STRUCTURE OF OPTIMAL TRAJECTORIES IN A NONLINEAR DYNAMIC MODEL WITH ENDOGENOUS DELAY

NATALI HRITONENKO AND YURI YATSENKO

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An exact solution is constructed to a nonlinear optimization problem in an integral dynamic model with delay. The problem involves the unknown duration of the delay and has important applications to the optimal replacement of capital equipment under technological change.

1. Introduction

One of the modern applications of integral equations is the replacement of capital equipment under technological change. Corresponding models are known as *vintage capital models* (VCMs), see [1, 2, 3, 4, 5, 13, 14, 17]. They focus on optimization of the equipment lifetime and can be expressed via special Volterra integral equations with delay (e.g., Corduneanu [5]). Existing results about endogenous equipment lifetime in VCMs include mainly the case of constant lifetime.

This paper is devoted to the construction of exact solutions to an *optimization problem* (OP) with endogenous equipment lifetime in the well-known Solow VCM [16]. The authors investigated the integral models with endogenous delay for various applied problems of economics, ecology, and engineering (see [6, 7, 8, 9, 10, 12, 19, 20] and the references therein). They provided an asymptotic analysis of the OP under study and discovered turnpike properties of the optimal equipment lifetime in [21] (see also [7, 8]). More complicated models with many inputs and outputs were investigated in [7, 10, 19].

The paper is organized as follows. The OP for the Solow model with endogenous capital lifetime is formulated in Section 2. Section 3 exposes preliminary results such as the condition for an extremum, gradient of the OP, and arising auxiliary nonlinear integral equation. In Section 4, the exact structure of optimal trajectories is established.

2. Statement of optimization problem

The OP under study consists of finding the functions $m(t)$ and $a(t)$, $t \in [t_0, T]$, $T < \infty$, which maximize the objective functional

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$$I = \int_{t_0}^{T} \rho(t) \left[\int_{a(t)}^{t} \beta(\tau,t) m(\tau) d\tau - \lambda(t) m(t) \right] dt \longrightarrow \max_{a,m}, \tag{2.1}$$

under the constraint equality

$$P(t) = \int_{a(t)}^{t} m(\tau) d\tau, \tag{2.2}$$

the constraint inequality

$$m_{\min}(t) \le m(t) \le M(t), \quad \text{where } m_{\min}(t) = \max\{0, P'(t)\}, \tag{2.3}$$

and the initial conditions

$$a(t_0) = a_0 < t_0, \qquad m(\tau) = m_0(\tau), \quad \tau \in [a_0, t_0]. \tag{2.4}$$

In mathematical economics, OP (2.1)–(2.4) describes the maximization of the net revenue (output minus investments) of an economic system in the Solow VCM [16]. The unknown variables are the investment $m(t)$ and the scrapping time $a(t)$ of obsolete capital, $t \in [t_0, T]$. Then $t - a(t)$ is the endogenous *lifetime of the capital* (the age of the oldest equipment still in use). The given characteristics are the *specific productivity* $\beta(\tau,t)$ (output per one worker on the equipment introduced at time τ), the specific cost $\lambda(t)$ of new equipment (per one worker), the total labour $P(t)$, the discounting factor $\rho(t)$, $0 < \rho(t) \le 1$, $\rho'(t) \le 0$, $t \in [t_0, T]$, and the investments $m_0(\tau)$ made on the prehistory interval $[a(t_0), t_0]$. The productivity $\beta(\tau,t)$ represents the *technological change* embodied in the new equipment *vintages* and strictly increases in τ (new machines are more efficient than the older ones).

We assume that the given functions β, λ, P, ρ, and M are Lipschitz continuous, m_0 is piecewise continuous, all these functions are positive and satisfy (2.2)–(2.4) at $t = t_0$.

3. Preliminary results

Presence of the unknown function a in integration limits determines the novelty of the OP. The investigation methods for such OPs were developed by Hritonenko and Yatsenko [7] and are based on common variation (perturbation) techniques of optimization theory (see, e.g., [11, 15, 18]). We introduce the gradient of functional (2.1) and express the extremum conditions in terms of the gradient.

3.1. The OP gradient. Let $m(t)$, $t \in [t_0, T]$, be the *independent control variable* of OP (2.1)–(2.4). Then the function $a(t)$, $t \in [t_0, T]$, is a dependent (*phase*) variable. As shown in Yatsenko [19], for any measurable control m that satisfies (2.3) almost everywhere (*a.e.*) on $[t_0, T]$, a unique a.e. continuous function $a(t) < t$, $t \in [t_0, T]$, exists which satisfies (2.2), (2.4) and a.e. has $a'(t) \ge 0$. In other words, the scrapping time $a(t)$ is monotonic and the scrapped equipment cannot be used again. The set of the measurable variables m that satisfy condition (2.3) is denoted by $\mathbf{U}$.

