LAPLACE TRANSFORM GENERATION THEOREMS AND LOCAL CAUCHY PROBLEMS

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We give new criterions to decide if some vector-valued function is a local Laplace transform and apply this to the theory of local Cauchy problems. This leads to an improvement of known results and new Hille-Yosida-type theorems for local convoluted semigroups.

1. Introduction

Let X be a Banach space, $x \in X$, $T > 0$, $h \in L_1([0,T],\mathbb{K})$, and $f(t) := \int_0^t h(s)ds$ the antiderivative of h.

Assume $A : D(A) \to X$ is some closed (linear) operator in X.

Consider the abstract Cauchy problem

(ACP) $Aw(t) = w'(t)$ if $0 \le t \le T$, $w(0) = x$, and $w \in C^1([0,T],X)$

and the f-regularized problem

(ACP_f) $Au(t) + f(t)x = u'(t)$ if $0 \le t \le T$, $u(0) = 0$, and $u \in C^1([0,T],X)$.

If w solves (ACP), then the convolution $u := w * f$ solves (ACP_f), since

$$u' = w' * f + f(\cdot)x = w * h \quad \text{on } [0,T]. \tag{1.1}$$

Any function u that solves (ACP_f) is called h-regularized solution of (ACP).

Now assume that $0 \in \operatorname{supp} h$, which means that h does not vanish on any interval $[0,\varepsilon)$, and which is equivalent to $0 \in \operatorname{supp} f$.

By the theorem of Titchmarsh-Foiaş (see [3]) the convolution operator

$$\begin{gathered} S_f : C([0,T],X) \longrightarrow C_0([0,T],X) = \{g \in C([0,T),X) \mid g(0) = 0\}, \\ S_f g = g * f, \end{gathered} \tag{1.2}$$

can be extended to an isometric isomorphism

$$\widetilde{S}_f : C^{[f]}([0,T],X) \longrightarrow C_0([0,T],X), \tag{1.3}$$

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where the space $C^{[f]}([0,T],X)$ of generalized functions is the completion of $C([0,T],X)$ with norm $\|g\| := \|S_f g\|_\infty$. On this construction see also [9]. Thus, if w solves (ACP), then $S_f w$ solves (ACP_f).

This gives a reason to extend the notion of solutions of (ACP): a generalized function $w \in C^{[f]}([0,T],X)$ is called h-generalized solution of

$$(\mathrm{ACP}_*)\ Aw(t) = w'(t) \text{ if } 0 \le t \le T,\ w(0) = x$$

if $u = \widetilde{S}_f w$ solves (ACP_f), that is, if $\widetilde{S}_f w$ is an h-regularized solution of (ACP).

The notation of generalized solutions was introduced by Cioranescu and Lumer [5, 6].

If w is an h-generalized solution of (ACP_*), there is a sequence $(v_n)_n \subset C([0,T], \overline{D}(A))$ with $\lim_v v_n = u'$ in $C([0,T],X)$ and with $\lim_n v_n^{[1]} = u$ in $C([0,T],\overline{D}(A))$ (where $\overline{D}(A)$ denotes the Banach space $D(A)$ with the graph norm).

Thus, $\lim_n v_n = u'$, and $\lim_n(Av_n^{[1]} + f(\cdot)x - v_n) = 0$ in $C([0,T],X)$.

If $(v_n)_n \subset C([0,T],X)$ converges uniformly and satisfies $v_n^{[1]}(t) \in D(A)$ for all t and $\lim_n(Av_n^{[1]} + f(\cdot)x - v_n) = 0$ uniformly, then $(v_n)_n$ is called h-approximate solution of (ACP). Thus to every h-generalized solution there is an h-approximate solution.

On the other hand, if $(v_n)_n$ is an h-approximate solution of (ACP), then $w_n = \widetilde{S}_h^{-1}(v_n - v_n(0))$ converges in $C^{[h]}([0,T],X)$ to some w, and $u = \widetilde{S}_f w = \lim_n v_n^{[1]}$ solves (ACP_f). On approximate solutions see [1, 2]. Consequently, the notations of generalized solution of (ACP_*), of approximate solutions of (ACP), and (classical) solutions of (ACP_f) are equivalent. That is the reason why it is interesting to study problems of type (ACP_f).

We first clarify the notations.

If X is some Banach space, $g \in L_1^{\mathrm{loc}}([0,\infty),X)$, and $\alpha, \tau > 0$, we let

$$g^{(-\alpha)}(t) := g^{[\alpha]}(t) := \frac{1}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} g(t-s)\,ds \tag{1.4}$$

be the αth integral of g. If there is some $h \in C([0,\tau],X)$ with $g = h^{[\alpha]}$, then let $g^{(\alpha)} = h$ be the αth derivative of g.

The finite Laplace transform of g on $[0,\tau]$ is given by

$$\hat{g}_\tau(\lambda) = g_\tau^\wedge(\lambda) := \int_0^\tau e^{-\lambda t} g(t)\,dt. \tag{1.5}$$

If $\alpha > 0$ and g is exponentially bounded for large arguments, then $g^{[\alpha]}$ is exponentially bounded for large arguments, and

$$\lambda^\alpha \int_0^\infty e^{-\lambda t} g^{[\alpha]}(t)\,dt = \int_0^\infty e^{-\lambda t} g(t)\,dt =: \hat{g}(\lambda) = g^\wedge(\lambda) \tag{1.6}$$

for all large $\mathrm{Re}\,\lambda$, which is the Laplace transform of g.

In fact we have, if $\tau > 0$,

$$\lambda^\alpha \int_0^\tau e^{-\lambda t} g^{[\alpha]}(t)\,dt \sim_\tau \int_0^\tau e^{-\lambda t} g(t)\,dt, \tag{1.7}$$

where

$$f(\lambda) \sim_\tau h(\lambda) \Longleftrightarrow \limsup_{\lambda\to\infty} \frac{\ln|f(\lambda)-h(\lambda)|}{\lambda} \le -\tau$$
$$\Longleftrightarrow \lim_{\lambda\to\infty} e^{-\delta\lambda}e^{\lambda\tau}|f(\lambda)-h(\lambda)| = 0 \quad \forall\delta>0. \tag{1.8}$$

This follows from

$$\lambda^\alpha \int_0^\tau e^{-\lambda t} g^{[\alpha]}(t)dt = \frac{\lambda^\alpha}{\Gamma(\alpha)}(p_{\alpha-1}*g)^\wedge_\tau(\lambda), \quad \text{where } p_\beta(s)=s^\beta,$$
$$\sim_\tau \frac{\lambda^\alpha}{\Gamma(\alpha)}(p_{\alpha-1})^\wedge(\lambda)\cdot g^\wedge_\tau(\lambda) = g^\wedge_\tau(\lambda). \tag{1.9}$$

In trying to get a nice theory how to solve abstract Cauchy problems

$$Au_x(t)+f(t)x = u_x'(t) \quad \text{if } 0\le t\le\tau,$$
$$u_x(0)=0, \tag{1.10}$$
$$u_x \in C^1([0,\tau],X),$$

with continuous inhomogeneity f, it seems to be natural first to consider inhomogeneities of type t^β, where $\beta\in\mathbb{N}$ or, more generally, $\beta\ge 0$. The reason for this is that if the problem is well posed on $[0,\tau]$ with this inhomogeneity, that is, A generates a β-integrated semigroup on $[0,\tau]$, then it is well posed with inhomogeneities $h^{[\beta+1]}$, where $h\in L_1([0,\tau],\mathbb{K})$, that is, A generates a local $h^{[\beta+1]}$-convoluted semigroup on $[0,\tau]$. This is shown in Section 2.

Section 3 starts with a generalization of the complex representation theorem, see [4, 12]. This will lead to an improvement of [7, Theorem II].

In Section 4, we generalize a representation theorem of Prüss [12] and develop a new Hille-Yosida-type theorem for integrated semigroups.

2. Integrated semigroups

In this section, we show that it is worthwhile to study integrated semigroups, that is, abstract Cauchy problems (ACP_f) with inhomogeneities $f(t)=t^\beta$.

If $\tau>0$, we say that a subspace $Y\subset L_1([0,\tau],\mathbb{K})$ satiesfies property (A) if Y is closed and there is some strictly decreasing null sequence $(\varepsilon_n)_n$ with $\mathbf{1}_{[0,\varepsilon_n]}\in Y$ for all n.

The smallest possible Y with property (A) is

$$Y = \left\{ \sum_{n=1}^\infty \alpha_n \mathbf{1}_{(\varepsilon_{n+1},\varepsilon_n)} \;\middle|\; (\alpha_n)_n \in \ell_\infty \right\}, \tag{2.1}$$

where $\mathbf{1}_{(a,b)}$ is the characteristic function on (a,b).

