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Counting Permutations by Simsun Successions

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Abstract. In this paper, we introduce the definitions of simsun succession statistics and simsun patterns. In addition to its original definition by Brenti, we give two more combinatorial interpretations of the q-Eulerian polynomials using simsun successions. We also present a bijection between permutations avoiding the simsun pattern 132 and set partitions.

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *descent* in π is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. We say that π contains no *double descents* if there is no index $i \in [n-2]$ such that $\pi(i) > \pi(i+1) > \pi(i+2)$. A permutation $\pi \in \mathfrak{S}_n$ is called *simsun* if for all k, the subword of π restricted to $[k]$ (in the order they appear in π) contains no double descents. For example, 35142 is simsun, but 35241 is not. Simsun permutations have been introduced by Simion and Sundaram in [28] and extensively studied in the literature (see [2,5,6,8–10,12,18] for instance). Let $\mathcal{RS}_n$ be the set of simsun permutations of length n. Simion and Sundaram [28, p. 267] showed that

$$\#\mathcal{RS}_n = E_{n+1},$$

where E_n is the nth Euler number, which is also the number of alternating permutations in $\mathfrak{S}_n$.

Let $\operatorname{des}(\pi)$ be the number of descents of π. Define

$$S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\operatorname{des}(\pi)} = \sum_{k=0}^{\lfloor n/2 \rfloor} S(n,k)x^k.$$

It follows from [5] that the coefficients $S(n,k)$ satisfy the recurrence relation

$$S(n,k) = (k+1)S(n-1,k) + (n-2k+1)S(n-1,k-1)$$

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with the initial conditions $S(0,0) = 1$ and $S(0,k) = 0$ for $k \geq 1$. In terms of generating functions, this is equivalent to

$$S_{n+1}(x) = (1+nx)S_n(x) + x(1-2x)S_n'(x)$$

with $S_0(x) = 1$. An *excedance* in $\pi \in \mathfrak{S}_n$ is an index $i \in [n-1]$ such that $\pi(i) > i$. Let $\mathrm{exc}(\pi)$ be the number of excedances of π. The classical *Eulerian polynomials* are defined by

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)+1} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}(\pi)+1} \quad \text{for } n \geq 1.$$

From [11, Proposition 2.7], we have

$$A_{n+1}(x) = x \sum_{k=0}^{\lfloor n/2 \rfloor} S(n,k)(2x)^k (1+x)^{n-2k}.$$

A *succession* in $\pi \in \mathfrak{S}_n$ is an index $i \in [n-1]$ such that $\pi(i+1) = \pi(i)+1$. The study of successions began in the 1940s (see [13, 22]), and there have been a lot of recent activities. There are many variants of successions, including circular successions [29] and cycle successions [20]. The succession statistic has also been studied on various combinatorial structures, such as compositions and words [14], and set partitions [19, 21]. For example, a succession in a partition of $[n]$ is an occurrence of two consecutive integers that appear in the same block. Following [26, p. 137, Exercise 108], the number of partitions of $[n]$ with no successions is $B(n-1)$, where $B(n)$ is the nth *Bell number*, which is also the number of partitions of $[n]$. Let $\mathcal{BS}_n$ be a subset of $\mathcal{RS}_n$ with the restriction that for any $\pi \in \mathcal{BS}_n$ and for all k, the subword of π restricted to $[k]$ (in the order they appear in π) does not contain successions. For example, $\mathcal{BS}_5 = \{25143, 21435, 24135, 24153, 52413\}$. Motivated by the following result, we shall explore the connections between the succession statistic for permutations and simsun permutations.

Proposition 1.1. *For $n \geq 1$, we have $S_n(x) = \sum_{\pi \in \mathcal{BS}_{n+2}} x^{\mathrm{des}(\pi)-1}$ and $\#\mathcal{BS}_n = E_{n-1}$.*

Proof. Let $r(n,k) = \#\{\pi \in \mathcal{BS}_{n+2} : \mathrm{des}(\pi) = k+1\}$. There are two ways in which permutations in $\mathcal{BS}_{n+2}$ with $k+1$ descents can be obtained from a permutation $\sigma \in \mathcal{BS}_{n+1}$:

(a) If $\mathrm{des}(\sigma) = k+1$, then we distinguish two subcases:

- (a_1) If $\sigma(n+1) = n+1$, then we can insert $n+2$ right after $\sigma(i)$, where i is a descent index; or
- (a_2) If $\sigma(n+1) < n+1$, let the index $j \in [n]$ be such that $\sigma(j) = n+1$. We insert $n+2$ right after $\sigma(i)$ if i is a descent index other than j, or insert $n+2$ at the end.

