem 1.1. Our approach is similar to that given in [6], and leads to a direct evaluation of the corresponding sum for every $n \in \mathbb{N}$. We obtain, in fact, the following generalization of the above result. Let μ denote the Möbius function and let $*$ be the convolution of arithmetic functions.

Theorem 1.2. *Let F be an arbitrary arithmetic function, let $s_j \in \mathbb{Z}$, χ_j be Dirichlet characters* $(\operatorname{mod} n)$ *with conductors d_j ($1 \le j \le m$) and λ_ℓ be additive characters as defined above, with $w_\ell \in \mathbb{Z}$ ($1 \le \ell \le k$). Then*

$$\begin{aligned}(1.4)\qquad &\sum_{a_1,\dots,a_m,b_1,\dots,b_k=1}^{n} F((a_1-s_1,\dots,a_m-s_m,b_1,\dots,b_k,n)) \\ &\qquad\qquad \times \chi_1(a_1)\cdots\chi_m(a_m)\lambda_1(b_1)\cdots\lambda_k(b_k) \\ &= \varphi(n)^m \chi_1^*(s_1)\cdots\chi_m^*(s_m) \sum_{\substack{e \mid (n/d_1,\dots,n/d_m,w_1,\dots,w_k) \\ (n/e,s_1\cdots s_m)=1}} \frac{e^k(\mu * F)(n/e)}{\varphi(n/e)^m},\end{aligned}$$

where χ_j^* are the primitive characters $(\operatorname{mod} d_j)$ *that induce χ_j ($1 \le j \le m$).*

We remark that the sum in the left hand side of identity (1.4) vanishes provided that there is an s_j such that $(s_j,d_j)>1$. If $F(n)=n$ ($n \in \mathbb{N}$), $m=1$ and $s_1=1$, then identity (1.4) reduces to (1.3). We also remark that the special case $F(n)=n$ ($n \in \mathbb{N}$), $m \ge 1$, $s_1=\dots=s_m=1$, $k \ge 1$, $w_1=\dots=w_k=0$ was considered in the quite recent preprint [2]. Several other special cases of formula (1.4) can be discussed.

See the papers [3–7] and the references therein for other generalizations and analogues of Menon's identity.

2. Proof

We need the following lemmas.

Lemma 2.1. *Let $n,d,e \in \mathbb{N}$, $d \mid n$, $e \mid n$ and let $r,s \in \mathbb{Z}$. Then*

$$\sum_{\substack{a=1 \\ (a,n)=1 \\ a \equiv r \ (\operatorname{mod} d) \\ a \equiv s \ (\operatorname{mod} e)}}^{n} 1 = \begin{cases} \dfrac{\varphi(n)}{\varphi(de)}(d,e) & \text{if } (r,d)=(s,e)=1 \text{ and } (d,e) \mid r-s, \\ 0 & \text{otherwise.} \end{cases}$$

In the special case $e = 1$ this is known in the literature, usually proved by the inclusion-exclusion principle. See, e.g., [1, Theorem 5.32]. Here we use a different approach, in the spirit of our paper.

Proof of Lemma 2.1. For each term of the sum, since $(a, n) = 1$, we have $(r, d) = (a, d) = 1$ and $(s, e) = (a, e) = 1$. Also, the given congruences imply $(d, e) \mid r - s$. We assume that these conditions are satisfied (otherwise the sum is empty and equals zero).