Theorem 2.1. *Let X be a real or complex Banach space, $A : D(A)\to X$ a closed linear operator, $\tau>0$, and $Y\subset L_1([0,\tau],\mathbb{K})$ some subspace with property (A).*

Let $\beta \geq 0$, and assume A generates a local $h^{[\beta+1]}$-convoluted semigroup on $[0,\tau]$ for all $h \in Y$.

Then $A_{\overline{D(A)}}$ generates a β-times-integrated semigroup on $[0,\tau]$.

Proof. If $x \in X$ and $h \in Y$, let $u_{x,h}$ denote the solution of the abstract Cauchy problem

$$\begin{gathered} Au_{x,h}(t) + h^{[\beta+1]}(t)x = u'_{x,h}(t) \quad \text{if } 0 \leq t \leq \tau, \\ u_{x,h}(0) = 0, \\ u_{x,h} \in C^1([0,\tau],X). \end{gathered} \tag{2.2}$$

Using the closedness of A and the uniqueness property of the abstract Cauchy problem, it is easy to see that the linear operator

$$\begin{gathered} R_h : X \longrightarrow C([0,\tau],X), \\ R_h x = u'_{x,h}, \end{gathered} \tag{2.3}$$

has closed graph for all $h \in Y$.

Moreover, for all $x \in X$, the linear operator

$$\begin{gathered} S_x : Y \longrightarrow C([0,\tau],X), \\ S_x h = u'_{x,h}, \end{gathered} \tag{2.4}$$

has closed graph.

From the uniform boundedness principle, it follows that

$$\sup_{\|x\| \leq 1} ||S_x|| < \infty; \tag{2.5}$$

consequently

$$\sup_{\|x\| \leq 1} \sup_{\|h\|_Y \leq 1} ||u'_{x,h}||_\infty < \infty. \tag{2.6}$$

Now, let $h_n := (1/\varepsilon_n) \cdot \mathbf{1}_{[0,\varepsilon_n]}$.

Then

$$\Gamma(\beta+1) \cdot h_n^{[\beta+1]}(t) = \begin{cases} \dfrac{1}{\varepsilon_n} \dfrac{t^{\beta+1}}{\beta+1} & \text{if } 0 \leq t \leq \varepsilon_n, \\ \dfrac{1}{\beta+1} \dfrac{t^{\beta+1} - (t-\varepsilon_n)^{\beta+1}}{\varepsilon_n} & \text{if } \varepsilon_n \leq t \leq \tau, \end{cases} \tag{2.7}$$

thus $\lim_{n\to\infty} h_n^{[\beta+1]}(t) = t^\beta / \Gamma(\beta+1)$ if $0 < t \leq \tau$.

Moreover, $\sup_n \|h_n^{[\beta+1]}\|_\infty < \infty$. From the dominated convergence theorem, it follows that

$$\int_0^\tau \left| h_n^{[\beta+1]}(t) - \frac{t^\beta}{\Gamma(\beta+1)} \right| dt \xrightarrow{n} 0. \tag{2.8}$$

Next, we show that

(a) $\lim_{\lambda,n\to\infty} e^{\lambda\tau}|(h_n^{[\beta+1]})_\tau^\wedge(\lambda)| = \infty$,

(b) $\liminf_{\lambda\to\infty}(\ln|(t^\beta/\Gamma(\beta+1))_\tau^\wedge(\lambda)|/\lambda) \geq 0$.

We prove (a). We have

$$\sup_{\lambda,n} e^{\lambda\tau}\left|\left(h_n^{[\beta+1]}\right)^\wedge(\lambda) - \left(h_n^{[\beta+1]}\right)_\tau^\wedge(\lambda)\right| < \infty \tag{2.9}$$

since

$$\left|\int_\tau^\infty e^{-\lambda t} h_n^{[\beta+1]}(t)\,dt\right| \leq \sup_n \left\|h_n^{[\beta+1]}\right\|_\infty \cdot \int_\tau^\infty e^{-\lambda t}\,dt. \tag{2.10}$$

Moreover,

$$e^{\lambda\tau}\left|\left(h_n^{[\beta+1]}\right)_\tau^\wedge(\lambda)\right| \geq e^{\lambda\tau}\left|\left(h_n^{[\beta+1]}\right)^\wedge(\lambda)\right| - e^{\lambda\tau}\left|\left(h_n^{[\beta+1]}\right)^\wedge(\lambda) - \left(h_n^{[\beta+1]}\right)_\tau^\wedge(\lambda)\right|. \tag{2.11}$$

Now, (a) follows from

$$\left(h_n^{[\beta+1]}\right)^\wedge(\lambda) = \frac{\widehat{h_n}(\lambda)}{\lambda^{\beta+1}} = \frac{1}{\lambda^{\beta+2}}\frac{1-e^{-\lambda\varepsilon_n}}{\varepsilon_n} \geq \frac{e^{-\lambda\varepsilon_n}}{\lambda^{\beta+1}} \geq \frac{e^{-\lambda\tau/2}}{\lambda^{\beta+1}} \quad \text{for all large } \lambda, n. \tag{2.12}$$

We prove (b). This follows from [10, Lemma 3.3.5] or [8, Lemma 2.6].

Now we can apply [11, Corollary 13, Theorem 14], which yields the desired results. □

Corollary 2.2. *Let X be a real or complex Banach space, $A : D(A) \to X$ a densely defined closed linear operator, $\tau > 0$, and $\beta \geq 0$.*

Then the following assertions are equivalent:

(1) *A generates a local β-times-integrated semigroup on $[0,\tau]$;*

(2) *A generates a local $h^{[\beta+1]}$-convoluted semigroup on $[0,\tau]$ whenever $h \in L_1([0,\tau],\mathbb{K})$;*

(3) *for one (for all) subspace(s) $Y \subset L_1([0,\tau],\mathbb{K})$ with property (A), the operator A generates a local $h^{[\beta+1]}$-convoluted semigroup on $[0,\tau]$ whenever $h \in Y$.*

Proof. The proof of (2) follows from (1) by a simple convolution argument even if $D(A)$ is not dense. □

3. The complex representation theorem and its application

In the following, we generalize the complex representation theorem to the local case. The global version can be found in [4] or [12, Proposition 0.2].

Theorem 3.1 (complex representation theorem). *Let X be a complex Banach space, $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_1,\ |\operatorname{Im} z| < F(\operatorname{Re} z - \omega_1)\}$.*

Furthermore, let $q : M \to X$ be a holomorphic function satisfying

$$\sup_{\lambda\in M} \|\lambda q(\lambda)\| < \infty. \tag{3.1}$$

Then for every $b > 0$ there is some function $g_b \in C([0,ab),X)$ such that $t \mapsto ((ab - t)/t^b)g_b(t)$ is bounded on $(0,ab)$ and with

$$q(\lambda) \sim_\xi \lambda^b \cdot (g_b)^\wedge_\xi(\lambda) \quad \forall \xi \in (0,ab). \tag{3.2}$$

Remark 3.2. If $0 < b' < b$, then $g_b = g_{b'}^{[b-b']}$ on $[0,ab')$.

Proof. Let $\omega > \omega_1$, $b > 0$, and $\Gamma := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \omega,\ |\operatorname{Im} z| \leq F(\operatorname{Re} z - \omega)\}$. Then $\Gamma = \overline{\Gamma} \subset M^0 = M$.

First, we note that

$$C_1^2 := \sup_{r>0} \frac{1 + \left(F'(r)\right)^2}{\left[(\omega + r)^2 + F(r)^2\right]^{b+1}} \cdot e^{2abr} < \infty. \tag{3.3}$$

Let $C_2 := \sup_{\lambda \in \Gamma} \|\lambda q(\lambda)\|$. We parameterize $\partial\Gamma$ by the following two functions:

$$\begin{aligned} \gamma^+ &: [0,\infty) \longrightarrow \mathbb{C}, \quad \gamma^+(t) = \omega + t + iF(t), \\ \gamma^- &: [0,\infty) \longrightarrow \mathbb{C}, \quad \gamma^-(t) = \omega + t - iF(t). \end{aligned} \tag{3.4}$$