In either case, there are $k+1$ ways to insert $n+2$, and we have $r(n-1,k)$ choices for σ. This gives $(k+1)r(n-1,k)$ possibilities.

(b) If $\mathrm{des}(\sigma)=k$, then we cannot insert $n+2$ immediately before or right after a descent index. Moreover, we cannot put $n+2$ at the end of σ. All of the remaining positions are allowed to insert $n+2$ so that the descent is increased by 1. Thus there are $n-2k+1$ ways to insert $n+2$, and we have $r(n-1,k-1)$ choices for σ. This gives $(n-2k+1)r(n-1,k-1)$ possibilities.

Therefore, $r(n,k)=(k+1)r(n-1,k)+(n-2k+1)r(n-1,k-1)$. Note that $\mathcal{BS}_2=\{21\}$. Thus $r(0,0)=1$ and $r(0,k)=0$ for $k\geq 1$. Hence the numbers $r(n,k)$ satisfy the same recurrence and initial conditions as $S(n,k)$, so they agree. □

This paper is organized as follows. In Section 2, we introduce the definition of simsun cycle succession, and give a combinatorial interpretation of the q-Eulerian polynomials introduced by Brenti [3]. In Section 3, we introduce the definition of simsun succession and give another combinatorial interpretation of the q-Eulerian polynomials. In Section 4, we present a bijection between permutations avoiding the simsun pattern 132 and set partitions.

2. q-Eulerian polynomials and simsun cycle successions

Recall that $\pi\in\mathfrak{S}_n$ can be written in the *standard cycle form*, where each cycle is written with its smallest entry first and the cycles are arranged in increasing order of their smallest entries. A *cycle succession* in π is an index $i\in[n-1]$ such that two consecutive entries i and $i+1$ appear in that order within one cycle of the standard cycle form of π (see [20]). For example, the permutation $(1,2,6)(3,5,4)$ contains one cycle succession.

Definition 2.1. A permutation $\pi\in\mathfrak{S}_n$, written in the standard cycle form, *avoids simsun cycle successions* if for any $k\in[n]$, the subword of π restricted to $[k]$ (in the order they appear in the standard cycle form of π) does not contain cycle successions.

For example, $\pi=(1543)(2)$ avoids simsun cycle successions, since any of the following subwords of π does not contain cycle successions:

$$(1),\ (1)(2),\ (13)(2),\ (143)(2),\ (1543)(2).$$

Let $\mathcal{CS}_n$ be the set of permutations in $\mathfrak{S}_n$ that avoid simsun cycle successions. In particular, $\mathcal{CS}_1=\{(1)\}$, $\mathcal{CS}_2=\{(1)(2)\}$, and $\mathcal{CS}_3=\{(1)(2)(3),(13)(2)\}$.

Brenti [3] considered a q-analog of the classical Eulerian polynomials defined by

$$A_0(x;q)=1,\quad A_n(x;q)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{exc}(\pi)}q^{\mathrm{cyc}(\pi)}\quad \text{for } n\geq 1,$$

where $\mathrm{cyc}(\pi)$ is the number of cycles of π. The first few q-Eulerian polynomials are

$$A_0(x;q) = 1, \quad A_1(x;q) = q, \quad A_2(x;q) = q(x+q), \quad A_3(x;q) = q\big(x^2 + (3q+1)x + q^2\big).$$

Clearly, $A_n(x) = xA_n(x;1)$ for $n \geq 1$. The real-rootedness of $A_n(x;q)$ has been studied in [1,17].

The first main result of this paper is the following.

Theorem 2.2. *For $n \geq 1$, we have*

$$qA_n(x;q) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}(\pi)} q^{\mathrm{cyc}(\pi)+1} = \sum_{\sigma \in \mathcal{CS}_{n+1}} x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)}. \tag{2.1}$$

In the rest of this section, we give a constructive proof of (2.1). Define

$$\mathcal{S}_{n,k,\ell} = \{\pi \in \mathfrak{S}_n : \mathrm{exc}(\pi) = k, \mathrm{cyc}(\pi) = \ell\},$$
$$\mathcal{CS}_{n,k,\ell} = \{\pi \in \mathcal{CS}_n : \mathrm{exc}(\pi) = k, \mathrm{cyc}(\pi) = \ell\}.$$

We now introduce two labelings for permutations which are written in the standard cycle form. The labels will be put as subscripts of entries of permutations.