As shown in Hritonenko and Yatsenko [7], the increment δI of functional (2.1) in OP (2.1)–(2.4) is of the form

$$\delta I = I(m+\delta m) - I(m) = \int_{t_0}^{T} I'(t)\delta m(t)dt + \delta^2 I, \tag{3.1}$$

where the *gradient of functional I* is

$$\begin{gathered} I'(t) = \int_{t}^{\bar{a}^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a(\tau),\tau)]d\tau - \lambda(t)\rho(t), \quad t \in [t_0, T), \\ \bar{a}^{-1}(t) = \begin{cases} a^{-1}(t), & t \in [t_0, a(T)], \\ T, & t \in [a(T), T), \end{cases} \end{gathered} \tag{3.2}$$

$a^{-1}(t)$ is the inverse function of $a(t)$, and the higher-order variation residual is

$$\delta^2 I = \int_{t_0}^{T} \rho(t) \int_{a(t)}^{a(t)+\delta a(t)} [\beta(a(t),t) - \beta(\tau,t)][m(\tau)+\delta m(\tau)]d\tau\, dt = O(|\delta m|^2). \tag{3.3}$$

According to (2.2), the admissible variations $\delta m(t)$, $\delta a(t)$, $t \in [t_0, T]$, of the functions $m(t)$, $a(t)$, $t \in [t_0, T]$, in formulas (3.1)–(3.3) satisfy the equality

$$\begin{gathered} \int_{\max\{a(t),t_0\}}^{t} \delta m(\tau)d\tau = \int_{a(t)}^{a(t)+\delta a(t)} [m(\tau) + \delta m_{\text{int}}(\tau)]d\tau, \\ \delta m_{\text{int}}(\tau) = \begin{cases} \delta m(\tau), & t \in (t_0, T), \\ 0, & t \in [a(t_0), t_0]. \end{cases} \end{gathered} \tag{3.4}$$

3.2. The necessary and sufficient condition for an extremum. In order for a function $m^*(t)$, $t \in [t_0, T]$, to be a solution of OP (2.1)–(2.4), it is necessary and sufficient that

$$\begin{gathered} I'(a^*;t) < 0 \quad \text{at } m^*(t) = m_{\min}(t), \\ I'(a^*;t) > 0 \quad \text{at } m^*(t) = M(t), \\ I'(a^*;t) \equiv 0 \quad \text{at } m_{\min}(t) \le m^*(t) \le M(t),\ t \in [t_0, T]. \end{gathered} \tag{3.5}$$

The proof is given by Hritonenko and Yatsenko [7, 8]. The proof of the necessary condition is standard for such OPs. The sufficiency follows from the convexity of the functional $I(m)$ that holds because $\beta(\tau,t)$ is monotonic in τ. We will now illustrate it. Using the mean value theorem, (3.3) can be rewritten as

$$\delta^2 I = \int_{t_0}^{T} \rho(t)[\beta(a(t),t) - \beta(a(t)+\chi(t),t)] \int_{a(t)}^{a(t)+\delta a(t)} [m(\tau)+\delta m(\tau)]d\tau\, dt, \tag{3.6}$$

where $0 < \chi(t) < \delta a(t)$. Let $\delta m(\tau) = m_1(\tau) - m_2(\tau) \geq 0$, $\tau \in [t_0, T)$, and $\delta m(\tau) > 0$, $\tau \in \Delta_m \subset [t_0, T)$. Then, in view of (3.4), the corresponding variation $\delta a(t) \geq 0$ at $t \in [t_0, T)$ and $\delta a(t) > 0$, at least, for $t \in \Delta = \{t : \Delta_m \cap [a(t), t] \neq \varnothing\}$, and $\int_{a(t)}^{a(t)+\delta a(t)} [m(\tau) + \delta m(\tau)]d\tau \geq 0$ at $t \in [t_0, T)$ and is positive on Δ. Also, $\beta(a(t), t) - \beta(a(t) + \chi(t), t) < 0$. Hence, the integrand in the last formula for $\delta^2 I$ is nonpositive on $[t_0, T)$ and is negative on some subset Δ of $[t_0, T)$, that is, $\delta^2 I < 0$. The case $\delta m(\tau) < 0$ leads to the same result. Therefore, the functional $I(m)$ is strictly convex.

If $m(t) = 0$ at some points $t \in [t_0, T)$, then in view of (2.2) the variation $\delta a(t)$ can be finite for an infinitesimal $\delta m(\tau)$, $\tau < t$. In this case, the functional $I(m)$ is not differentiable, and expression (3.2) does not represent the gradient of functional (2.1). However, conditions (3.5) are still valid in this case because of the convexity of the functional $I(m)$ [14, 15]. The case $m = 0$ is natural in economics and is also presented below.

3.3. Dual integral-functional equation. As follows from (3.5), the integral-functional equation $I'(a; t) = 0$, $t \in [t_0, T)$, $T \leq \infty$, or

$$\int_t^{a^{-1}(t)} \rho(\tau)[\beta(t, \tau) - \beta(a(\tau), \tau)]d\tau = \lambda(t)\rho(t), \quad t \in [t_0, T), \tag{3.7}$$

with respect to the unknown function a plays an important role in a qualitative analysis of the OP solutions. In accordance with the economic content, we consider only its monotonic solutions $a(t) < t$.

Equation (3.7) generates a set of solutions $a_T(t)$ for a finite interval $[t_0, T]$. The given functions β, λ, ρ need to satisfy some strict conditions for the existence of the solutions a_T on large intervals $[t_0, T]$, $T \gg t_0$. The existence and uniqueness of the infinite solution $\tilde{a}(t)$, $t \in [t_0, \infty)$, has been investigated in [6, 7] for various combinations of exponential, power, and logarithmic functions β and λ.