Let $\gamma_n^+ := \gamma^+|_{[0,n]}$ and $\gamma_n^- := \gamma^-|_{[0,n]}$. Then the sequence

$$\begin{gathered} g_n^+ : [0,\infty) \longrightarrow X, \\ g_n^+(t) = \frac{1}{2\pi i} \int_{\gamma_n^+} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda, \end{gathered} \tag{3.5}$$

of continuous functions converges uniformly on $[0,q'b]$ whenever $q' \in (0,a)$, since if $t \in [0,q'b]$ and $n > m$, we obtain

$$\begin{aligned} 2\pi \left\| g_n^+(t) - g_m^+(t) \right\| &\leq \left\| \int_m^n e^{t\gamma^+(r)} \frac{q(\gamma^+(r))}{\gamma^+(r)^b} \cdot \frac{d}{dr}\gamma^+(r) dr \right\| \\ &\leq C_2 \int_m^n e^{t(\omega+r)} \frac{\sqrt{1 + F'(r)^2}}{\left[(\omega + r)^2 + F(r)^2\right]^{(b+1)/2}} dr \\ &\leq C_1 C_2 e^{t\omega} \int_m^n e^{tr} e^{-abr} dr \\ &\leq C_1 C_2 e^{t\omega} \int_m^n e^{rb(q'-a)} dr. \end{aligned} \tag{3.6}$$

Thus the function

$$\begin{gathered} g^+ : [0,ab) \longrightarrow X, \\ g^+(t) = \frac{1}{2\pi i} \int_{\gamma^+} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda, \end{gathered} \tag{3.7}$$

is continuous. In the same way we see that the function

$$
\begin{gathered}
g^{-}:[0,ab)\longrightarrow X,\\
g^{-}(t)=\frac{1}{2\pi i}\int_{-\gamma^{-}}e^{\lambda t}\frac{q(\lambda)}{\lambda^{b}}d\lambda=-\frac{1}{2\pi i}\int_{\gamma^{-}}e^{\lambda t}\frac{q(\lambda)}{\lambda^{b}}d\lambda,
\end{gathered}
\tag{3.8}
$$

is continuous.

Let $g(t):=g^{+}(t)+g^{-}(t)$ if $t\in[0,ab)$. We show that $t\mapsto((ab-t)/t^{b})g(t)$ is bounded on $(0,ab)$. To this end, let $R>0$ and consider the three paths

$$
\begin{aligned}
\alpha:[R,\infty)\longrightarrow\mathbb{C},&\quad \alpha(r)=\omega+r-iF(r),\\
\beta:[R,\infty)\longrightarrow\mathbb{C},&\quad \alpha(r)=\omega+r+iF(r),\\
\gamma:[-F(\omega+R),F(\omega+R)]\longrightarrow\mathbb{C},&\quad \gamma(r)=\omega+R+ir.
\end{aligned}
\tag{3.9}
$$

Then, if $0\leq t<ab$,

$$
g(t)=\frac{1}{2\pi i}\int_{\beta+\gamma-\alpha}e^{\lambda t}\frac{q(\lambda)}{\lambda^{b}}d\lambda. \tag{3.10}
$$

We have

$$
\begin{gathered}
\left\|\int_{\beta}e^{\lambda t}\frac{q(\lambda)}{\lambda^{b}}d\lambda\right\|\leq C_1C_2e^{\omega t}\int_{R}^{\infty}e^{(t-ab)r}dr=C_1C_2e^{\omega t}\frac{e^{(t-ab)R}}{ab-t},\\
\left\|\int_{-\alpha}e^{\lambda t}\frac{q(\lambda)}{\lambda^{b}}d\lambda\right\|\leq C_1C_2e^{\omega t}\frac{e^{(t-ab)R}}{ab-t}.
\end{gathered}
\tag{3.11}
$$

Finally,

$$
\begin{aligned}
\left\|\int_{\gamma}e^{\lambda t}\frac{q(\lambda)}{\lambda^{b}}d\lambda\right\|&\leq\int_{-\infty}^{\infty}e^{t(\omega+R)}C_2\frac{dr}{\left[(\omega+R)^2+r^2\right]^{(b+1)/2}}\\
&=\frac{C_2e^{t(\omega+R)}}{(\omega+R)^{b}}\int_{-\infty}^{\infty}\frac{ds}{\left(1+s^2\right)^{(b+1)/2}}.
\end{aligned}
\tag{3.12}
$$

If we let $R:=1/t$, we obtain the desired result.

Next, we show that, if $\lambda\in\Gamma$,

$$
\frac{q(\lambda)}{\lambda^{b}}=\frac{1}{2\pi i}\lim_{n\to\infty}\int_{\gamma_n^{+}-\gamma_n^{-}}\frac{(1/\mu^{b})q(\mu)}{\lambda-\mu}d\mu. \tag{3.13}
$$

To this end, consider a path β_R in $\mathbb{C}$ consisting of a part of a circle with center $\omega\in\mathbb{C}$ and radius R which connects a point on γ^{+} with a point on γ^{-}. Its parameterization is given by

$$
\beta_R:[-\psi,\psi]\longrightarrow\mathbb{C},\quad \beta_R(\varphi)=\omega+\mathrm{R}e^{i\varphi}, \tag{3.14}
$$

with some $\psi\in(0,\pi/2)$ depending on R.

By Cauchy's formula we have to show that

$$\lim_{R\to\infty}\int_{\beta_R}\frac{(1/\mu^b)q(\mu)}{\mu-\lambda}d\mu=0. \tag{3.15}$$

But, if R is large enough,

$$\begin{aligned}\left\|\int_{\beta_R}\frac{(1/\mu^b)q(\mu)}{\mu-\lambda}d\mu\right\| &\le \int_{-\pi/2}^{\pi/2}\frac{\|q(\omega+\mathrm{Re}^{i\varphi})\|}{|\omega+\mathrm{Re}^{i\varphi}|^b\cdot|\lambda-(\omega+\mathrm{Re}^{i\varphi})|}R\,d\varphi\\ &\le C_2\int_{-\pi/2}^{\pi/2}\frac{R}{|\omega+\mathrm{Re}^{i\varphi}|^{b+1}}d\varphi\xrightarrow{R\to\infty}0.\end{aligned} \tag{3.16}$$

Consequently, if $\xi\in(0,ab)$ and $\lambda\in\Gamma$,

$$\begin{aligned}\int_0^\xi e^{-\lambda t}g(t)dt &= \frac{1}{2\pi i}\int_0^\xi e^{-\lambda t}\lim_{n\to\infty}\int_{\gamma_n^+-\gamma_n^-}e^{\mu t}\frac{q(\mu)}{\mu^b}d\mu\,dt\\ &=\frac{1}{2\pi i}\lim_n\int_{\gamma_n^+-\gamma_n^-}\int_0^\xi e^{(\mu-\lambda)t}dt\frac{q(\mu)}{\mu^b}d\mu\\ &=\frac{1}{2\pi i}\lim_n\int_{\gamma_n^+-\gamma_n^-}\frac{e^{(\mu-\lambda)\xi}}{\mu-\lambda}\frac{q(\mu)}{\mu^b}d\mu\\ &\quad+\frac{1}{2\pi i}\lim_n\int_{\gamma_n^+-\gamma_n^-}\frac{(1/\mu^b)q(\mu)}{\lambda-\mu}d\mu.\end{aligned} \tag{3.17}$$

It remains to show that, for all $\sigma>0$,

$$e^{-\lambda\sigma}e^{\lambda\xi}\lim_n\left\|\int_{\gamma_n^+-\gamma_n^-}\frac{e^{(\mu-\lambda)\xi}}{\mu-\lambda}\frac{q(\mu)}{\mu^b}d\mu\right\|\xrightarrow{\lambda\to\infty}0. \tag{3.18}$$

But, if $\lambda>\omega$,

$$\begin{aligned}e^{-\lambda\sigma}\left\|\int_{\gamma_n^+}\frac{e^{\mu\xi}}{\mu-\lambda}\frac{q(\mu)}{\mu^b}d\mu\right\| &\le e^{-\lambda\sigma}\int_0^n\frac{e^{\xi\,\mathrm{Re}\,\gamma^+(t)}}{|\gamma^+(r)-\lambda|}\cdot\frac{\|q(\gamma^+(r))\|}{|\gamma^+(r)|^b}\cdot\left|\frac{d}{dr}\gamma^+(r)\right|dr\\ &\le C_1C_2e^{-\lambda\sigma}e^{\xi\omega}\int_0^n\frac{e^{\xi r}}{|\gamma^+(r)-\lambda|}e^{-abr}dr\\ &\le C_1C_2\left(\sup_{r\ge0}\frac{1}{|\gamma^+(r)-\lambda|}\right)e^{\xi\omega}e^{-\lambda\sigma}\int_0^\infty e^{(\xi-ab)r}dr\xrightarrow{\lambda\to\infty}0,\end{aligned} \tag{3.19}$$

where if λ is large enough,

$$\sup_{r\ge0}\frac{1}{|\gamma^+(r)-\lambda|}\le 1. \tag{3.20}$$

In the same way we obtain

$$e^{-\lambda\sigma}\left\|\int_{\gamma_n^-}\frac{e^{\mu\xi}}{\mu-\lambda}\frac{q(\mu)}{\mu^b}d\mu\right\|\xrightarrow{\lambda\to\infty}0. \tag{3.21}$$

This shows

$$q(\lambda)\sim_\xi \lambda^b\widehat{g}_\xi(\lambda)\quad \forall\xi\in(0,ab). \tag{3.22}$$

Finally, if $0<b'<b$, then $g_b=g_{b'}^{[b-b']}$ on $[0,ab')$.