- Let $\pi \in \mathcal{S}_{n,k,\ell}$, and $i_1, i_2, \ldots, i_k$ be the excedances of π, in the order of their appearances in π (in the standard cycle form). We put the subscript label u_r after i_r, for each $r = 1, 2, \ldots, k$. For the remaining entries, we put the subscript labels $v_1, v_2, \ldots, v_{n-k}$ from left to right.

- Let $\pi \in \mathcal{CS}_{n,k,\ell}$, and $i_1, i_2, \ldots, i_k$ be the excedances of π, in the order of their appearances in π (in the standard cycle form). We put the subscript label p_r after i_r for each $r = 1, 2, \ldots, k$. For the remaining entries other than n, we put the subscript labels $q_1, q_2, \ldots, q_{n-k-1}$ from left to right.

In the following discussion, we always add labels to permutations in $\mathcal{S}_{n,k,\ell}$ and $\mathcal{CS}_{n,k,\ell}$. As an example, consider the permutation $\pi = (135)(26)(4)$. If we say that $\pi \in \mathcal{S}_{6,3,3}$, then π is labeled as $(1_{u_1}3_{u_2}5_{v_1})(2_{u_3}6_{v_2})(4_{v_3})$; if we say that $\pi \in \mathcal{CS}_{6,3,3}$, then π is labeled as $(1_{p_1}3_{p_2}5_{q_1})(2_{p_3}6)(4_{q_2})$.

Now we start to construct a bijection Φ between $\mathcal{S}_{n,k,\ell}$ and $\mathcal{CS}_{n+1,k,\ell+1}$. When $n = 1$, we have $\mathcal{S}_{1,0,1} = \{(1)\}$ and $\mathcal{CS}_{2,0,2} = \{(1)(2)\}$. Set $\Phi((1_{v_1})) = (1_{q_1})(2)$. This gives a bijection between $\mathcal{S}_{1,0,1}$ and $\mathcal{CS}_{2,0,2}$. Let $n = m$ and assume that the bijections Φ have been constructed between $\mathcal{S}_{m,k,\ell}$ and $\mathcal{CS}_{m+1,k,\ell+1}$ for all k and ℓ. Consider the case $n = m+1$. For a permutation $\pi \in \mathcal{S}_{m,k,\ell}$ and $\sigma = \Phi(\pi) \in \mathcal{CS}_{m+1,k,\ell+1}$, we consider the following three cases:

(i) If $\widehat{\pi}$ is obtained from π by inserting the entry $m+1$ to the position of π with label u_r, then we insert $m+2$ to the position of σ with label p_r to form $\widehat{\sigma} = \Phi(\widehat{\pi})$. In this case, $\operatorname{exc}(\widehat{\pi}) = \operatorname{exc}(\widehat{\sigma}) = k$ and $\operatorname{cyc}(\widehat{\pi}) + 1 = \operatorname{cyc}(\widehat{\sigma}) = \ell + 1$.

(ii) If $\widehat{\pi}$ is obtained from π by inserting the entry $m+1$ to the position of π with label v_j, then we insert $m+2$ to the position of σ with label q_j to form $\widehat{\sigma} = \Phi(\widehat{\pi})$. In this case, $\operatorname{exc}(\widehat{\pi}) = \operatorname{exc}(\widehat{\sigma}) = k+1$ and $\operatorname{cyc}(\widehat{\pi}) + 1 = \operatorname{cyc}(\widehat{\sigma}) = \ell + 1$.

(iii) If $\widehat{\pi}$ is obtained from π by appending $(m+1)$ to π as a new cycle, then we append $(m+2)$ to σ as a new cycle to form $\widehat{\sigma} = \Phi(\widehat{\pi})$. In this case, $\operatorname{exc}(\widehat{\pi}) = \operatorname{exc}(\widehat{\sigma}) = k$ and $\operatorname{cyc}(\widehat{\pi}) + 1 = \operatorname{cyc}(\widehat{\sigma}) = \ell + 2$.

By induction, we see that Φ is the desired bijection between $\mathcal{S}_{n,k,\ell}$ and $\mathcal{CS}_{n+1,k,\ell+1}$ for all k and ℓ, hence it gives a constructive proof of (2.1).