Using the property of the Möbius function, the given sum, say S, can be written as

$$S = \sum_{\substack{a=1 \\ a\equiv r \pmod d \\ a\equiv s \pmod e}}^{n} \sum_{\delta\mid(a,n)} \mu(\delta) = \sum_{\delta\mid n} \mu(\delta) \sum_{\substack{j=1 \\ \delta j\equiv r \pmod d \\ \delta j\equiv s \pmod e}}^{n/\delta} 1. \tag{2.1}$$

Let $\delta \mid n$ be fixed. The linear congruence $\delta j \equiv r \pmod d$ has solutions in j if and only if $(\delta, d) \mid r$, equivalent to $(\delta, d) = 1$, since $(r, d) = 1$. Similarly, the congruence $\delta j \equiv s \pmod e$ has solutions in j if and only if $(\delta, e) \mid s$, equivalent to $(\delta, e) = 1$, since $(s, e) = 1$. These two congruences have common solutions in j due to the condition $(d, e) \mid r - s$. Furthermore, if j_1 and j_2 are solutions of these simultaneous congruences, then $\delta j_1 \equiv \delta j_2 \pmod d$ and $\delta j_1 \equiv \delta j_2 \pmod e$. Since $(\delta, d) = 1$, this gives $j_1 \equiv j_2 \pmod{[d, e]}$. We deduce that there are

$$N = \frac{n}{\delta[d, e]}$$

solutions $(\bmod\, n/\delta)$ and the last sum in (2.1) is N. This gives

$$S = \frac{n}{[d, e]} \sum_{\substack{\delta\mid n \\ (\delta,de)=1}} \frac{\mu(\delta)}{\delta} = \frac{n}{[d, e]} \cdot \frac{\varphi(n)/n}{\varphi(de)/(de)} = \frac{\varphi(n)}{\varphi(de)}(d, e). \qquad \square$$

The next lemma is a known result. See, e.g., [6] for its (short) proof.

Lemma 2.2. *Let $n \in \mathbb{N}$ and χ be a primitive character* $(\bmod\, n)$*. Then for any $e \mid n$, $e < n$ and any $s \in \mathbb{Z}$,*

$$\sum_{\substack{a=1 \\ a\equiv s \pmod e}}^{n} \chi(a) = 0.$$

Now we prove

Lemma 2.3. *Let χ be a Dirichlet character* $(\bmod\, n)$ *with conductor d ($n \in \mathbb{N}$, $d \mid n$) and let $e \mid n$, $s \in \mathbb{Z}$. Then*

$$\sum_{\substack{a=1 \\ a\equiv s \pmod e}}^{n} \chi(a) = \begin{cases} \dfrac{\varphi(n)}{\varphi(e)}\chi^*(s) & \text{if } d \mid e \text{ and } (s, e) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where χ^* is the primitive character $(\bmod\, d)$ *that induces χ.*

Proof. We can assume $(a,n)=1$ in the sum. If $a\equiv s \pmod{e}$, then $(s,e)=(a,e)=1$. Given the Dirichlet character $\chi \pmod{n}$, the primitive character $\chi^* \pmod{d}$ that induces χ is defined by

$$\chi(a)=\begin{cases}\chi^*(a) & \text{if } (a,n)=1,\\ 0 & \text{if } (a,n)>1.\end{cases}$$

We deduce

$$T:=\sum_{\substack{a=1\\ a\equiv s \ (\mathrm{mod}\, e)}}^{n}\chi(a)=\sum_{\substack{a=1\\ (a,n)=1\\ a\equiv s \ (\mathrm{mod}\, e)}}^{n}\chi^*(a)=\sum_{r=1}^{d}\chi^*(r)\sum_{\substack{a=1\\ (a,n)=1\\ a\equiv r \ (\mathrm{mod}\, d)\\ a\equiv s \ (\mathrm{mod}\, e)}}^{n}1,$$

where the inner sum is evaluated in Lemma 2.1. Since $(s,e)=1$, as mentioned above, we have

$$T=\sum_{\substack{r=1\\ (r,d)=1\\ (d,e)\mid r-s}}^{d}\chi^*(r)\frac{\varphi(n)}{\varphi(de)}(d,e)=\frac{\varphi(n)}{\varphi(de)}(d,e)\sum_{\substack{r=1\\ (r,d)=1\\ r\equiv s \ (\mathrm{mod} (d,e))}}^{d}\chi^*(r)=\frac{\varphi(n)}{\varphi(de)}(d,e)\chi^*(s),$$

by Lemma 2.2 in the case $(d,e)=d$, that is $d\mid e$. We conclude that

$$T=\frac{\varphi(n)}{\varphi(de)}d\chi^*(s)=\frac{\varphi(n)}{\varphi(e)}\chi^*(s).$$