We have

$$\begin{aligned} q(\lambda)&\sim_\xi \lambda^{b'}\cdot(g_{b'})_\xi^\wedge(\lambda)\quad \forall\xi\in(0,ab'),\\ q(\lambda)&\sim_\xi \lambda^{b}\cdot(g_{b})_\xi^\wedge(\lambda)\quad \forall\xi\in(0,ab), \end{aligned} \tag{3.23}$$

thus if $\xi\in(0,ab')$,

$$(g_b)_\xi^\wedge(\lambda)\sim_\xi \frac{1}{\lambda^{b-b'}}(g_{b'})_\xi^\wedge(\lambda)\sim_\xi\left(g_{b'}^{[b-b']}\right)_\xi^\wedge(\lambda). \tag{3.24}$$

The Phragmén Doetsch inversion formula (see, e.g., [4]) shows that $g_b=g_{b'}^{[b-b']}$ on $[0,ab')$. □

Theorem 3.3. *Let $\alpha\in\mathbb{R}$, $\omega,c,a>0$, $F(t)=c(e^{at}-1)$, and $M:=\{z\in\mathbb{C}\mid \operatorname{Re}z>\omega,\ |\operatorname{Im}z|<F(\operatorname{Re}z-\omega)\}$.*

Suppose $p:[0,\infty)\to\mathbb{C}$ is locally integrable with $|p(t)|\le \text{const}\cdot e^{\omega t}$ for all large t and with

$$\liminf_{\lambda\to\infty}\frac{\ln|\widehat{p}(\lambda)|}{\lambda}\ge 0, \tag{3.25}$$

$$\sup_{\lambda\in M}|\lambda|^{\alpha-1}\cdot|\widehat{p}(\lambda)|<\infty. \tag{3.26}$$

Furthermore, let X be a complex Banach space, let $A:D(A)\to X$ be linear with $M\subset\rho(A)$, and let

$$\sup_{\lambda\in M}|\lambda|^{\alpha}\cdot\|\widehat{p}(\lambda)R(\lambda,A)\|<\infty. \tag{3.27}$$

Then, for all $m\in(-\infty,\alpha-1)$, the operator A generates a local $p^{(m-1)}$-convoluted semigroup on $[0,a(\alpha-m-1))$.

Remark 3.4. (i) Condition (3.25) can be omitted if one then assumes in addition that

$$\limsup_{\lambda\to\infty}\frac{\ln\|R(\lambda,A)\|}{\lambda}\le 0. \tag{3.28}$$

(ii) From condition (3.25), it follows that

$$\limsup_{\lambda\to\infty} \frac{\ln|\hat{p}(\lambda)|}{\lambda} = 0, \tag{3.29}$$

which is equivalent to $0 \in \operatorname{supp} p$, that is, p does not vanish on any interval $[0,\varepsilon)$, $\varepsilon > 0$.
This can be shown by using the Phragmén Doetsch inversion formula.

Proof. We first show that p is $C_0^{(m-1)}$ on $[0,a(\alpha-m-1))$ if $m \geq 1$, that is, there is some $g \in C([0,a(\alpha-m-1)),\mathbb{C})$ with $g(0)=0$ and $p = g^{[m-1]}$.

To this end, define $\tilde{q} : M \to \mathbb{C}$ by $\tilde{q}(\lambda) = \lambda^{\alpha-2}\hat{p}(\lambda)$.

Then we have $\sup_{\lambda\in M}|\lambda\tilde{q}(\lambda)| < \infty$. Letting $b := \alpha - m - 1$, the complex representation theorem shows that there is some $g \in C([0,a(\alpha-m-1)),\mathbb{C})$ with $g(0)=0$ such that

$$\tilde{q}(\lambda) \sim_\xi \lambda^{\alpha-m-1}\hat{g}_\xi(\lambda) \quad \forall \xi \in (0,a(\alpha-m-1)), \tag{3.30}$$

thus

$$\hat{p}(\lambda) \sim_\xi \frac{1}{\lambda^{m-1}}\hat{g}_\xi(\lambda) \sim_\xi \left(g^{[m-1]}\right)^\wedge_\xi(\lambda). \tag{3.31}$$

The Phragmén Doetsch inversion formula shows that $p^{[1]} = g^{[m]}$ on $(0,a(\alpha-m-1))$.

Next, we define the holomorphic function

$$q : M \longrightarrow \mathcal{L}(X,\overline{D}(A)), \qquad q(\lambda) = \lambda^{\alpha-2}\hat{p}(\lambda)R(\lambda,A), \tag{3.32}$$

where $\overline{D}(A)$ is the Banach space $D(A)$ with the norm $\|x\|_{\overline{D}(A)} = \|x\| + \|Ax\|$.

If $\lambda \in M$, we obtain

$$\begin{aligned}
||\lambda q(\lambda)|| &\leq |\lambda|^{\alpha-1}|\hat{p}(\lambda)|\left(||R(\lambda,A)|| + ||\lambda R(\lambda,A) - \mathrm{id}||\right)\\
&\leq |\lambda|^{\alpha-1}|\hat{p}(\lambda)|\left(||R(\lambda,A)|| + |\lambda|\cdot||R(\lambda,A)|| + 1\right)\\
&\leq |\lambda|^{\alpha-1}|\hat{p}(\lambda)| + |\lambda|^{\alpha-1}|\hat{p}(\lambda)|\cdot(1+|\lambda|)\cdot||R(\lambda,A)||.
\end{aligned} \tag{3.33}$$

From conditions (3.26) and (3.27), it follows that

$$\sup_{\lambda\in M}||\lambda q(\lambda)|| < \infty. \tag{3.34}$$

Letting $b := \alpha - m - 1$, the complex representation theorem yields some continuous function $H : [0,a(\alpha-m-1)) \to \mathcal{L}(X,\overline{D}(A))$ with $H(0)=0$ and with

$$q(\lambda) \sim_\xi \lambda^{\alpha-m-1}\hat{H}_\xi(\lambda) \quad \forall \xi \in (0,a(\alpha-m-1)). \tag{3.35}$$

Consequently, if $\xi \in (0,a(\alpha-m-1))$ and $x \in X$,

$$\lambda\left(p^{(m-1)}\right)^\wedge_\xi(\lambda)R(\lambda,A)x \sim_\xi \lambda^m\hat{p}_\xi(\lambda)R(\lambda,A)x \tag{3.36}$$

since $\|R(\lambda,A)\| \sim_0 0$ by conditions (3.25) and (3.27), and since $(p^{(m-1)})^\wedge_\xi(\lambda) \sim_\xi \lambda^{m-1}\hat{p}_\xi(\lambda)$.

Moreover,

$$\lambda^{\alpha-2}\widehat{p}_{\xi}(\lambda)R(\lambda,A)x \sim_{\xi} q(\lambda)x \sim_{\xi} \lambda^{\alpha-m-1}(H(\cdot)x)^{\wedge}_{\xi}(\lambda). \tag{3.37}$$

Thus,

$$\begin{aligned}\lambda\left(p^{(m-1)}\right)^{\wedge}_{\xi}(\lambda)R(\lambda,A)x &\sim_{\xi} \lambda^{m-\alpha+2}\lambda^{\alpha-2}\widehat{p}_{\xi}(\lambda)R(\lambda,A)x \\ &\sim_{\xi} \lambda(H(\cdot)x)^{\wedge}_{\xi}(\lambda) \\ &\sim_{\xi} \int_0^{\tau} e^{-\lambda t}d_t(H(t)x).\end{aligned} \tag{3.38}$$

Therefore, by the theory of local convoluted semigroups, the unique solution of the abstract Cauchy problem

$$\begin{gathered}Au_x(t)+p^{(m-1)}(t)x = u_x'(t) \quad \forall t \in [0,a(\alpha-m-1)), \\ u_x(0)=0, \\ u_x \in C^1([0,a(\alpha-m-1)),X),\end{gathered} \tag{3.39}$$

is given by

$$u_x(t) = H(t)x. \tag{3.40}$$

□

We now state and prove a lemma which we will need frequently in the sequel.

Lemma 3.5. *Let $\tau > 0$, $\beta > 0$, let $h : [0,\infty) \to [0,\infty)$ be some function, and* $\Gamma := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0,\ |\operatorname{Im} z| \le h(\operatorname{Re} z)\}$.

