# WHICH SOLUTIONS OF THE THIRD PROBLEM FOR THE POISSON EQUATION ARE BOUNDED?

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This paper deals with the problem $\Delta u = g$ on $G$ and $\partial u/\partial n + uf = L$ on $\partial G$. Here, $G \subset \mathbb{R}^m$, $m > 2$, is a bounded domain with Lyapunov boundary, $f$ is a bounded nonnegative function on the boundary of $G$, $L$ is a bounded linear functional on $W^{1,2}(G)$ representable by a real measure $\mu$ on the boundary of $G$, and $g \in L_2(G) \cap L_p(G)$, $p > m/2$. It is shown that a weak solution of this problem is bounded in $G$ if and only if the Newtonian potential corresponding to the boundary condition $\mu$ is bounded in $G$.

Suppose that $G \subset \mathbb{R}^m$, $m > 2$, is a bounded domain with Lyapunov boundary (i.e., of class $C^{1+\alpha}$). Denote by $n(y)$ the outer unit normal of $G$ at $y$. If $f, g, h \in C(\partial G)$ and $u \in C^2(\operatorname{cl} G)$ is a classical solution of

$$\begin{aligned} \Delta u &= g \quad \text{on } G, \\ \frac{\partial u}{\partial n} + uf &= h \quad \text{on } \partial G, \end{aligned} \tag{1}$$

then Green's formula yields

$$\int_G \nabla u \cdot \nabla v \, d\mathcal{H}_m + \int_{\partial G} u f v \, d\mathcal{H}_{m-1} = \int_{\partial G} h v \, d\mathcal{H}_{m-1} - \int_G g v \, d\mathcal{H}_m \tag{2}$$

for each $v \in \mathcal{D}$, the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$. Here, $\partial G$ denotes the boundary of $G$ and $\operatorname{cl} G$ is the closure of $G$; $\mathcal{H}_k$ is the $k$-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. Denote by $\mathcal{D}(G)$ the set of all functions from $\mathcal{D}$ with the support in $G$.

For an open set $V \subset \mathbb{R}^m$, denote by $W^{1,2}(V)$ the collection of all functions $f \in L_2(V)$, the distributional gradient of which belongs to $[L_2(V)]^m$.

*Definition 1.* Let $f \in L_\infty(\mathcal{H})$, $g \in L_2(G)$ and let $L$ be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathcal{D}(G)$. We say that $u \in W^{1,2}(G)$ is a weak solution


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in $W^{1,2}(G)$ of the third problem for the Poisson equation

$$\begin{aligned}\Delta u &= g \quad \text{on } G,\\ \frac{\partial u}{\partial n} + uf &= L \quad \text{on } \partial G,\end{aligned} \tag{3}$$

if

$$\int_G \nabla u \cdot \nabla v \, d\mathcal{H}_m + \int_{\partial G} ufv \, d\mathcal{H} = L(v) - \int_G gv \, d\mathcal{H}_m \tag{4}$$

for each $v \in W^{1,2}(G)$.

Denote by $\mathscr{C}'(\partial G)$ the Banach space of all finite signed Borel measures with support in $\partial G$ with the total variation as a norm. We say that the bounded linear functional $L$ on $W^{1,2}(G)$ is representable by $\mu \in \mathscr{C}'(\partial G)$ if $L(\varphi) = \int \varphi \, d\mu$ for each $\varphi \in \mathfrak{D}$. Since $\mathfrak{D}$ is dense in $W^{1,2}(G)$, the operator $L$ is uniquely determined by its representation $\mu \in \mathscr{C}'(\partial G)$.

For $x, y \in \mathbb{R}^m$, denote

$$h_x(y) = \begin{cases} (m-2)^{-1}A^{-1}|x-y|^{2-m} & \text{for } x \neq y,\\ \infty & \text{for } x = y, \end{cases} \tag{5}$$

where $A$ is the area of the unit sphere in $\mathbb{R}^m$. For the finite real Borel measure $\nu$, denote

$$\mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y) \, d\nu(y) \tag{6}$$

the Newtonian potential corresponding to $\nu$, for each $x$ for which this integral has sense.