Here and thereafter we assume that

$$\begin{gathered} \beta(\tau, t) = \beta_0 \exp(c_1 \tau), \quad \lambda(t) = \lambda_0 \exp(c_2 t), \quad \rho(t) = \exp(-c_3 t), \\ c_1, \beta_0, \lambda_0 > 0, \quad c_2 \leq c_1 < c_3, \quad \beta_0(c_3 - c_1) > \lambda_0 c_3 (c_3 - c_2) \exp[(c_2 - c_1)t_0]. \end{gathered} \tag{3.8}$$

According to [6, 7], (3.7) has a unique solution $\tilde{a}(t)$, $d\tilde{a}/dt > 0$, $t \in [t_0, \infty)$, such that

(1) if $c_1 > c_2$, then $t - \tilde{a}(t) \to 0$ at $t \to \infty$;
(2) if $c_1 = c_2$, then $\tilde{a}(t) \equiv t - L$, $t \in [t_0, \infty)$, where the constant L is determined from the following nonlinear equation

$$c_3 \exp(-c_1 L) - c_1 \exp(-c_3 L) = (c_3 - c_1)\left(1 - \frac{c_3 \lambda_0}{\beta_0}\right). \tag{3.9}$$

In particular, $L \approx [2\lambda_0/\beta_0/c_1]^{1/2}$ for $0 \leq c_1 < c_3 \ll 1$. Equation (3.7) has also a set of solutions $a_T(t)$, $da_T/dt > 0$, for any interval $t \in [t_0, T]$ such that all the solutions $a_T(\cdot)$ approach the unique solution $\tilde{a}(\cdot)$ at $T - t_0 \to \infty$.

3.4. Asymptotics of OP solutions. The study of the asymptotic behavior of OP solutions at large $T - t_0 \gg 1$ was provided by Yatsenko [19], Hritonenko and Yatsenko [7, 8, 10] where a convergence of the optimal trajectories a^* to (3.7) solution $\tilde{a}$ on the infinite interval $[t_0, \infty)$ was established. Namely, under some assumptions [7, 8], the solution a^* to OP (2.1)–(2.4) strives to $\tilde{a}$ at $T \to \infty$ on an asymptotically largest part of the interval $[t_0, T]$, that is, for any $\varepsilon > 0$, the time T_0 exists such that for any $T \geq T_0$ the condition $|a^*(t) - \tilde{a}(t)| < \varepsilon$ is true on some subset $\Delta \subset [t_0, T]$ such that $\text{mes}(\Delta)/(T - t_0) \to 1$ for $T \to \infty$. In economics, such phenomena are known as turnpike properties.

Next, the exact structure of the solution (m^*, a^*) of OP (2.1)–(2.4), (3.8) is studied.

4. Structure of OP solutions

Lemma 4.1. *There is an instant Θ, $t_0 \leq \Theta < T$, such that $I'(a^*; t) < 0$ and the OP solution is the minimum possible from Θ onward: $m^*(t) \equiv m_{\min}(t)$, $a^*(t) \equiv a_{\min}(t)$ for $t \in (\Theta, T]$, where $m_{\min}$ is determined by (2.3).*

The proof follows directly from the analysis of expression (3.2) at t close to T. Lemma 4.1 shows that the OP (2.1)–(2.4), (3.8) possesses a "*zero-investment period*" *at the end of the planning horizon*, which is a common effect in various finite-horizon OPs of mathematical economics.

We construct an exact analytic solution to the nonlinear OP (2.1)–(2.4), (3.8). The technique is essentially based on Lemma 4.1 and the special structure of the expression (3.2) for the gradient $I'(a; t)$, $t \in [t, T]$. Namely, $I'(a; t)$ does not depend explicitly on the independent unknown control m. It allows us to use the following approach to construct the OP solution. We start the construction of the solution a^* from the right end of the horizon $[t_0, T]$ because Lemma 4.1 gives the clear clue about its behavior: $a^*(t) = a_{\min}(t)$ on some interval $(\Theta, T]$, $\Theta \geq t_0$. Then, if $\Theta > t_0$, we try to build the solution a^* recurrently from the right to the left, adjusting $I'(a^*; t)$ to zero and keeping its value zero where it is possible. The corresponding solution $m^*(t)$, $t \in [t_0, T]$, is determined from (2.2). Finally, we will verify that the constructed solution satisfies the extremum conditions (3.5).

4.1. Analysis of dual equation. In order to implement the above-described structure of the OP solution, we need to solve equation $I'(a; t) \equiv 0$ on the interval $[t_0, T]$ with the initial condition at the right end:

$$a(t) \equiv a(\Theta) = \text{const}, \quad t \in (\Theta, T]. \tag{4.1}$$

In case (3.8), the OP gradient (3.2) is determined as

$$I'(a; t) = \beta_0 \int_t^{\bar{a}^{-1}(t)} e^{-c_3 \tau} \left[e^{c_1 t} - e^{c_1 a(\tau)} \right] d\tau - \lambda_0 e^{(-c_3 + c_2) t}, \quad t \in [t_0, T]. \tag{4.2}$$

The differentiation of equation $I'(a; t) \equiv 0$ leads to the following expression:

$$t - a(t) = -\frac{1}{c_1} \ln \left\{ \frac{1 - \lambda_0}{\beta_0 (c_3 - c_2) e^{(c_2 - c_1) t} + (c_1/c_3) \left[e^{-c_3(\bar{a}^{-1}(t) - t)} - 1 \right]} \right\}, \quad t \in \Delta. \tag{4.3}$$

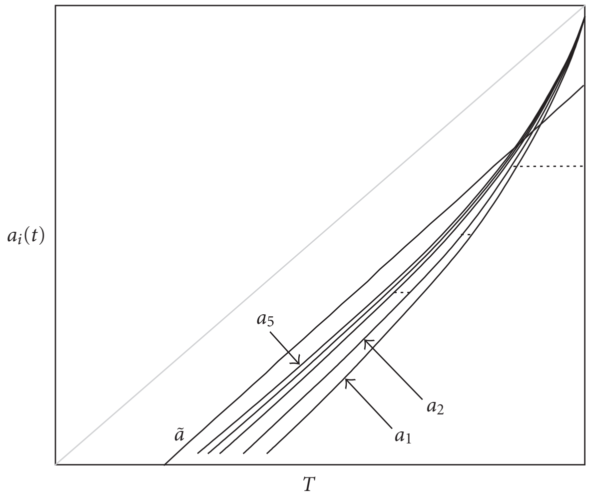

Figure 4.1. The trajectories a_i, $i = 1,2,3,\dots$, can keep the zero value of the OP gradient and are the only interior parts of the optimal trajectory a.