Then

$$\lim_{\substack{\operatorname{Re}\lambda\to\infty \\ \lambda\in\Gamma}} \lambda^{\beta}e^{-\lambda\tau} = 0 \Longleftrightarrow \lim_{t\to\infty} e^{-(\tau/\beta)t}h(t) = 0. \tag{3.41}$$

In one of these cases, even if $\beta \ge 0$,

$$\lim_{\substack{\operatorname{Re}\lambda\to\infty \\ \lambda\in\Gamma}} \lambda^{\beta+1}\int_{\tau}^{\infty} e^{-\lambda t}\left(d\cdot t^{\beta}+g^{[\beta+1]}(t)\right)dt = 0 \tag{3.42}$$

whenever $d \in \mathbb{C}$ and $g \in L_1^{\mathrm{loc}}([0,\infty),\mathbb{C})$ is exponentially bounded for large arguments.

Moreover, if $\beta > 0$,

$$\limsup_{\substack{\operatorname{Re}\lambda\to\infty \\ \lambda\in\Gamma}} |\lambda^{\beta}e^{-\lambda\tau}| < \infty \Longleftrightarrow \limsup_{t\to\infty} e^{-(\tau/\beta)t}h(t) < \infty. \tag{3.43}$$

Proof. We prove (3.41).

"If" part. If $\lambda \in \Gamma$, we obtain

$$\begin{aligned} \left|\lambda^\beta e^{-\lambda\tau}\right|^{2/\beta} &= \left(|\operatorname{Im}\lambda|^2 + |\operatorname{Re}\lambda|^2\right) e^{-(2\tau/\beta)\operatorname{Re}\lambda} \\ &\le \left(h(\operatorname{Re}\lambda) e^{-(\tau/\beta)\operatorname{Re}\lambda}\right)^2 + |\operatorname{Re}\lambda|^2 e^{-(2\tau/\beta)\operatorname{Re}\lambda}. \end{aligned} \tag{3.44}$$

"Only if" part. Suppose there are some $\varepsilon > 0$ and numbers $x_n \in \mathbb{R}$ with $x_n \to \infty$ and with $|e^{-(\tau/\beta)x_n} h(x_n)| \ge \varepsilon$ for all n. Then $z_n := x_n + i\varepsilon e^{(\tau/\beta)x_n}$ is in Γ, and

$$0 = \lim_n \left| z_n^\beta e^{-z_n\tau} \right|^{2/\beta} = \lim_n \left(x_n^2 + \varepsilon^2 e^{(2\tau/\beta)x_n}\right) e^{-(2\tau/\beta)x_n} = \varepsilon^2, \tag{3.45}$$

which is a contradiction. This shows (3.41).

Now let $f(t) := dt^\beta + g^{[\beta+1]}(t)$. Then $f'(t) = d\beta t^{\beta-1} + g^{[\beta]}(t)$ is L_1^{loc} and exponentially bounded for large arguments.

Partial integration shows that

$$\lambda^{\beta+1} \int_\tau^\infty e^{-\lambda t} f(t)\,dt = \lambda^\beta e^{-\lambda\tau} f(\tau) + \lambda^\beta e^{-\lambda\tau} \int_0^\infty e^{-\lambda w} f'(w+\tau)\,dw. \tag{3.46}$$

□

COROLLARY 3.6. *Let $\omega, a, c, \nu > 0$, $\mu \ge 0$, $\beta \in (\mu/a, \mu/a + \nu]$, $F(t) = c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega,\ |\operatorname{Im} z| < F(\operatorname{Re} z - \omega)\}$.*

Furthermore, let X be a complex Banach space, let $A : D(A) \to X$ be linear with $M \subset \rho(A)$, with

$$\limsup_{\lambda\to\infty} \frac{\ln \|R(\lambda, A)\|}{\lambda} \le 0, \tag{3.47}$$

and with

$$\sup_{\lambda\in M} \left\| \lambda^{\beta-\nu+1} e^{-\lambda\mu} R(\lambda, A) \right\| < \infty. \tag{3.48}$$

Then A generates a ν-times-integrated semigroup on $[0, a\beta - \mu)$.

Proof. Apply Theorem 3.3 to $m := 0$, $\alpha := \beta + 1$, and $p(t) := \mathbf{1}_{(\mu,\infty)}(t) \cdot (t-\mu)^{\nu-1}$, where $\mathbf{1}_{(\mu,\infty)}$ is the characteristic function on (μ, ∞). Then we have $\hat{p}(\lambda) = e^{-\lambda\mu}(\Gamma(\nu)/\lambda^\nu)$ if $\operatorname{Re}\lambda > 0$.

Condition (3.25) is omitted, see Remark 3.4.

Condition (3.26) is equivalent to $\sup_{\lambda\in M} |\lambda^{\beta-\nu} e^{-\lambda\mu}| < \infty$. This follows from Lemma 3.5. Here we need that $\beta \le \mu/a + \nu$.

Condition (3.27) is equivalent to $\sup_{\lambda\in M} \|\lambda^{\beta+1-\nu} e^{-\lambda\mu} R(\lambda, A)\| < \infty$.

By Theorem 3.3 the operator A generates a local $p^{[1]}$-convoluted semigroup on $[0, a\beta)$.

If $x \in X$, let u_x denote the (unique) solution of the abstract Cauchy problem

$$\begin{aligned} Au_x(t) + p^{[1]}(t)x &= u_x'(t) \quad \text{if } 0 \le t < a\beta, \\ u_x(0) &= 0, \\ u_x &\in C^1([0,a\beta),X). \end{aligned} \tag{3.49}$$

The uniqueness property follows from the Ljubic uniqueness theorem.

Then $u_x(t) = 0$ if $0 \le t \le \mu$, since $p^{[1]}(t) = 0$ if $0 \le t \le \mu$. Thus if $x \in X$, then

$$\begin{aligned} v_x &: [0,a\beta-\mu) \longrightarrow X, \\ v_x(t) &= u_x(t+\mu), \end{aligned} \tag{3.50}$$

is the (unique) solution of the abstract Cauchy problem

$$\begin{aligned} Av_x(t) + \frac{t^\nu}{\nu \cdot x} &= v_x'(t) \quad \text{if } 0 \le t < a\beta - \mu, \\ v_x(0) &= 0, \\ v_x &\in C^1([0,a\beta-\mu),X). \end{aligned} \tag{3.51}$$

□

Corollary 3.7. *Let X be a complex Banach space, let $A : D(A) \to X$ be linear, $\omega, c, a > 0$, $F(t) = c(e^{at}-1)$, and $M = \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega,\ |\operatorname{Im} z| < F(\operatorname{Re} z - \omega)\}$.*

Assume that $M \subset \rho(A)$ and

$$\sup_{\lambda \in M} \frac{\|R(\lambda,A)\|}{|\lambda|^\gamma} < \infty \quad \text{for some } \gamma \in [-1,\infty). \tag{3.52}$$

Then, for all $\varepsilon > 0$, the operator A generates a $(\gamma+1+\varepsilon)$-integrated semigroup on $[0,a\varepsilon)$.

Proof. Apply Corollary 3.6 to $\mu := 0$, $\nu := \gamma+\varepsilon+1$, and $\beta := \varepsilon$. □

Remark 3.8. Corollary 3.7 improves [7, Theorem II] if $D(A)$ is dense.

To see this we abbreviate "Cioranescu/Lumer" as "(CL)" and refer to the notation in [7].

In that theorem, put $\Phi^{(CL)}(r) := \ln r$, $\alpha^{(CL)} := 1/a$, $\gamma^{(CL)} := \gamma$, $K^{(CL)} := h^{[\gamma+\varepsilon+1]}$, and $l^{(CL)} := \gamma+\varepsilon+1$. Then $\sigma^{(CL)} = 1$ and $\chi^{(CL)} = 0$.

If ω is large enough, then $M \subset \Gamma^{(CL)}_{\alpha^{(CL)},\beta^{(CL)}}$.

To apply [7, Theorem II] we have to make the assumptions that $\gamma > -1$, that $h^{[\gamma+\varepsilon+1]}$ is exponentially bounded, and that $\widehat{K^{(CL)}} \ne 0$ for all λ with large real part. Then, by [7, Theorem II], A generates a local $h^{[\gamma+\varepsilon+2]}$-convoluted semigroup on $[0,\tau^{(CL)}) = [0,(l^{(CL)} - \gamma^{(CL)} - \sigma^{(CL)})/\alpha^{(CL)}) = [0,a\varepsilon)$.

Example 3.9. Let $X = C([0,1],\mathbb{C})$, let $g \in X$ with $g(t) > 0$ for all $t \in [0,1]$, and let $D(A) = \{f \in C^1([0,1],\mathbb{C}) \mid f(0) = 0\}$.

Then the operator $Ah = -gh'$ generates a $(1+\varepsilon)$-integrated semigroup on $[0,\infty)$ for all $\varepsilon > 0$.