We denote by $\mathscr{C}'_b(\partial G)$ the set of all $\mu \in \mathscr{C}'(\partial G)$ for which $\mathcal{U}\mu$ is bounded on $\mathbb{R}^m \setminus \partial G$.

Remark that $\mathscr{C}'_b(\partial G)$ is the set of all $\mu \in \mathscr{C}'(\partial G)$ for which there is a polar set $M$ such that $\mathcal{U}\mu(x)$ is meaningful and bounded on $\mathbb{R}^m \setminus M$, because $\mathbb{R}^m \setminus \partial G$ is finely dense in $\mathbb{R}^m$ (see [1, Chapter VII, Sections 2, 6], [7, Theorems 5.10 and 5.11]) and $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set. Remark that $\mathcal{H}_{m-1}(M) = 0$ for each polar set $M$ (see [7, Theorem 3.13]). (For the definition of polar sets, see [4, Chapter 7, Section 1]; for the definition of the fine topology, see [4, Chapter 10].)

Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ to $\partial G$.

Lemma 2. *Let $\mu \in \mathscr{C}'(\partial G)$. Then the following assertions are equivalent:*

(1) $\mu \in \mathscr{C}'_b(\partial G)$,
(2) $\mathcal{U}\mu$ *is bounded in* $G$,
(3) $\mathcal{U}\mu \in L_\infty(\mathcal{H})$.

*Proof.* (2)⇒(3). Since $\partial G$ is a subset of the fine closure of $G$ by [1, Chapter VII, Sections 2, 6] and [7, Theorems 5.10 and 5.11], $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set $M$, and $\mathcal{H}_{m-1}(M) = 0$ by [4, Theorem 7.33] and [7, Theorem 3.13], then we obtain that $\mathcal{U}\mu \in L_\infty(\mathcal{H})$.

(3)⇒(1). Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of $\mu$. For $z \in G$, denote by $\mu_z$ the harmonic measure corresponding to $G$ and $z$. If $y \in \partial G$ and $z \in G$, then

$$\int_{\partial G} h_y(x)\, d\mu_z(x) = h_y(z) \tag{7}$$

by [7, pages 264, 299]. Using Fubini's theorem, we get

$$\int \mathcal{U}\mu^+\, d\mu_z = \int_{\partial G}\int_{\partial G} h_y(x)\, d\mu_z(x)\, d\mu^+(y) = \int_{\partial G} h_y(z)\, d\mu^+(y) = \mathcal{U}\mu^+(z). \tag{8}$$

Similarly, $\int \mathcal{U}\mu^-\, d\mu_z = \mathcal{U}\mu^-(z)$. Since $\mathcal{U}\mu \in L_\infty(\mathcal{H})$, $\mu_z$ is a nonnegative measure with the total variation 1 (see [4, Lemma 8.12]) which is absolutely continuous with respect to $\mathcal{H}$ by [2, Theorem 1], then we obtain that $|\mathcal{U}\mu(z)| \le \|\mathcal{U}\mu\|_{L_\infty(\mathcal{H})}$.

If $z \in \mathbb{R}^m \setminus \operatorname{cl} G$, choose a bounded domain $V$ with smooth boundary such that $\operatorname{cl} G \cup \{z\} \subset V$. Repeating the previous reasonings for $V \setminus \operatorname{cl} G$, we get $|\mathcal{U}\mu(z)| \le \|\mathcal{U}\mu\|_{L_\infty(\mathcal{H})}$. □

Lemma 3. *Let $f \in L_\infty(\mathcal{H})$ and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Then $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$. Moreover, there is a bounded linear functional $L$ on $W^{1,2}(G)$ representable by $\mu \in \mathcal{C}_b'(\partial G)$ such that $\mathcal{U}(g\mathcal{H}_m)$ is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation*

$$\Delta u = -g \quad \text{on } G, \qquad \frac{\partial u}{\partial n} + uf = L \quad \text{on } \partial G. \tag{9}$$

*Proof.* Suppose first that $g$ is nonnegative. Since $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m)$ by [3, Theorem A.6], the energy $\int g\mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m < \infty$. According to [7, Theorem 1.20], we have

$$\int \left|\nabla \mathcal{U}(g\mathcal{H}_m)\right|^2 d\mathcal{H}_m = \int g\mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m < \infty, \tag{10}$$

and therefore $\mathcal{U}(g\mathcal{H}_m) \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.1.4]).