If $I'(a;t) \equiv 0$, $t \in \Delta \subset [t_0,T]$, then the functions $a(t)$ and $a^{-1}(t)$ satisfy (4.3) on Δ. Since $a(t) < t$, then $a^{-1}(t) > t$ and expression (4.3) is a recurrent relation from the right to the left.

It appears that there is a discrete set of special trajectories a_i, $i = 1,2,\dots$, such that a function $a(t)$ should coincide with one of the trajectories $a_i(t)$ on $\Delta \subset [t_0,T]$ in order to produce $I'(a;t) \equiv 0$ at $t \in \Delta$. Namely, knowing $a(t) \equiv a(\Theta)$, $t \in (\Theta,T]$, we can determine $a_1(t)$ from (4.3) on the interval $[a(\Theta),\Theta]$, then determine $a_2(t)$ on the interval $[a_1(a(\Theta)),a(\Theta)]$, and so on. The trajectories $a_i(t)$ depend only on the constant T and functions β, λ, ρ. Several first trajectories $a_i(t)$ calculated at $T = 40$, $c_1 = 0.47$, $c_2 = 0.47$, $c_3 = 0.5$, $\beta_0 = 1$, and $\lambda_0 = 1.9$ are shown on Figure 4.1 with solid lines. At large i and $T-t$, they are close to the solution $\tilde{a}$ of (3.7) on the infinite interval $[t_0,\infty)$, indicated by a gray line.

Lemma 4.2. *The trajectories*

$$a_1(t) = t + \frac{1}{c_1}\ln\left\{\frac{1-\lambda_0}{\beta_0(c_3-c_2)e^{(c_2-c_1)t} + (c_1/c_3)\left[e^{-c_3(T-t)}-1\right]}\right\}, \quad t < T, \tag{4.4}$$

$$a_{i+1}(t) = t + \frac{1}{c_1}\ln\left\{\frac{1-\lambda_0}{\beta_0(c_3-c_2)e^{(c_2-c_1)t} + (c_1/c_3)\left[e^{-c_3(a_i^{-1}(t)-t)}-1\right]}\right\}, \quad t < T,\ i = 2,3,\dots, \tag{4.5}$$

have the following properties:

(1) *if $I'(a_i;t)$ is constant at $t \in [t',t'']$, $t'' \le T$, then $I'(a_{i+1};t)$ is constant at $t \in [a_i(t'), a_i(t'')]$, $t'' \le T$, $i \ge 1$;*
(2) *$a_i(t) < a_{i+1}(t)$, $t \in [t_0,T]$;*

(3) $da_i(t)/dt > 1$, $t \in [t_0, T]$, *at* $c_1 \geq c_2$;
(4) $a_1(t) \to t + \ln[1 - \lambda_0/\beta_0(c_3 - c_2)\exp(c_2 - c_1)t - c_1/c_3]/c_1$ *as* $T - t \to \infty$.

The proof follows from the analysis of the recurrent relation (4.3).

So, if a function a satisfies equation $I'(a;t) \equiv 0$ on the interval $[t_0, T]$ with the initial condition (4.1) at the right end $(\Theta, T]$, it should coincide with the trajectories $a_i(t)$, $i = 1, 2, 3, \dots$, on the intervals $[a(\Theta), \Theta], [a(a(\Theta)), a(\Theta)], [a(a(a(\Theta))), a(a(\Theta))], \dots$, and jump from $a_i(t)$ to $a_{i+1}(t)$. As follows from (4.5) and Figure 4.1, such a solution is *discontinuous*. Next, we use this solution to build a *continuous quasisolution* to the OP.

4.2. Structure of the OP solutions. We define the *quasisolution* to OP (2.1)–(2.4) as a *continuous* monotonic function $a_q(t)$, $t \in [t_0, T]$, that satisfies the extremum condition (3.5) and does not necessarily satisfy the initial condition $a(t_0) = a_0$ in (2.4).

We assume here and thereafter that

$$P'(t) \geq 0, \quad t \in [t_0, T]. \tag{4.6}$$

This condition ensures that the quasisolution a_q (if it exists) does not depend on m. Indeed, in view of (2.4),

$$m(a(t))a'(t) = m(t) - P'(t) \geq 0 \tag{4.7}$$

for any admissible m. So, the boundary-valued regime $m(t) = m_{\min}(t)$ at $[t_1, t_2] \subset [t_0, T]$ means $a'(t) \equiv 0$, $a(t) \equiv a(t_1)$, and $m(t) = P'(t) \geq 0$ for $t \in [t_1, t_2]$. Hence, the $a_{\min}(t)$, $t \in [t_1, t_2]$, depends on the value $a_{\min}(t_1)$ only and does not depend on $m_{\min}$.