Example 2.3. Given $\pi = (135)(2)(4) \in \mathcal{S}_{5,2,3}$. The correspondence between π and $\Phi(\pi)$ is built up as follows:

$$\begin{aligned}
(1_{v_1}) &\iff (1_{q_1})(2);\\
(1_{v_1})(2_{v_2}) &\iff (1_{q_1})(2_{q_2})(3);\\
(1_{u_1}3_{v_1})(2_{v_2}) &\iff (1_{p_1}4)(2_{q_1})(3_{q_2});\\
(1_{u_1}3_{v_1})(2_{v_2})(4_{v_3}) &\iff (1_{p_1}4_{q_1})(2_{q_2})(3_{q_3})(5);\\
(1_{u_1}3_{u_2}5_{v_1})(2_{v_2})(4_{v_3}) &\iff (1_{p_1}4_{p_2}6)(2_{q_1})(3_{q_2})(5_{q_3}).
\end{aligned}$$

3. Simsun successions

As a variant of simsun cycle successions, we introduce the following definition.

Definition 3.1. We say that a permutation π, written in word structure, *avoids simsun successions* if for any k, the subword of π restricted to $[k]$ (in the order they appear in π) does not contain successions.

For example, the permutation $\pi = 321465$ contains a simsun succession, since π restricted to $[5]$ equals 32145 and it contains a succession. Let $\mathcal{AS}_n$ denote the set of permutations in $\mathfrak{S}_n$ that avoid simsun successions. In particular, $\mathcal{AS}_1 = \{1\}$, $\mathcal{AS}_2 = \{21\}$, and $\mathcal{AS}_3 = \{213, 321\}$.

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. We say that an element $\pi(i)$ is a *left-to-right minimum* of π if $\pi(i)$ is the smallest entry among $\pi(1), \pi(2), \ldots, \pi(i)$. Let $\operatorname{lrmin}(\pi)$ be the number of left-to-right minima of π. For example, $\operatorname{lrmin}(\mathbf{3}\mathbf{2}4\mathbf{1}) = 3$. Let $\operatorname{asc}(\pi)$ be the number of ascents of $\pi \in \mathfrak{S}_n$, i.e., the number of indices $i \in [n-1]$ such that $\pi(i) < \pi(i+1)$. Suppose that $\sigma \in \mathcal{AS}_n$ with left-to-right minima $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$, where $i_1 < i_2 < \cdots < i_k$.

Let σ' be the permutation obtained from σ by inserting a left parenthesis before each left-to-right minimum, and then inserting right parentheses at the end of σ and before each left parenthesis except the first one. It is clear that $\sigma' \in \mathcal{CS}_n$. By reordering the cycles, σ' can be written in standard cycle form. For example, if $\sigma = 3241$, then $\sigma' = (3)(24)(1) = (1)(24)(3)$. Note that $\mathrm{asc}(\sigma) = \mathrm{exc}(\sigma')$ and $\mathrm{lrmin}(\sigma) = \mathrm{cyc}(\sigma)$. Combining this with Theorem 2.2, we get the second main result of this paper.

Theorem 3.2. *For $n \geq 1$, we have*

$$qA_n(x;q) = \sum_{\sigma \in \mathcal{AS}_{n+1}} x^{\mathrm{asc}(\sigma)} q^{\mathrm{lrmin}(\sigma)}.$$

The number of peaks of permutations is certainly among the most important combinatorial statistics. See, e.g., [7,15,16,24] and the references therein. An *interior peak* in π is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. A *left peak* in $\pi \in \mathfrak{S}_n$ is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\mathrm{pk}(\pi)$ (resp. $\mathrm{lpk}(\pi)$) be the number of interior peaks (resp. left peaks) of π. Similarly, a *valley* in π is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) > \pi(i) < \pi(i+1)$. Let $\mathrm{val}(\pi)$ be the number of valleys of π. Clearly, interior peaks and valleys are equidistributed over $\mathfrak{S}_n$.

Along the same lines as the proof of (2.1), it is easy to verify the following result.

Theorem 3.3. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{val}(\pi)+1} y^{\mathrm{lpk}(\pi)} = \sum_{\sigma \in \mathcal{AS}_{n+1}} x^{\mathrm{lpk}(\pi)} y^{\mathrm{val}(\pi)}. \tag{3.1}$$

Motivated by the study of longest increasing subsequences, Stanley [25] initiated a study of the longest alternating subsequences. An *alternating subsequence* of $\pi \in \mathfrak{S}_n$ is a subsequence $\pi(i_1), \pi(i_2), \ldots, \pi(i_k)$ satisfying $\pi(i_1) > \pi(i_2) < \pi(i_3) > \pi(i_4) < \cdots$, where $i_1 < i_2 < \cdots < i_k$. Let $\mathrm{as}(\pi)$ be the length (number of terms) of the longest alternating subsequence of a permutation $\pi \in \mathfrak{S}_n$. Note that $\mathrm{as}(\pi) = \mathrm{val}(\pi) + \mathrm{lpk}(\pi) + 1$. Taking $x = y$ in (3.1), we get the following result.