Since $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$, $f \in L_\infty(\mathcal{H})$ and the trace operator is a bounded operator from $W^{1,2}(G)$ to $L_2(\mathcal{H})$ by [8, Theorem 3.38], then the operator

$$L(\varphi) = \int_G \nabla\varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f\varphi\, d\mathcal{H}_{m-1} - \int_G g\varphi\, d\mathcal{H}_m \tag{11}$$

is a bounded linear functional on $W^{1,2}(G)$.

According to [7, Theorem 4.2], there is a nonnegative $\nu \in \mathcal{C}'(\partial G)$ such that $\mathcal{U}\nu = \mathcal{U}(g\mathcal{H}_m)$ on $\mathbb{R}^m \setminus \operatorname{cl} G$. Choose a bounded domain $V$ with smooth boundary such that $\operatorname{cl} G \subset V$. Since $\mathcal{U}\nu$ is bounded in $V \setminus \operatorname{cl} G \subset \mathbb{R}^m \setminus \operatorname{cl} G$, Lemma 2 yields that $\nu \in \mathcal{C}_b'(\partial(V \setminus \operatorname{cl} G))$. Therefore, $\nu \in \mathcal{C}_b'(\partial G)$. According to [13, Lemma 4], there is $\tilde{\nu} \in \mathcal{C}_b'(\partial G)$ such that

$$\int_{\mathbb{R}^m \setminus \operatorname{cl} G} \nabla\varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m = \int_{\mathbb{R}^m \setminus \operatorname{cl} G} \nabla\varphi \cdot \nabla \mathcal{U}\nu\, d\mathcal{H}_m = \int_{\partial G} \varphi\, d\tilde{\nu} \tag{12}$$

for each $\varphi \in \mathcal{D}$. Let $\mu = \tilde{\nu} - f\mathcal{U}(g\mathcal{H}_m)\mathcal{H}$. Since $\mathcal{U}(f\mathcal{U}(g\mathcal{H}_m)\mathcal{H}) \in \mathcal{C}(\mathbb{R}^m)$ by [6, Corollary 2.17 and Lemma 2.18] and $\mathcal{U}(f\mathcal{U}(g\mathcal{H}_m)\mathcal{H})(x) \to 0$ as $|x| \to \infty$, we have $f\mathcal{U}(g\mathcal{H}_m)\mathcal{H} \in \mathcal{C}_b'(\partial G)$. Therefore, $\mu \in \mathcal{C}_b'(\partial G)$.

If $\varphi \in \mathcal{D}$, then $\varphi = \mathcal{U}((-\Delta\varphi)\mathcal{H}_m)$ by [3, Theorem A.2]. According to [7, Theorem 1.20],

$$\begin{aligned}\int_{\mathbb{R}^m} \nabla\varphi \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\,d\mathcal{H}_m &= \int_{\mathbb{R}^m} \nabla\mathcal{U}((-\Delta\varphi)\mathcal{H}_m) \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\,d\mathcal{H}_m \\ &= \int_{\mathbb{R}^m} g\mathcal{U}((-\Delta\varphi)\mathcal{H}_m)\,d\mathcal{H}_m \\ &= \int_{\mathbb{R}^m} g\varphi\,d\mathcal{H}_m.\end{aligned} \tag{13}$$

Since $\mathcal{H}_m(\partial G) = 0$,

$$\begin{aligned}&\int_G \nabla\varphi \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\,d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f\varphi\,d\mathcal{H}_{m-1} \\ &\quad = \int_G g\varphi\,d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f\varphi\,d\mathcal{H}_{m-1} \\ &\qquad - \int_{\mathbb{R}^m\setminus \mathrm{cl}\,G} \nabla\varphi \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\,d\mathcal{H}_m \\ &\quad = \int_G g\varphi\,d\mathcal{H}_m + \int_{\partial G} \varphi\,d\mu.\end{aligned} \tag{14}$$