Because of Lemma 4.2, we can separate the interval $[t_0, T]$ of any finite length into the parts $[a_q(\Theta), \Theta], [a_q(a_q(\Theta)), a_q(\Theta)], [a_q(a_q(a_q(\Theta))), a_q(a_q(\Theta))], \dots$, and assign the intended quasisolution a_q to the trajectories $a_i(t)$, $i = 1, 2, 3, \dots$. To obtain *a continuous quasisolution*, we connect the separate pieces of a_i with boundary-valued trajectories $a_{\min}$ as it is illustrated on Figure 4.1 with the dashed lines. The full implementation of the explained scheme is provided below.

Lemma 4.3 (on OP quasisolution). *Under conditions (3.8) and (4.6), there exists a* quasisolution $a_q(t)$ *to the finite-horizon OP (2.1)–(2.4):*

$$a_q(t) = \begin{cases} a_i(\alpha_i), & I'(t) < 0,\ t \in (\alpha_i, \beta_i) \\ a_i(t), & I'(t) = 0,\ t \in [\beta_{i+1}, \alpha_i), \end{cases} \quad i = 1, 2, \dots,\ t \in [t_0, T), \tag{4.8}$$

where the parameters α_i, β_i, $i = 1, 2, 3, \dots$, *are uniquely determined,* $\beta_{i+1} < \alpha_i$, $\alpha_i < \beta_i$, $\beta_1 = T$, *and the trajectories* a_i, $i = 1, 2, 3, \dots$, *are determined in Lemma 4.2.*

Proof. The construction of the quasisolution a_q starts from the right end T of the horizon $[t_0, T]$. In view of Lemma 4.1, the gradient $I'(a_q;t) < 0$ on some "zero-investment" interval $(\Theta, T]$ to the left of T. Hence, $a_q(t)$ is minimum possible, $a_q(t) = a_{\min}(t) \equiv a_q(T)$, and $a_q^{-1}(t) \equiv T$, $t \in [\Theta, T]$. After increasing $I'(a_q;t)$ up to zero at $t = \Theta$, we keep $I'(a_q;t) = 0$ to the left of Θ. Lemma 4.2 shows that the only way to implement this is to keep $a_q(t)$ on the curve $a_1(t)$ at $t < \Theta$. So, we need to find the point $(\Theta, a_1(\Theta))$ on the curve $a_1(t)$

that satisfies $I'(a_q;\Theta) = 0$. To show that such Θ exists, we investigate the asymptotic of function $I'(a_q;\Theta)$. The substitution of $a_q(t)$ and $a_q^{-1}(t)$, $t \in [\Theta, T]$, into (4.2) leads to

$$\begin{aligned} I'(a_q;\theta) &= \int_\theta^T e^{-c_3\tau}\left[e^{c_1\theta} - e^{c_1 a(\theta)}\right]d\tau - \frac{\lambda_0}{\beta_0 e^{(-c_3+c_2)\theta}} \\ &= e^{(-c_3+c_1)\theta}\left\{\frac{\left[1 - e^{c_1[a(\theta)-\theta]}\right]\left[1 - e^{-c_3(T-\theta)}\right]}{c_3} - \frac{\lambda_0}{\beta_0 e^{(c_2-c_1)\theta}}\right\}. \end{aligned} \tag{4.9}$$

Substituting the asymptotic expression of $a_1(t)$ at $T - t \to \infty$ from Lemma 4.2 for $a(\Theta)$, we obtain that

$$\begin{aligned} I'(a_q;\theta) &= e^{(-c_3+c_1)\theta}\left\{\left[\frac{\lambda_0}{\beta_0(c_3-c_2)e^{(c_2-c_1)\theta}} - \frac{\lambda_0}{\beta_0 c_1} + \frac{c_1}{c_3}\right]\frac{\left[1 - e^{-c_3(T-\theta)}\right]}{c_3} - \frac{\lambda_0}{\beta_0 e^{(c_2-c_1)\theta}}\right\} \\ &= \frac{e^{(-c_3+c_1)\theta}}{c_3}\left\{\left[\frac{\lambda_0}{\beta_0 c_3 e^{(c_2-c_1)\theta}} + \frac{c_1}{c_3}\left(\frac{1-\lambda_0}{\beta_0 c_3}\right)\right]\left[1 - e^{-c_3(T-\theta)}\right] - \frac{\lambda_0}{\beta_0 e^{(c_2-c_1)\theta} c_3}\right\} \\ &= \frac{e^{(-c_3+c_1)\theta}}{c_3}\left\{\left[-\frac{c_2-c_1}{c_3} + \frac{c_2}{c_3}\left(1 - \frac{\lambda_0}{\beta_0} c_3 e^{(c_2-c_1)\theta}\right)\right]\left[1 - e^{-c_3(T-\theta)}\right]\right. \\ &\qquad \left. - \frac{\lambda_0}{\beta_0} e^{(c_2-c_1)\theta} c_3 e^{-c_3(T-\theta)}\right\}. \end{aligned} \tag{4.10}$$

Using the inequalities in (3.8), we obtain $I'(a_q;\Theta) > 0$ for $T - \Theta \gg 1$. Since $I'(a_q;T) < 0$ and $I'(a_q;\Theta)$ is continuous, at $T - t_0 \gg 1$ a unique moment $\alpha_1 = \Theta$ exists that satisfies the equality $I'(a_q;\Theta) = 0$.