If $\lambda \in \mathbb{C}$, we have

$$R(\lambda, A)f(t) = e^{-\lambda G(t)} \int_0^t \frac{e^{\lambda G(s)} f(s)}{g(s)} ds, \tag{3.53}$$

where $G(t) = \int_0^t (ds/g(s))ds$.

The assertion follows from Corollary 3.7 since $\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)\| < \infty$.

Corollary 3.10. *Let X be a complex Banach space, let $A : D(A) \to X$ be linear, $\omega, c, a > 0$, $F(t) = c(e^{at} - 1)$, and $M = \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega,\ |\operatorname{Im} z| < F(\operatorname{Re} z - \omega)\}$.*

Assume that $M \subset \rho(A)$, that

$$\limsup_{\lambda\to\infty} \frac{\ln ||R(\lambda, A)||}{\lambda} \le 0, \tag{3.54}$$

and that

$$\sup_{\lambda \in M} ||e^{-\lambda\mu} R(\lambda, A)|| < \infty \quad \text{for some } \mu \ge 0. \tag{3.55}$$

Then for all $\varepsilon > \mu/a$, the operator A generates a $(\varepsilon + 1)$-integrated semigroup on $[0, a\varepsilon - \mu)$.

Proof. Apply Corollary 3.6 to $\nu := 1 + \varepsilon$ and $\beta := \varepsilon$. □

Theorem 3.11. *Let $\beta \ge 0$, $\tau > 0$, $g \in L_1([0, \tau], \mathbb{K})$, $c \in \mathbb{K}\setminus\{0\}$, and $f(t) := ct^\beta + g^{[\beta+1]}(t)$.*

Furthermore, let $h : [0, \infty) \to [0, \infty)$ be some function with $h(t) = o(e^{(\tau/\beta)t})$ as $t \to \infty$ (this condition is always fulfilled if $\beta = 0$. In this case Γ is supposed to be some right half-plane in $\mathbb{C}$), and let

$$\Gamma := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \le h(\operatorname{Re} z)\}. \tag{3.56}$$

Finally, let X be some real or complex Banach space and let $A : D(A) \to X$ be some linear and closed operator which generates a local f-convoluted semigroup on $[0, \tau]$.

Then there is some $\omega > 0$ such that $R(\lambda, A)$ exists if $\lambda \in \Gamma$ with $\operatorname{Re}\lambda > \omega$, and

$$\sup_{\substack{\operatorname{Re}\lambda>\omega\\ \lambda\in\Gamma}} \operatorname{Re}\lambda \cdot \frac{||R(\lambda, A)||}{|\lambda|^\beta} < \infty. \tag{3.57}$$

Proof. From Lemma 3.5, it follows that

$$\lim_{\substack{\operatorname{Re}\lambda\to\infty\\ \lambda\in\Gamma}} \lambda^{\beta+1} \widehat{f}_\tau(\lambda) = c \cdot \Gamma(\beta + 1). \tag{3.58}$$

Thus, again by Lemma 3.5,

$$\left|\lambda e^{\lambda\tau} \widehat{f}_\tau(\lambda)\right| = \left|\lambda^{-\beta} e^{\lambda\tau}\right| \cdot \left|\lambda^{\beta+1} \widehat{f}_\tau(\lambda)\right| \longrightarrow \infty \quad \text{if } \operatorname{Re}\lambda \longrightarrow \infty,\ \lambda \in \Gamma, \tag{3.59}$$

which also holds if $\beta = 0$.

From the theory of local convoluted semigroups (see, e.g., [8, 11]), it follows that there is some $\omega > 0$ such that $R(\lambda, A)$ exists if $\mathrm{Re}\lambda > \omega$, $\lambda \in \Gamma$, and that

$$||R(\lambda,A)|| \leq \frac{\text{const}}{\mathrm{Re}\lambda \cdot |\lambda \widehat{f}_\tau(\lambda)|}. \tag{3.60}$$

To be more precise, let u_x denote the solution of the abstract Cauchy problem

$$\begin{gathered} Au_x(t) + f(t)x = u_x'(t) \quad \text{if } 0 \leq t \leq \tau, \\ u_x(0) = 0, \\ u_x \in C^1([0,\tau], X). \end{gathered} \tag{3.61}$$

Let $Hx := Au_x(\tau)$ and $J_\lambda := \lambda \widehat{f}_\tau(\lambda) \cdot \mathrm{id} - e^{-\lambda\tau}H$. From

$$\frac{J_\lambda}{\lambda \widehat{f}_\tau(\lambda)} = \mathrm{id} - \frac{H}{\lambda e^{\lambda\tau} \widehat{f}_\tau(\lambda)}, \tag{3.62}$$

it follows that J_λ is invertible if $\lambda \in \Gamma$ and $\mathrm{Re}\lambda$ is large, and

$$\left(\frac{J_\lambda}{\lambda \widehat{f}_\tau(\lambda)}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{H}{\lambda e^{\lambda\tau} \widehat{f}_\tau(\lambda)}\right)^k. \tag{3.63}$$

Thus, $\|J_\lambda^{-1}\| \leq 2/|\lambda \widehat{f}_\tau(\lambda)|$ if $\lambda \in \Gamma$ and $\mathrm{Re}\lambda$ is large.

From

$$(\lambda - A)\int_0^\tau e^{-\lambda t} du_x(t) = J_\lambda x \quad \forall x \in X, \lambda \in \mathbb{K}, \tag{3.64}$$

it follows that $R(\lambda, A)$ exists, and

$$||R(\lambda,A)|| \leq \frac{\text{const}}{\mathrm{Re}\lambda \cdot |\lambda \widehat{f}_\tau(\lambda)|}. \tag{3.65}$$

□

Example 3.12. Let $X = C([0,1],\mathbb{C})$ and $D(A) = \{f \in C^2([0,1],\mathbb{C}) \mid f(0) = f'(0) = 0\}$. Then $Ah = -h''$ does not generate a local β-integrated semigroup on $[0,\tau]$, independent of $\beta \geq 0$ and $\tau > 0$.

We have, if $\lambda \in \mathbb{C}$,

$$R(-\lambda^2, A)f(t) = \begin{cases} \dfrac{1}{\lambda}\displaystyle\int_0^t \sinh(\lambda s) f(t-s)\,ds & \text{if } \lambda \neq 0, \\ \displaystyle\int_0^t s f(t-s)\,ds & \text{if } \lambda = 0. \end{cases} \tag{3.66}$$

Let $\Gamma := \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \leq 4/3 \cdot |\mathrm{Re}\, z|\}$ and consider $\mu(\alpha) = -\alpha + 2i\alpha$ if $\alpha > 0$.

Then $-\mu^2 = 3\alpha^2 + 4i\alpha^2 \in \Gamma$, and

$$\frac{||R(-\mu^2, A)||}{|\mu|^{2\beta}} \geq \frac{||R(-\mu^2, A)(\mathbf{1})||}{|\mu|^{2\beta}} \xrightarrow{\alpha \to \infty} \infty, \tag{3.67}$$

where $\mathbf{1}(t) = 1$.

THEOREM 3.13. *Let X be a complex Banach space and $A : D(A) \to X$ a linear operator. Then the following assertions are equivalent.*

(1) *There are $\beta \geq 0$ and $\xi > 0$ such that A generates a β-times integrated semigroup on $[0, \xi)$.*

(2) *There are $c, \omega, a > 0$ such that if $F(t) := c(e^{at} - 1)$ and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| \leq F(\operatorname{Re} z - \omega)\}$, the following is valid:*

$$\begin{gathered} M \subset \rho(A), \\ \sup_{\lambda \in M} \left\| \frac{e^{-\lambda\mu}}{\lambda^{\widetilde{\beta}}} R(\lambda, A) \right\| < \infty \quad \textit{for some } \mu, \widetilde{\beta} \in \mathbb{R}, \\ \limsup_{\lambda \to \infty} \frac{\ln ||R(\lambda, A)||}{\lambda} \leq 0. \end{gathered} \tag{3.68}$$

(3) *There are $\omega, \delta > 0$ such that if $h(t) := e^{\delta t}$ and $\Gamma := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| \leq h(\operatorname{Re} z)\}$, the following is valid:*

$$\begin{gathered} \Gamma \subset \rho(A), \\ \sup_{\lambda \in \Gamma} \left\| \frac{R(\lambda, A)}{\lambda^{\overline{\beta}}} \right\| < \infty \quad \textit{for some } \overline{\beta} \in \mathbb{R}. \end{gathered} \tag{3.69}$$

Proof. (1)⇒(3). Choose $\delta \in (0, \xi/\beta)$, where $\xi/\beta := \infty$ if $\beta = 0$. Apply Theorem 3.11 to $g := 0$, $c := 1$, $\overline{\beta} := \beta$ and choose $\tau \in (\beta\delta, \xi)$. Then $h(t) = e^{\delta t}$ fulfills $h(t) = o(e^{(\tau/\beta)t})$, thus, by Theorem 3.11, there is some $\omega > 0$ such that $R(\lambda, A)$ exists if $\operatorname{Re}\lambda > \omega$, $\lambda \in \Gamma$, and

$$\sup_{\substack{\operatorname{Re}\lambda > \omega \\ \lambda \in \Gamma}} \frac{||R(\lambda, A)||}{|\lambda|^{\overline{\beta}}} < \infty. \tag{3.70}$$

Here we needed the uniqueness property on $[0, \tau]$, see [10, Theorem 3.5.1].