□

Lemma 4. *Let $f \in L_\infty(\mathcal{H})$ and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Let $L$ be a bounded linear functional on $W^{1,2}(G)$ representable by $\mu \in \mathcal{C}'(\partial G)$. If $u \in L_\infty(G) \cap W^{1,2}(G)$ is a weak solution in $W^{1,2}(G)$ of problem (3), then $\mu \in \mathcal{C}'_b(\partial G)$.*

*Proof.* Let $w = u - \mathcal{U}(g\mathcal{H}_m)$. According to Lemma 3, there is a bounded linear functional $\tilde{L}$ on $W^{1,2}(G)$ representable by $\nu \in \mathcal{C}'_b(\partial G)$ such that $w$ is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{aligned}\Delta w = 0 \quad &\text{on } G, \\ \frac{\partial w}{\partial n} + wf = L - \tilde{L} \quad &\text{on } \partial G.\end{aligned} \tag{15}$$

Fix $x \in G$. Choose a sequence $G_j$ of open sets with $C^\infty$ boundary such that $\mathrm{cl}\,G_j \subset G_{j+1} \subset G$, $x \in G_1$, and $\cup G_j = G$. Fix $r > 0$ such that $\Omega_{2r}(x) \subset G_1$. Choose an infinitely differentiable function $\psi$ such that $\psi = 0$ on $\Omega_r(x)$ and $\psi = 1$ on $\mathbb{R}^m \setminus \Omega_{2r}(x)$. According to Green's identity,

$$\begin{aligned}w(x) &= \lim_{j\to\infty}\left[\int_{\partial G_j} h_x(y)\frac{\partial w(y)}{\partial n}\,d\mathcal{H}_{m-1}(y) - \int_{\partial G_j} w(y)n(y)\cdot\nabla h_x(y)\,d\mathcal{H}_{m-1}(y)\right] \\ &= \lim_{j\to\infty}\left[\int_{G_j} \nabla w(y)\cdot\nabla\big(h_x(y)\psi(y)\big)\,d\mathcal{H}_m(y)\right. \\ &\qquad \left. - \int_{G_j} \nabla\big(w(y)\psi(y)\big)\cdot\nabla h_x(y)\,d\mathcal{H}_m(y)\right]\end{aligned}$$

$$\begin{aligned}&= \int_G \nabla w(y) \cdot \nabla (h_x(y)\psi(y)) d\mathcal{H}_m(y) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) d\mathcal{H}_m(y) \\ &= \mathcal{U}(\mu - \nu - f w\mathcal{H})(x) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) d\mathcal{H}_m(y).\end{aligned} \tag{16}$$

According to [16, Theorem 2.3.2], there is a sequence of infinitely differentiable functions $w_n$ such that $w_n \to w\psi$ in $W^{1,2}(G)$. According to [6, Section 2],

$$\begin{aligned}w(x) &= \mathcal{U}(\mu - \nu - f w\mathcal{H})(x) - \lim_{n\to\infty} \int_G \nabla w_n(y) \cdot \nabla h_x(y) d\mathcal{H}_m(y) \\ &= \mathcal{U}(\mu - \nu - f w\mathcal{H})(x) - \lim_{n\to\infty} \int_{\partial G} w_n(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y).\end{aligned} \tag{17}$$

Since the trace operator is a bounded operator from $W^{1,2}(G)$ to $L_2(\mathcal{H})$ by [8, Theorem 3.38], we obtain

$$w(x) = \mathcal{U}(\mu - \nu - f w\mathcal{H})(x) - \int_{\partial G} w(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y). \tag{18}$$