We construct the next piece of a_q on the interval $[\alpha_2, \alpha_1]$. Because of the symmetry of inverse functions, the inverse $a_q^{-1}(t)$, $t \in (a_1(a_1(\Theta)), T]$, is already defined by $a_q(t)$, $t \in (a_1(\Theta), T]$. We put $a_q(t) = a_1(t)$ to keep $I'(a_q;t) \equiv 0$ to the left of α_1 on some interval $[\beta_2, \alpha_1]$. So, the trajectory $a_q(t)$ has to leave $a_1(t)$ at some point β_2, before $a_q^{-1}(t)$ jumps from T to $a_1^{-1}(t)$ at $t = a_1(\Theta)$. To the left of β_2, $a_q(t)$ follows the boundary minimum trajectory $a_q(t) = a_q(\beta_2) = a_1(\beta_2)$ until it reaches the second line $a_2(t)$ at some point $\alpha_2 < \beta_2$. The points α_2 and β_2 are found from the condition $I'(a_q;\alpha_2) = 0$ on the new curve $a_2(t)$. To show that the point α_2 exists, we estimate that the gradient $I'(a_q;t) > 0$ at $\alpha_2'' = a_1(\Theta)$ and $I'(a_q;t) < 0$ at the point $\alpha_2' < \alpha_2''$ such that $\beta_2' = a_1(\Theta)$ and $a_2(\alpha_2') = a_1(\beta_2')$. Because of the continuity of $I'(a_q;t)$ in t, a unique moment α_2, $\alpha_2' < \alpha_2 < \alpha_2''$, exists such that $I'(a_q;\alpha_2) = 0$.

We show that $I'(a_q;t)$ is less than 0 when the quasisolution $a_q(t)$ leaves the curve $a_1(t)$ at $t = \beta_2$ and until it reaches $a_2(t)$ at $t = \alpha_2$. By construction, the gradient $I'(a_q;t) \equiv 0$ on interval $[\beta_2, \alpha_1]$. We investigate its derivative $d[I'(a_q;t)]/dt$ in t:

$$\begin{aligned} \frac{d[I'(a;t)]}{dt} &= -\frac{c_1}{c_3}\left[e^{-c_3 a^{-1}(t)} - e^{-c_3 t}\right]e^{c_1 t} + \left[e^{c_1 a(t)} - e^{c_1 t}\right]e^{-c_3 t} \\ &\quad + \lambda_0(c_3 - c_2)e^{-(c_3-c_2)t}, \quad t \in [t_0, T). \end{aligned} \tag{4.11}$$

At the beginning, we consider a small neighborhood of the instant $t = \beta_2$. Here $a_q(t) > a_1(t)$ at $t < \beta_2$, hence $d[I'(a_q;t)]/dt > d[I'(a_1;t)]/dt = 0$ in view of the last formula. Similarly, $d[I'(a_q;t)]/dt < d[I'(a_2;t)]/dt = 0$ at some neighborhood $t > \alpha_2$. Therefore, $I'(a_q; t) < 0$ on interval (α_2, β_2).

The previous part contains a complete iteration in constructing the quasisolution a_q. At the beginning, $a_q(t) = a_1(t)$, $t \in (\beta_2, \alpha_1)$, hence $a_q^{-1}(t) = a_1^{-1}(t)$ at $t < a_1(\beta_2)$. According to Lemma 4.2, the new curve is $a_2(t)$ on some interval to the left of β_2. The trajectory a_q is minimum possible $a_q(t) = a_q(\beta_2) = a_1(\beta_2)$ until it intersects a_2 at some point $\alpha_2 < \beta_2$. Then the corresponding a_q^{-1} may be found, the iteration may be repeated, and so on. The "switch" points α_i, β_i, $a_i(\alpha_i) = a_{i-1}(\beta_i)$, $i = 1, 2, \ldots$, where the quasisolution $a_q(t)$ leaves the old curve $a_i(t)$ for the new one, are uniquely determined from the equation $I'(a_q;\alpha_i) = 0$ on the new curve a_{i+1}.

Finally, we verify that the quasisolution a_q satisfies the extremum conditions (3.5). Namely, $a_q(t)$, $t \in [t_0, T]$, is constructed in such a way that $I'(a_q;t) < 0$ on (α_i, β_i) or where $a_q = a_{\min} \equiv \text{const}$, and $I'(a_q;t) = 0$ on (β_{i+1}, α_i) or where $a_q \equiv a_i$. □

If a quasisolution a_q exists, then the optimal trajectory a^* will coincide with it except for an initial finite interval $[t_0, \mu)$. At the initial interval $[t_0, \mu)$, the OP solution will be boundary-valued: $m^* \equiv m_{\min}$ or $m^* \equiv M$. The corresponding $m^*(t)$, $t \in [t_0, T]$, is defined from (2.2) and always depends on the initial condition m_0.

The explicit formula (4.8) for the OP quasisolution a_q allows us to prove the following result.

Theorem 4.4 (on the structure of the OP solution). *Under conditions (3.8) and (4.6), OP (2.1)–(2.4) has the unique solution (m^*, a^*) of the following form:*

$$m^*(t) = \begin{cases} m_{\min}(t) \text{ or } M(t), & t \in [t_0, \mu), \\ m_q(t), & t \in [\mu, T), \end{cases}$$
$$a^*(t) = \begin{cases} a_\mu(t), & t \in [t_0, \mu), \\ a_q(t), & t \in [\mu, T), \end{cases} \tag{4.12}$$

where

$$a_\mu(t) = \begin{cases} a_{\min}(t) & \text{if } a_0 > \tilde{a}(t_0), \\ a_{\max}(t) & \text{if } a_0 < \tilde{a}(t_0), \end{cases} \tag{4.13}$$

and a_q is the quasisolution determined by Lemma 4.3. The function m_q is found from (2.2) at $a \equiv a_q$, the functions $a_{\min}$ and $a_{\max}$ are defined from (2.2) and correspond to the minimum $m \equiv m_{\min}$ and maximum $m \equiv M$, and the instant μ is determined from the condition $a_\mu(\mu) = a_q(\mu)$.