Corollary 3.4. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{as}(\pi)} = \sum_{\pi \ in \mathcal{AS}_{n+1}} x^{\mathrm{as}(\pi)-1}.$$

4. Permutations avoiding the simsun pattern 132 and set partitions

In this section, containment and avoidance will always refer to consecutive patterns. Let m and n be two positive integers with $m \leq n$, and let $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$. We say that π contains τ as a *consecutive pattern* if it has a subsequence of consecutive entries order-isomorphic to τ. A permutation π avoids a pattern τ if π does not contain τ.

Definition 4.1. Let $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$. We say that π *avoids the simsun pattern* τ if for any k, the subword of π restricted to $[k]$ (in the order they appear in π) does not contain the consecutive pattern τ.

Let $\mathcal{SP}_n(\tau)$ denote the set of permutations in $\mathfrak{S}_n$ that avoid the simsun pattern τ. In particular, $\mathcal{SP}_n(321) = \mathcal{RS}_n$. Using the reverse map, we get $\#\mathcal{SP}_n(321) = \#\mathcal{SP}_n(123) = E_{n+1}$. In the following, we study the relationship between $\mathcal{SP}_n(132)$ and set partitions of $[n]$.

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. Let $\mathrm{suc}(\pi)$ be the number of successions of π. A *right peak* in π is an entry $\pi(i)$ with $i \in \{2, 3, \ldots, n\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(n+1) = 0$. Let $\mathrm{rpk}(\pi)$ be the number of right peaks of π. An *inversion* of π is a pair $(\pi(i), \pi(j))$ such that $i < j$ and $\pi(i) > \pi(j)$. Let $\mathrm{inv}(\pi)$ be the number of inversions of π. An *exterior double descent* in π is an entry $\pi(i)$ such that $\pi(i-1) > \pi(i) > \pi(i+1)$, where $i \in [n]$ and we take $\pi(0) = +\infty$ and $\pi(n+1) = 0$. Let $\mathrm{exddes}(\pi)$ be the number of exterior double descents of π. For example, $\mathrm{suc}(42315) = 1$, $\mathrm{rpk}(42315) = 2$, $\mathrm{inv}(42315) = 5$ and $\mathrm{exddes}(42315) = 1$.

A *partition* σ of $[n]$, written $\sigma \vdash [n]$, is a collection of pairwise disjoint nonempty subsets (called *blocks*) of $[n]$ whose union is $[n]$. Let Π_n denote the family of all set partitions of $[n]$ and let $l(\sigma)$ be the number of blocks of σ. As usual, we always write $\sigma = B_1/B_2/\cdots/B_k$, where we list the blocks in the standard order $\min B_1 < \min B_2 < \cdots < \min B_k$. Let $\sigma = B_1/B_2/\cdots/B_k$. For $c \in B_s$ and $d \in B_t$, we say that the pair (c, d) is a *free rise* of σ if $c < d$, where $1 \le s < t \le k$. Let $\mathrm{fr}(\sigma)$ be the number of free rises of σ. A *singleton* of a partition is a block with exactly one element (see [27] for instance). Let $\mathrm{single}(\sigma)$ be the number of singletons of σ. We say that a block is *non-singleton* if it contains at least two elements. Let $\mathrm{nsingle}(\sigma)$ be the number of non-singletons of σ. Let $\mathrm{suc}(\sigma)$ be the number of successions of σ, i.e., occurrences of two consecutive integers appear in the same block of σ. For example, $\mathrm{suc}(\{1,2,3\}/\{4\}/\{5\}) = 2$, $\mathrm{single}(\{1,2,3\}/\{4\}/\{5\}) = 2$, $\mathrm{nsingle}(\{1,2,3\}/\{4\}/\{5\}) = 1$, and $\mathrm{fr}(\{1,2,3\}/\{4\}/\{5\}) = 7$.

Now we present the third main result of this paper.

Theorem 4.2. *For $n \ge 1$, we have*

$$\sum_{\pi \in \mathcal{SP}_n(132)} x^{\mathrm{des}(\pi)+1} y^{\mathrm{rpk}(\pi)} z^{\mathrm{exddes}(\pi)} p^{\mathrm{suc}(\pi)} q^{\mathrm{inv}(\pi)}$$
$$= \sum_{\sigma \in \prod_n} x^{l(\sigma)} y^{\mathrm{nsingle}(\sigma)} z^{\mathrm{single}(\sigma)} p^{\mathrm{suc}(\sigma)} q^{\mathrm{fr}(\sigma)}.$$

In the following, we shall present a constructive proof of Theorem 4.2.