If $d\nmid e$, then $T=0$. □

Proof of Theorem 1.2. Let V denote the given sum. By using the identity $F(n)=\sum_{e\mid n}(\mu*F)(e)$, we have

$$\begin{aligned}V&=\sum_{a_1,\ldots,a_m,b_1,\ldots,b_k=1}^{n}\chi_1(a_1)\cdots\chi_m(a_m)\lambda_1(b_1)\cdots\lambda_k(b_k)\sum_{e\mid(a_1-s_1,\ldots,a_m-s_m,b_1,\ldots,b_k,n)}(\mu*F)(e)\\ &=\sum_{e\mid n}(\mu*F)(e)\sum_{\substack{a_1=1\\ a_1\equiv s_1 \ (\mathrm{mod}\, e)}}^{n}\chi_1(a_1)\cdots\sum_{\substack{a_m=1\\ a_m\equiv s_m \ (\mathrm{mod}\, e)}}^{n}\chi_m(a_m)\sum_{\substack{b_1=1\\ e\mid b_1}}^{n}\lambda_1(b_1)\cdots\sum_{\substack{b_k=1\\ e\mid b_k}}^{n}\lambda_k(b_k).\end{aligned}$$

Here for every $1\le \ell\le k$,

$$\sum_{\substack{b_\ell=1\\ e\mid b_\ell}}^{n}\lambda_\ell(b_\ell)=\sum_{c_\ell=1}^{n/e}\exp(2\pi i w_\ell c_\ell/(n/e))=\begin{cases}\frac{n}{e} & \text{if } \frac{n}{e}\mid w_\ell,\\ 0 & \text{otherwise,}\end{cases}$$

and using Lemma 2.3 we deduce that

$$V=\chi_1^*(s_1)\cdots\chi_m^*(s_m)\sum{}'(\mu*F)(e)\left(\frac{\varphi(n)}{\varphi(e)}\right)^m\left(\frac{n}{e}\right)^k,$$

where the sum $\sum'$ is over $e \mid n$ such that $d_j \mid e$, $(e, s_j) = 1$ for all $1 \leq j \leq m$ and $n/e \mid w_\ell$ for all $1 \leq \ell \leq k$. Interchanging e and n/e, the sum is over e such that $e \mid n/d_j$, $(n/e, s_j) = 1$ for all $1 \leq j \leq m$ and $e \mid w_\ell$ for all $1 \leq \ell \leq k$. This completes the proof. $\square$

Acknowledgments

This work was supported by the European Union, co-financed by the European Social Fund EFOP-3.6.1.-16-2016-00004.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1976.

[2] M. Chen, S. Hu and Y. Li, *On Menon-Sury's identity with several Dirichlet characters*, arXiv:1807.07241 [math.NT].

[3] Y. Li, X. Hu and D. Kim, *A generalization of Menon's identity with Dirichlet characters*, Int. J. Number Theory, accepted.

[4] ———, *A Menon-type identity with multiplicative and additive characters*, Taiwanese J. Math., accepted.

[5] L. Tóth, *Menon's identity and arithmetical sums representing functions of several variables*, Rend. Semin. Mat. Univ. Politec. Torino **69** (2011), no. 1, 97–110.

[6] ———, *Menon-type identities concerning Dirichlet characters*, Int. J. Number Theory **14** (2018), no. 4, 1047–1054.

[7] X.-P. Zhao and Z.-F. Cao, *Another generalization of Menon's identity*, Int. J. Number Theory **13** (2017), no. 9, 2373–2379.

László Tóth
Department of Mathematics, University of Pécs, Ifjúság útja 6, 7624 Pécs, Hungary
E-mail address: `ltoth@gamma.ttk.pte.hu`