Proof. The proof consists of two steps: (a) the construction of an admissible solution (m^*, a^*) and (b) the verification of its optimality. During the first step, the optimal solution a^* is obtained by the adjustment of the quasisolution a_q to the initial conditions (2.4). Two cases are possible.

Case 1. $a_0 > a_q(t_0)$. We choose $m^*(t) = m_{\min}(t)$ at $t > t_0$ and move on the corresponding $a_{\min}(t) \equiv a_0$ until it crosses the trajectory a_q. Then the point of interception is μ and the line of movement is $a_\mu(t)$, $t \in [t_0, \mu]$. According to (4.7), $a_\mu(t) = a_0$, $t \in [t_0, \mu]$, and the corresponding gradient

$$\begin{aligned} I'(a^*;t) &= \int_t^{a^{*-1}(t)} e^{-c_3\tau}\Big[e^{c_1 t} - e^{c_1 a^*(\tau)}\Big]d\tau - \lambda_0 e^{(-c_3+c_2)t} \\ &= \int_t^{a_q^{-1}(t)} e^{-c_3\tau}\Big[e^{c_1 t} - e^{c_1 a_q(\tau)}\Big]d\tau - \lambda_0 e^{(-c_3+c_2)t} \\ &\quad + \int_t^{\mu} e^{-c_3\tau}\Big[e^{c_1 a_q(\tau)} - e^{c_1 a_0}\Big]d\tau \\ &= \int_t^{\mu} e^{-c_3\tau}\Big[e^{c_1 a_q(\tau)} - e^{c_1 a_0}\Big]d\tau < 0, \quad t \in [t_0, \mu], \end{aligned} \tag{4.14}$$

that is, the pair (a^*, m^*) satisfies the extremum conditions (3.5) at $t \in [t_0, \mu]$. Later on at $t > \mu$, the solution $a^*(t)$ coincides with the quasisolution $a_q(t)$. The corresponding $m^*(t)$, $t \in [\mu, T]$, is determined from (4.7), hence, it will be 0 on (α_i, β_i) where $I'(a^*;t) < 0$ and an internal value between $m_{\min}$ and M from the domain (2.3) on (β_{i+1}, α_i) where $I'(a^*;t) = 0$, $i = 1, 2, 3, \ldots$. Therefore, according to Lemma 4.2, the pair (4.12) is a solution to OP (2.1)–(2.4).

Case 2. $a_0 < a_q(t_0)$ is investigated similarly. In this case $m(t) = M$ brings $a_\mu(t)$ up to the point of its interception with the trajectory a_q and $I'(a^*;t) > 0$ on $t \in [t_0, \mu)$. □

The dynamics of OP solution (a^*, m^*) and the corresponding gradient $I'(t)$ are depicted in Figure 4.2 for the case $P'(t) = 0$. The restriction $P' = 0$ is selected for simplicity only (then two boundary-valued regimes $a'_{\min} \equiv 0$ and $m_{\min} \equiv 0$ coincide).

Theorem 4.4 shows that the irregularities in the optimal controls m^* and a^* are caused by the initial and final conditions of the OP. First, the "imperfect" initial condition $a(t_0) = a_0 \neq \tilde{a}(t_0)$ *on the left end* $t = t_0$ *of* $[t_0, T]$ causes the appearance of an initial boundary-valued section $m(t) \equiv 0$ or M, $t \in [t_0, \mu]$, in the optimal trajectory m^*. The control $m^*(t)$ is determined by (2.2) as $m^*(t) = P'(t) + m(a^*(t))da^*/dt$ from the left to the right, starting with the initial condition (2.4). This formula reproduces the jump in m^* throughout the whole horizon $[t_0, T]$ (when we reach the interval $[a^{-1}(t_0), a^{-1}(\mu)]$, later on the interval $[a^{-1}(a^{-1}(t_0)), a^{-1}(a^{-1}(\mu))]$, and so on). This phenomenon was earlier analyzed in [2].

Secondly, the optimal trajectory a^* also has irregular sections $[\alpha_i, \beta_i]$ where $a^{*'}(t) \equiv 0$. They represent the impact of the zero-investment period $(\alpha_1, T]$ at *the right end of* $[t_0, T]$ on optimal trajectories. When we reach such a section, then $m^* = m_{\min} = P'$. Thus, the optimal control m^* has *two groups* of the *replacement echoes* on the planning horizon $[t_0, T]$: (a) *the echoes* caused by the "imperfect" initial condition $a(t_0) = a_0 \neq \tilde{a}(t_0)$ at $t = t_0$; (b) the "*zero-investment*" *echoes* caused by the "zero-investment period" $(\alpha_1, T]$