It is well known that the number of partitions of $[n]$ with exactly k blocks is the *Stirling number of the second kind* $\left\{ n \atop k \right\}$. Recently, Chen et al. [4] presented a grammatical labeling of partitions of $[n]$: For $\sigma \in \Pi_n$, we label a block of σ by letter b and label the partition itself by letter a, and the weight of a partition is defined to be the product of its labels. Hence $w(\sigma) = ab^k$ if $l(\sigma) = k$. They deduced that $\sum_{\sigma\in\Pi_n} w(\sigma) = a\sum_{k=0}^{n} \left\{ n \atop k \right\} b^k$. As a variant of the grammatical labeling, we introduce the following labelings on partitions and permutations.

- Let $\sigma = B_1/B_2/\cdots/B_k$ be a partition of $[n]$. Then we label B_i by the letter b_{k+1-i}, where $1 \le i \le k$. Moreover, we put letter a at the end of σ.

- Suppose that $\pi \in \mathcal{SP}_n(132)$ with $k-1$ descents, where $1 \le k \le n$. Let $i_1 < i_2 < \cdots < i_{k-1}$ be the descent indices of π. We put the subscript labels s_r after $\pi(i_r)$, where $1 \le r \le k-1$. Moreover, we put the subscript label s_k at the end of π and the subscript label t at the front of π.

For example, the partition $\{1,3\}/\{2,4,5\}$ and the permutation 42315 are labeled as follows:

$$\{1,3\}_{b_2}/\{2,4,5\}_{b_1}a; \quad {}_t4_{s_1}23_{s_2}15_{s_3}.$$

For $1 \le k \le n$ and $0 \le \ell \le \binom{n}{2}$, we define $\Pi_{n,k,\ell} = \{\sigma \in \Pi_n : l(\sigma) = k, \mathrm{fr}(\sigma) = \ell\}$ and

$$\mathcal{SP}_{n,k,\ell}(132) = \{\pi \in \mathcal{SP}_n(132) : \mathrm{des}(\pi) = k-1, \mathrm{inv}(\pi) = \ell\}.$$

In the following discussion, we always add labels to partitions and permutations.

Now we construct a bijection Ψ between $\mathcal{SP}_{n,k,\ell}(132)$ and $\Pi_{n,k,\ell}$. When $n = 1$, we have $\mathcal{SP}_{1,1,0}(132) = \{1\}$ and $\Pi_{1,1,0} = \{\{1\}\}$. The bijection between $\mathcal{SP}_{1,1,0}(132)$ and $\Pi_{1,1,0}$ is given by ${}_t1_{s_1} \Longleftrightarrow \{1\}_{b_1}a$. When $n = 2$, if the entry 2 is inserted to the position with label t of ${}_t1_{s_1}$, then we append the block $\{2\}$ to $\{1\}_{b_1}a$; if the entry 2 is inserted to the position with label s_1 of ${}_t1_{s_1}$, then we insert the element 2 into the block $\{1\}$. In other words, the bijection Ψ is given by

$$\begin{aligned} {}_t2_{s_1}1_{s_2} &\Longleftrightarrow \{1\}_{b_2}/\{2\}_{b_1}a; \\ {}_t12_{s_1} &\Longleftrightarrow \{12\}_{b_1}a. \end{aligned}$$

It should be noted that the block with label b_1 consists of the entries of the corresponding permutation lying before the label s_1, and the block with label b_2 (if exists) consists of the entries lying between the labels s_1 and s_2.

For the induction step, assume that $n = m \ge 2$, and the bijection Ψ has been constructed between $\mathcal{SP}_{m,k,\ell}(132)$ and $\Pi_{m,k,\ell}$ for all k and ℓ. Consider the case $n = m+1$. Suppose that $\pi \in \mathcal{SP}_{m,k,\ell}(132)$ and $\widehat{\pi}$ is obtained from π by inserting the entry $m+1$ into

π. Set $\Psi(\pi) = \sigma$. Suppose further that the block of σ with label b_1 consists of the entries of π lying before the label s_1, and for $1 < i \leq k$, the block of σ with label b_i consists of the entries of π lying between the labels s_{i-1} and s_i. Consider the following two cases:

(i) If the entry $m+1$ is inserted to the position with label t of π, then we append the block $\{m+1\}$ to σ. In this case, $\mathrm{des}(\widehat{\pi}) = \mathrm{des}(\pi)+1 = k$ and $\mathrm{inv}(\widehat{\pi}) = \mathrm{inv}(\pi)+m = \ell+m$. Moreover, $l(\Psi(\widehat{\pi})) = l(\sigma)+1 = k+1$ and $\mathrm{fr}(\Psi(\widehat{\pi})) = \mathrm{fr}(\sigma)+m = \ell+m$.