(3)⇒(2). Choose $a := \delta$, $c := 1$, and numbers $\mu \geq 0$, $\widetilde{\beta} \in \mathbb{R}$ such that $\delta \cdot (\overline{\beta} - \widetilde{\beta}) \leq \mu$. For example, take $\mu := 0$ and $\widetilde{\beta} := \overline{\beta}$.

Then $M \subset \Gamma$, and if $\lambda \in M$,

$$\left\| \frac{e^{-\lambda\mu}}{\lambda^{\widetilde{\beta}}} R(\lambda, A) \right\| = \left| \lambda^{\overline{\beta} - \widetilde{\beta}} e^{-\lambda\mu} \right| \cdot \left\| \frac{R(\lambda, A)}{\lambda^{\overline{\beta}}} \right\|. \tag{3.71}$$

If $\overline{\beta} - \widetilde{\beta} > 0$, then

$$\limsup_{\substack{\mathrm{Re}\lambda \to \infty \\ \lambda \in M}} \left| \lambda^{\overline{\beta} - \widetilde{\beta}} e^{-\lambda\mu} \right| < \infty, \tag{3.72}$$

by Lemma 3.5, since

$$\limsup_{t \to \infty} e^{-(\mu/(\overline{\beta} - \widetilde{\beta}))t} h(t) < \infty. \tag{3.73}$$

(2)⇒(1). First, let $\widetilde{\beta} \geq -\mu/a - 1$. Choose $\nu > \mu/a + 1 + \widetilde{\beta}$ and $\widehat{\beta} := \nu - \widetilde{\beta} - 1$. Then $\widehat{\beta} \in (\mu/a, \mu/a + \nu]$ and

$$\left\| \lambda^{\widehat{\beta} - \nu + 1} e^{-\lambda\mu} R(\lambda, A) \right\| \leq \left\| \frac{e^{-\lambda\mu}}{\lambda^{\widetilde{\beta}}} R(\lambda, A) \right\| \quad \forall \lambda \in M,\ \mathrm{Re}\lambda > \omega. \tag{3.74}$$

Corollary 3.6 shows that A generates a local ν-times integrated semigroup on $[0, a(\nu - 1 - \widetilde{\beta}) - \mu)$.

Second, assume $\widetilde{\beta} \leq -\mu/a - 1$. Let $\nu > 0$ and $\widehat{\beta} = \mu/a + \nu$. Then $\widehat{\beta} \in (\mu/a, \mu/a + \nu]$ and, if $\omega > 1$,

$$\left\| \lambda^{\widehat{\beta} - \nu + 1} e^{-\lambda\mu} R(\lambda, A) \right\| \leq \left\| \frac{e^{-\lambda\mu}}{\lambda^{\widetilde{\beta}}} R(\lambda, A) \right\| \quad \forall \lambda \in M,\ \mathrm{Re}\lambda > \omega, \tag{3.75}$$

since $\widetilde{\beta} + \widehat{\beta} - \nu + 1 \leq 0$.

Corollary 3.6 shows that A generates a local ν-times integrated semigroup on $[0, a\nu)$. □

4. The theorem of Prüss and its application

Next we generalize a result from Prüss [12, Theorem 0.4] to the local case. In its original form, the function q has to be defined on some right half-plane and then is the Laplace transform of some continuous and exponentially bounded function.

This will lead to a new Hille-Yosida-type theorem for integrated semigroups.

Theorem 4.1. *Let X be a complex Banach space, $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \mathrm{Re}\, z > \omega_1,\ |\mathrm{Im}\, z| < F(\mathrm{Re}\, z - \omega_1)\}$.*

Furthermore, let $q : M \to X$ be a holomorphic function satisfying

$$\sup_{\lambda \in M} \|\lambda q(\lambda)\| < \infty, \qquad \sup_{\lambda \in M} \|\lambda^2 q'(\lambda)\| < \infty. \tag{4.1}$$

Then there is some $H \in C((0, a), X)$ such that $t \mapsto (t - a)H(t)$ is bounded on $(0, a)$ and with

$$q(\lambda) \sim_{\xi} \widehat{H}_{\xi}(\lambda) \quad \forall \xi \in (0, a). \tag{4.2}$$

Proof. From the complex representation theorem it follows that there are functions $g,h \in C([0,a),X)$ such that $g(t)/t$ and $h(t)/t$ are in $L_\infty^{\mathrm{loc}}([0,a),X)$ and with

$$\begin{aligned} q(\lambda) &= \lambda \int_0^\xi e^{-\lambda t} g(t)\,dt + \varepsilon_\xi(\lambda), \\ \lambda q'(\lambda) &= \lambda \int_0^\xi e^{-\lambda t} h(t)\,dt + \psi_\xi(\lambda), \end{aligned} \tag{4.3}$$

for all $\xi \in (0,a)$, where ε_ξ and ψ_ξ are functions satisfying

$$\limsup_{\lambda\to\infty} \frac{\ln\|\varepsilon_\xi(\lambda)\|}{\lambda} \le -\xi, \qquad \limsup_{\lambda\to\infty} \frac{\ln\|\psi_\xi(\lambda)\|}{\lambda} \le -\xi. \tag{4.4}$$

Thus, if $\eta > \omega > \omega_1$, then

$$\begin{aligned} \int_\omega^\eta q'(\lambda)\,d\lambda &= \int_\omega^\eta \int_0^\xi e^{-\lambda t} h(t)\,dt\,d\lambda + \int_\omega^\eta \frac{\psi_\xi(\lambda)}{\lambda}\,d\lambda \\ &= \int_0^\xi \frac{e^{-\omega t} - e^{-\eta t}}{t} h(t)\,dt + \int_\omega^\eta \frac{\psi_\xi(\lambda)}{\lambda}\,d\lambda \\ &\xrightarrow{\eta\to\infty} \int_0^\xi \frac{e^{-\omega t} h(t)}{t}\,dt + \int_\omega^\infty \frac{\psi_\xi(\lambda)}{\lambda}\,d\lambda. \end{aligned} \tag{4.5}$$

Thus, $y := \lim_{\eta\to\infty} q(\eta)$ exists, and

$$y - \omega \int_0^\xi e^{-\omega t} g(t)\,dt - \varepsilon_\xi(\omega) = \int_0^\xi \frac{e^{-\omega t} h(t)}{t}\,dt + \int_\omega^\infty \frac{\psi_\xi(\lambda)}{\lambda}\,d\lambda \tag{4.6}$$

for all $\omega > \omega_1$ and all $\xi \in (0,a)$.

It is not hard to see that for all $\delta > 0$ we have

$$e^{-\omega\delta} e^{\omega\xi} \left\| \int_\omega^\infty \frac{\psi_\xi(\lambda)}{\lambda}\,d\lambda \right\| \xrightarrow{\omega\to\infty} 0, \tag{4.7}$$

that is,

$$\limsup_{\omega\to\infty} \frac{\ln\left\| \int_\omega^\infty (\psi_\xi(\lambda)/\lambda)\,d\lambda \right\|}{\lambda} \le -\xi. \tag{4.8}$$

Thus

$$\int_0^\xi e^{-\omega t}(y - g(t))\,dt \sim_\xi \int_0^\xi \frac{e^{-\omega t}}{\omega} \cdot \frac{h(t)}{t}\,dt \sim_\xi \int_0^\xi e^{-\omega t} H^{[1]}(t)\,dt, \tag{4.9}$$

where $H(t) := h(t)/t$.