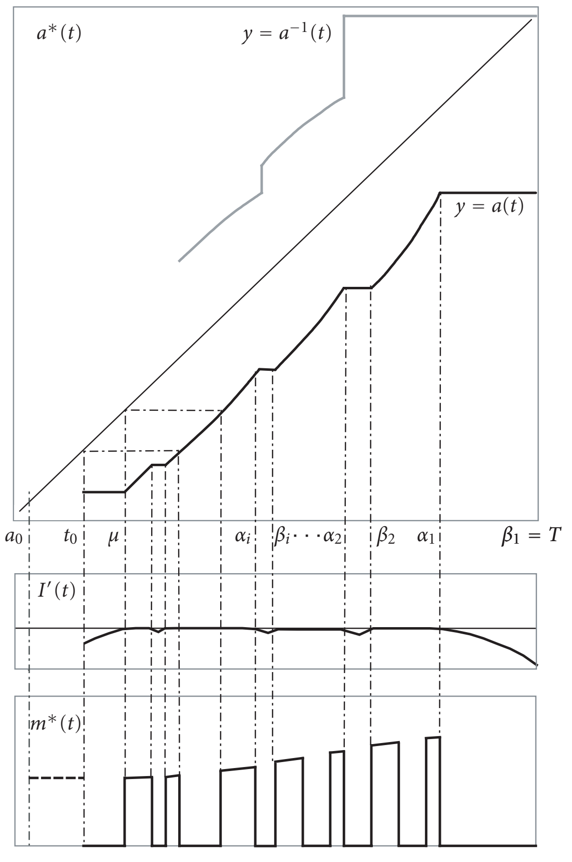

Figure 4.2. The solution a and m and the gradient of OP (2.1)–(2.4).

at *the right end of* $[t_0, T]$. The "zero-investment" echoes propagate backward throughout the whole horizon $[t_0, T]$ starting from the right end of $[t_0, T]$.

5. Conclusion

The constructed exact solution to Solow VCM considerably develops the mathematical theory of VCMs. Investigation of applied OPs usually involves a combination of analytic, approximate, and simulation methods. The construction of exact solutions is important to every applied mathematical problem, especially to nonlinear optimal control problems because of their high analytic and computational complexity. Even the existence of a solution is usually an open question and the exact solution automatically solves the existence problem.

The established structure of the exact OP solutions provides a new insight into the optimal dynamics of the capital renovation process. An important feature of this process is that the optimal trajectories do not possess irregularities of an arbitrary small length similar to "vibration controls" or generalized functions (see [11, 18] and others). While such behavior of optimal controls is natural for technical applications, it causes essential problems in economic interpretation.

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References

[1] J. Benhabib and A. Rustichini, *Vintage capital, investment, and growth*, J. Econom. Theory **55** (1991), no. 2, 323–339.
[2] R. Boucekkine, M. Germain, and O. Licandro, *Replacement echoes in the vintage capital growth model*, J. Econom. Theory **74** (1997), no. 2, 333–348.
[3] J. Campbell, *Entry, exit, embodied technology, and business cycles*, Rev. Econ. Dynam. **1** (1998), no. 2, 371–408.
[4] T. Cooley, J. Greenwood, and M. Yorukoglu, *The replacement problem*, J. Monet. Econ. **40** (1997), no. 3, 457–499.
[5] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.
[6] N. Hritonenko and Yu. Yatsenko, *Integral-functional equations for optimal renovation problems*, Optimization **36** (1996), no. 3, 249–261.
[7] ______, *Modeling and Optimization of the Lifetime of Technologies*, Applied Optimization, vol. 4, Kluwer Academic Publishers, Dordrecht, 1996.
[8] ______, *Turnpike theorems in an integral dynamic model of economic restoration*, Cybernet. Systems Anal. **33** (1997), no. 2, 259–273.
[9] ______, *Mathematical Modeling in Economics, Ecology and the Environment*, Applied Optimization, vol. 34, Kluwer Academic Publishers, Dordrecht, 1999.
[10] ______, *Applied Mathematical Modelling of Engineering Problems*, Applied Optimization, vol. 81, Kluwer Academic Publishers, Massachusetts, 2003.
[11] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*, Studies in Mathematics and Its Applications, vol. 6, North-Holland Publishing, Amsterdam, 1979.
[12] V. V. Ivanov and Ju. P. Jacenko, *Problem of optimal control for integral dynamic models with controllable prehistory*, System Modelling and Optimization (Budapest, 1985), Lecture Notes in Control and Inform. Sci., vol. 84, Springer, Berlin, 1986, pp. 350–357.
[13] B. Jovanovic, *Vintage capital and inequality*, Rev. Econ. Dynam. **1** (1998), no. 2, 497–530.
[14] J. M. Malcomson, *Replacement and the rental value of capital equipment subject to obsolescence*, J. Econom. Theory **10** (1975), no. 1, 24–41.
[15] R. T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series, no. 28, Princeton University Press, New Jersey, 1970.
[16] R. M. Solow, J. Tobin, C. von Weizsacker, and M. Yaari, *Neoclassical growth with fixed factor proportions*, Rev. Econom. Stud. **33** (1966), 79–115.
[17] O. van Hilten, *The optimal lifetime of capital equipment*, J. Econom. Theory **55** (1991), no. 2, 449–454.
[18] J. Warga, *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.
[19] Yu. Yatsenko, *Integral Models of Systems with Controllable Memory*, Naukova Dumka, Kiev, 1991.
[20] ______, *Volterra integral equations with unknown delay time*, Methods Appl. Anal. **2** (1995), no. 4, 408–419.

[21] Yu. Yatsenko and N. Hritonenko, *Optimization in integral model of developing systems*, Optimization **31** (1994), no. 2, 179–192.

Natali Hritonenko: Department of Mathematics, Prairie View A&M University, Prairie View, TX 77446-4189, USA
E-mail address: natalia_hritonenko@pvamu.edu

Yuri Yatsenko: College of Business & Economics, Houston Baptist University, 7502 Founder Road, Houston, TX 77074-3298, USA
E-mail address: yyatsenko@hbu.edu