(ii) If the entry $m+1$ is inserted to the position with label s_i of π, then we insert the element $m+1$ into the block with label b_i of σ. In this case, $\mathrm{des}(\widehat{\pi})+1 = l(\Psi(\widehat{\pi}))$ and $\mathrm{inv}(\widehat{\pi}) = \mathrm{fr}(\Psi(\widehat{\pi}))$. More precisely, we distinguish two subcases:

(c_1) if $i = k$, then $\mathrm{des}(\widehat{\pi}) = \mathrm{des}(\pi) = k-1$, $\mathrm{inv}(\widehat{\pi}) = \mathrm{inv}(p) = \ell$, $l(\widehat{\pi}) = l(\sigma) = k$ and $\mathrm{fr}(\widehat{\pi}) = \mathrm{fr}(\sigma) = \ell$.

(c_2) if $1 \leq i < k$ and the label s_i lies right after $\pi(j)$, then $\pi(j) > \pi(j+1)$. By the induction hypothesis, there are $m-j$ elements in the union of the blocks with labels $b_{i+1}, b_{i+2}, \ldots, b_k$ of σ. Therefore, $\mathrm{des}(\widehat{\pi}) = \mathrm{des}(\pi) = k-1$, $\mathrm{inv}(\widehat{\pi}) = \ell+m-j$, $l(\widehat{\pi}) = l(\sigma) = k$ and $\mathrm{fr}(\widehat{\pi}) = \mathrm{fr}(\sigma)+m-j = \ell+m-j$.

After the above step, we label the obtained permutations and partitions accordingly. It is clear that the block of $\Psi(\widehat{\pi})$ with label b_1 consists of the entries of $\widehat{\pi}$ lying before the label s_1, and for $1 < i \leq k$, the block of $\Psi(\widehat{\pi})$ with label b_i consists of the entries of $\widehat{\pi}$ lying between the labels s_{i-1} and s_i, and the block of $\Psi(\widehat{\pi})$ with label b_{k+1} (if it exists) consists of the entries of $\widehat{\pi}$ lying between the labels s_k and s_{k+1}. By induction, we see that Ψ is the desired bijection between $\mathcal{SP}_{n,k,\ell}(132)$ and $\Pi_{n,k,\ell}$ for all k and ℓ. Using Ψ, we see that if $\pi(i)$ is a right peak of π, then $\pi(i-1)$ and $\pi(i)$ are in the same block and $\pi(i)$ is the largest element of that block. If $\pi(i)$ is an exterior double descent of π, then $\{\pi(i)\}$ is a singleton of $\Psi(\pi)$. Moreover, if i is a succession of π, then $\pi(i)$ and $\pi(i+1)$ must be in the same block of $\Psi(\pi)$.

Furthermore, we define a map $\varphi\colon \Pi_n \to \mathcal{SP}_n(132)$ as follows: For $\sigma = B_1/B_2/\cdots/B_k \in \Pi_n$, let $\sigma^r = B_k/B_{k-1}/\cdots/B_1$. Let $\varphi(\sigma)$ be a permutation obtained from σ^r by erasing all of the braces of blocks and bars of σ^r. For example, if $\sigma = \{1\}_{b_4}/\{2,4\}_{b_3}/\{3,5,7\}_{b_2}/\{6\}_{b_1}a$, then $\varphi(\sigma) = {}_t6_{s_1}357_{s_2}24_{s_3}1_{s_4}$. Combining this with Ψ, we see that φ is also a bijection between $\mathcal{SP}_{n,k,\ell}(132)$ and $\Pi_{n,k,\ell}$ and $\mathcal{SP}_{n,k,\ell}(132) = \{\varphi(\sigma) : \sigma \in \Pi_{n,k,\ell}\}$. It is clear that if B_i is a non-singleton of σ with the largest element m, then m is a right peak of $\varphi(\sigma)$. If $\{c\}$ is a singleton of σ, then c is an exterior double descent of $\varphi(\sigma)$. If d and $d+1$ appear in different blocks of σ, then we have $\sigma = \cdots/\{\ldots,d,\ldots\}/\cdots/\{\ldots,d+1,\ldots\}/\cdots$, or $\sigma = \cdots/\{e,\ldots,d+1,\ldots\}/\cdots/\{\ldots,d,\ldots\}/\cdots$, where $e < d$. Thus $\varphi(\sigma) = \cdots(d+1)\cdots d\cdots$, or $\varphi(\sigma) = \cdots d\cdots e\cdots(d+1)\cdots$. Therefore, there exists an index i such that $\varphi(\sigma)(i) = d$