Consequently, $y - g(t) = H^{[1]}(t)$ for all $0 \le t < a$. This shows that $y = 0$ and

$$g(t) = -\int_0^t \frac{h(t)}{t}\,dt \quad \forall 0 \le t < a. \tag{4.10}$$

Thus

$$q(\lambda) \sim_\xi \int_0^\xi e^{-\lambda t} H(t)dt. \tag{4.11}$$

□

Theorem 4.2. *Let* $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_1, \ |\operatorname{Im} z| < F(\operatorname{Re} z - \omega_1)\}$, *and let* $p : (0,\infty) \to \mathbb{C}$ *be some continuous function satisfying* $|p(t)| \le \text{const} \cdot e^{\omega_1 t}$ *for all* $t > 0$ *and*

$$\liminf_{\lambda \to \infty} \frac{\ln |\hat{p}(\lambda)|}{\lambda} \ge 0, \tag{4.12}$$

$$\sup_{\lambda \in M} |\lambda \hat{p}(\lambda)| < \infty, \tag{4.13}$$

$$\sup_{\lambda \in M} \left|\lambda^2 \frac{d}{d\lambda} \hat{p}(\lambda)\right| < \infty. \tag{4.14}$$

Let X *be a complex Banach space,* $A : D(A) \to X$ *linear with* $M \subset \rho(A)$,

$$\sup_{\lambda \in M} |\lambda^3 \hat{p}(\lambda)| \cdot ||R(\lambda, A)^2|| < \infty, \tag{4.15}$$

$$\sup_{\lambda \in M} |\lambda^2 \hat{p}(\lambda)| \cdot ||R(\lambda, A)|| < \infty, \tag{4.16}$$

$$\sup_{\lambda \in M} \left|\lambda^3 \frac{d}{d\lambda} \hat{p}(\lambda)\right| \cdot ||R(\lambda, A)|| < \infty. \tag{4.17}$$

Then the following assertions hold:

(1) *A generates a local $p^{[1]}$-convoluted semigroup on* $[0, a)$;
(2) *the abstract Cauchy problem*

$$\begin{gathered} Av(t) + p(t)x = v'(t) \quad \text{if } 0 < t < a, \\ v(0) = 0, \\ v \in C^1((0,a), X) \cap C([0,a), X), \end{gathered} \tag{4.18}$$

has a unique solution v_x for all $x \in X$;

(3) *if p can be continuously extended in 0, then $A_{\overline{D(A)}}$ generates a local p-convoluted semigroup on* $[0, a)$.

Remark 4.3. Condition (4.12) can be omitted if one then assumes in addition that

$$\limsup_{\lambda \to \infty} \frac{\ln ||R(\lambda, A)||}{\lambda} \le 0. \tag{4.19}$$

Proof. Consider the holomorphic function

$$q : M \longrightarrow \mathcal{L}(X, \overline{D}(A)), \quad q(\lambda) = \hat{p}(\lambda) R(\lambda, A). \tag{4.20}$$

Then $\lambda^2 q'(\lambda) = -\lambda^2 \widehat{p}(\lambda)R(\lambda,A)^2 + \lambda^2(d\widehat{p}(\lambda)/d\lambda)R(\lambda,A)$, thus

$$
\begin{aligned}
\|\lambda^2 q'(\lambda)\| &\leq \|\lambda^2\widehat{p}(\lambda)R(\lambda,A)^2\| + \left\|\lambda^2\frac{d\widehat{p}(\lambda)}{d\lambda}R(\lambda,A)\right\| \\
&\quad + \left\|(\lambda R(\lambda,A)-\mathrm{id})\left(\lambda^2\widehat{p}(\lambda)R(\lambda,A) - \lambda^2\frac{d\widehat{p}(\lambda)}{d\lambda}\cdot\mathrm{id}\right)\right\| \\
&\leq \|\lambda^2\widehat{p}(\lambda)R(\lambda,A)^2\| + \left\|\lambda^2\frac{d\widehat{p}(\lambda)}{d\lambda}R(\lambda,A)\right\| + \|\lambda^3\widehat{p}(\lambda)R(\lambda,A)^2\| \\
&\quad + \left\|\lambda^3\frac{d\widehat{p}(\lambda)}{d\lambda}R(\lambda,A)\right\| + \|\lambda^2\widehat{p}(\lambda)R(\lambda,A)\| + \left|\lambda^2\frac{d\widehat{p}(\lambda)}{d\lambda}\right|.
\end{aligned}
\tag{4.21}
$$

It follows from the assumptions that

$$\sup_{\lambda\in M}\|\lambda^2 q'(\lambda)\| < \infty. \tag{4.22}$$

Moreover,

$$\|\lambda q(\lambda)\| \leq \|\lambda\widehat{p}(\lambda)R(\lambda,A)\| + \|\lambda\widehat{p}(\lambda)(\lambda R(\lambda,A)-\mathrm{id})\|, \tag{4.23}$$

thus

$$\sup_{\lambda\in M}\|\lambda q(\lambda)\| < \infty. \tag{4.24}$$

From Theorem 4.1 it follows that there is some $H \in C((0,a),\mathscr{L}(X,\overline{D}(A)))$ such that $t \mapsto (t-a)H(t)$ is bounded on $(0,a)$ and with

$$\widehat{p}(\lambda)R(\lambda,A) \sim_\xi \int_0^\xi e^{-\lambda t}H(t)\,dt \quad \forall \xi \in (0,a). \tag{4.25}$$

From conditions (4.12) and (4.16), it follows that $\limsup_{\lambda\to\infty}(\ln\|R(\lambda,A)\|/\lambda) \leq 0$, thus

$$
\begin{aligned}
\lambda\left(p^{[1]}\right)^{\wedge}_{\xi}(\lambda)R(\lambda,A)x &\sim_\xi \widehat{p}_\xi(\lambda)R(\lambda,A)x \\
&\sim_\xi \int_0^\xi e^{-\lambda t}H(t)x\,dt \\
&\sim_\xi \int_0^\xi e^{-\lambda t}d_t\left(H^{[1]}(t)x\right).
\end{aligned}
\tag{4.26}
$$

This shows (1). To be precise, the solution of the abstract Cauchy problem

$$
\begin{gathered}
Au_x(t) + p^{[1]}(t)x = u_x'(t), \quad 0 \leq t < a, \\
u_x(0) = 0, \\
u_x \in C^1([0,a),X),
\end{gathered}
\tag{4.27}
$$

is $u_x(t) = \int_0^t H(s)x\,ds$.

Further on, we define $v_x : [0,a) \to X$ by

$$v_x(t) = \begin{cases} 0 & \text{if } t = 0, \\ H(t)x & \text{if } 0 < t < a. \end{cases} \tag{4.28}$$

Since $Av_x^{[1]} = (AH(\cdot)x)^{[1]}$ is in $C^1((0,a),X)$, we have $v_x \in C^1((0,a),X)$, and since A is closed, we obtain

$$Av_x(t) + p(t)x = v_x'(t) \quad \text{if } 0 < t < a. \tag{4.29}$$

This shows (2).

Finally if $x \in D(A)$, then

$$\begin{gathered} A_{\overline{D(A)}} v_x(t) + p(t)x = v_x'(t) \quad \text{if } 0 \le t < a, \\ v_x(0) = 0, \\ v_x \in C^1\Big([0,a), \overline{D(A)}\Big). \end{gathered} \tag{4.30}$$

The result follows if we can show that

$$\sup_{0 \le t \le \xi} \sup_{\substack{y \in D(A) \\ \|y\| \le 1}} ||v_y'(t)||_X < \infty \qquad \forall \xi \in (0,a). \tag{4.31}$$

If $0 < t \le \xi$ and $y \in D(A)$ with $\|y\| \le 1$, we obtain

$$\begin{aligned} ||v_y'(t)|| &\le ||Av_y(t) + p(t)y|| = ||AH(t)y + p(t)y|| \\ &\le ||AH(t)||_{\mathcal{L}(X)} + \|p\|_{L_\infty[0,\xi]}. \end{aligned} \tag{4.32}$$

□

Corollary 4.4. *Let $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_1,\ |\operatorname{Im} z| < F(\operatorname{Re} z - \omega_1)\}$.*

Suppose X is a complex Banach space, $A : D(A) \to X$ is linear with $M \subset \rho(A)$, and that there is some $\gamma \ge 0$ with

$$\begin{gathered} \sup_{\lambda \in M} \frac{||R(\lambda,A)^2||}{|\lambda|^{\gamma-2}} < \infty, \\ \sup_{\lambda \in M} \frac{||R(\lambda,A)||}{|\lambda|^{\gamma-1}} < \infty. \end{gathered} \tag{4.33}$$

Then the following assertions hold:

(1) *A generates a $(\gamma+1)$-integrated semigroup on $[0,a)$;*

(2) *the abstract Cauchy problem*

$$\begin{gathered}
Av(t)+t^{\gamma}\cdot x=v'(t) \quad \text{if } 0<t<a,\\
v(0)=0,\\
v\in C^1((0,a),X)\cap C([0,a),X),
\end{gathered} \tag{4.34}$$

has a unique solution v_x *for all* $x\in X$;

(3) $A_{\overline{D(A)}}$ *generates a* γ*-integrated semigroup on* $[0,a)$.

Proof. Let $p(t)=t^{\gamma}$ and apply Theorem 4.2. □

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