and $\varphi(\sigma)(i+1) = d+1$ if and only if d and $d+1$ appear in the same block of σ. In conclusion, using the bijections Ψ and φ, we get a constructive proof of Theorem 4.2.

Example 4.3. Given $\pi = 42351 \in \mathcal{SP}_{5,3,6}(132)$. The correspondence between π and $\Psi(\pi)$ is done as follows:

$$\begin{aligned}
{}_t1_{s_1} &\Longleftrightarrow \{1\}_{b_1}a;\\
{}_t2_{s_1}1_{s_2} &\Longleftrightarrow \{1\}_{b_2}/\{2\}_{b_1}a;\\
{}_t23_{s_1}1_{s_2} &\Longleftrightarrow \{1\}_{b_2}/\{2,3\}_{b_1}a;\\
{}_t4_{s_1}23_{s_2}1_{s_3} &\Longleftrightarrow \{1\}_{b_3}/\{2,3\}_{b_2}/\{4\}_{b_1}a;\\
{}_t4_{s_1}235_{s_2}1_{s_3} &\Longleftrightarrow \{1\}_{b_3}/\{2,3,5\}_{b_2}/\{4\}_{b_1}a.
\end{aligned}$$

Let $B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$ be the Stirling polynomials. Taking $y = z = p = q = 1$ in Theorem 4.2 leads to the following.

Corollary 4.4. *For $n \geq 1$, we have*

$$B_n(x) = \sum_{\pi \in \mathcal{SP}_n(132)} x^{\mathrm{des}(\pi)+1}.$$

By using the reverse and complement maps, it is clear that

$$B_n(x) = \sum_{\pi \in \mathcal{SP}_n(231)} x^{\mathrm{asc}(\pi)+1} = \sum_{\pi \in \mathcal{SP}_n(312)} x^{\mathrm{asc}(\pi)+1} = \sum_{\pi \in \mathcal{SP}_n(213)} x^{\mathrm{des}(\pi)+1}.$$

Using the bijection Ψ and [26, p. 137, Exercise 108], we get the following result.

Proposition 4.5. *The number of permutations in $\mathcal{SP}_n(132)$ with no successions is $B(n-1)$.*

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. We say that an element $\pi(i)$ is a *left-to-right maximum* of π if $\pi(i)$ is the largest entry among $\pi(1), \pi(2), \ldots, \pi(i)$. Let $\mathrm{lrmax}(\pi)$ be the number of left-to-right maxima of π. For example, $\mathrm{lrmax}(2314) = 3$. Let $\sigma = B_1/B_2/\cdots/B_k$ be a partition of $[n]$. Following [23], we define a_i to be the number of $c \in B_i$ with $c > \min B_{i-1}$, where $2 \leq i \leq k$. Let

$$\widehat{\mathrm{Des}}(\sigma) = \{2^{a_2}, 3^{a_3}, \ldots, k^{a_k}\}$$

be the *dual descent multiset* of p, where i^d means that i is repeated d times. For example, $\widehat{\mathrm{Des}}(\{1,3,5\}/\{2\}/\{4,6,7\}) = \{2^1, 3^3\}$. Let $\mathrm{dudes}(\sigma) = \#\widehat{\mathrm{Des}}(\sigma)$. Using the bijections Ψ and φ, it is easy to verify the following result.

Proposition 4.6. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathcal{SP}_n(132)} x^{\mathrm{lrmax}(\pi)} = \sum_{\sigma \in \Pi_n} x^{n-\mathrm{dudes}(\sigma)}.$$

Let $D(\pi) = \{i : \pi(i) > \pi(i+1)\}$ be the descent set of π. The *major index* of π is the sum of the descents: $\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i$. Along the same lines as the proof of [26, Eq. (1.41)], it is routine to check that

$$\sum_{\pi \in \mathcal{SP}_n(132)} x^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathcal{SP}_n(132)} x^{\binom{n}{2} - \mathrm{maj}(\pi)}.$$

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