Since $w \in L_\infty(G)$ by Lemma 3, the trace of $w$ is an element of $L_\infty(\mathcal{H})$. Since

$$\begin{aligned}&\left| \int_{\partial G} w(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) \right| \\ &\quad \le \|w\|_{L_\infty(\mathcal{H})} \int_{\partial G} |n(y) \cdot \nabla h_x(y)| d\mathcal{H}_{m-1}(y) \\ &\quad \le \|w\|_{L_\infty(\mathcal{H})} \left[ \sup_{z\in\partial G} \int_{\partial G} |n(y) \cdot \nabla h_z(y)| d\mathcal{H}_{m-1}(y) + \frac{1}{2} \right] < \infty\end{aligned} \tag{19}$$

by [6, Lemma 2.15 and Theorem 2.16] and the fact that $\partial G$ is of class $C^{1+\alpha}$, the function

$$x \longmapsto \int_{\partial G} w(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) \tag{20}$$

is bounded in $G$. Since $\mathcal{U}\nu$ is bounded in $G$ and $\mathcal{U}(f w\mathcal{H})$ is bounded in $G$ by [6, Corollary 2.17 and Lemma 2.18], the function $\mathcal{U}\mu$ is bounded in $G$ by (18). Thus, $\mu \in \mathcal{C}'_b(\partial G)$ by Lemma 2. □

*Notation 5.* Let $X$ be a complex Banach space and $T$ a bounded linear operator on $X$. We denote by $\operatorname{Ker} T$ the kernel of $T$, by $\sigma(T)$ the spectrum of $T$, by $r(T)$ the spectral radius of $T$, by $X'$ the dual space of $X$, and by $T'$ the adjoint operator of $T$. Denote by $I$ the identity operator.

THEOREM 6. *Let $X$ be a complex Banach space and $K$ a compact linear operator on $X$. Let $Y$ be a subspace of $X'$ and $T$ a closed linear operator from $Y$ to $X$ such that $y(Tx) = x(Ty)$ for each $x, y \in Y$. Suppose that $K'(Y) \subset Y$ and $KTy = TK'y$ for each $y \in Y$. Let $\alpha \in C \setminus \{0\}$, $\operatorname{Ker}(K' - \alpha I)^2 = \operatorname{Ker}(K' - \alpha I) \subset Y$, and $\{\beta \in \sigma(K'); (\beta - \alpha) \cdot \alpha \le 0\} \subset \{\alpha\}$. If $x, y \in X$, $(K' - \alpha I)x = y$, then $x \in Y$ if and only if $y \in Y$.*

*Proof.* If $x \in Y$, then $y \in Y$. Suppose that $y \in Y$. Since $K$ is a compact operator, the operator $K'$ is a compact operator by [14, Chapter IV, Theorem 4.1]. Suppose first that $\alpha \in \sigma(K')$. Since $K'$ is compact, then $\alpha$ is a pol of the resolvent by [5, Satz 50.4]. Since

$$\operatorname{Ker}(K' - \alpha I)^2 = \operatorname{Ker}(K' - \alpha I), \tag{21}$$

the ascent of $(K' - \alpha I)$ is equal to 1. Since $\alpha$ is a pol of the resolvent and the ascent of $(K' - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space $X'$ is the direct sum of $\operatorname{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X')$ and the descent of $(K' - \alpha I)$ is equal to 1. Since the descent of $(K' - \alpha I)$ is equal to 1, we have

$$(K' - \alpha I)^2(X') = (K' - \alpha I)(X'). \tag{22}$$

Since the space $X'$ is the direct sum of $\operatorname{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X') = (K' - \alpha I)^2(X')$, the operator $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$. If $\alpha \notin \sigma(K')$, then the space $X'$ is the direct sum of $\operatorname{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X')$, and the operator $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$. Therefore, there are $x_1 \in \operatorname{Ker}(K' - \alpha I) \subset Y$ and $x_2 \in (K' - \alpha I)(X')$ such that $x_1 + x_2 = x$. We have $(K' - \alpha I)x_2 = y$.

Denote by $Z$ the closure of $Y$. Since $K'(Y) \subset Y$, we obtain $K'(Z) \subset Z$. Denote by $K'_Z$ the restriction of $K'$ to $Z$. Then $K'_Z$ is a compact operator in $Z$. Since $\operatorname{Ker}(K' - \alpha I)^2 \subset Y$, we have

$$\operatorname{Ker}\left(K'_Z - \alpha I\right)^2 = \operatorname{Ker}(K' - \alpha I)^2 = \operatorname{Ker}(K' - \alpha I) = \operatorname{Ker}\left(K'_Z - \alpha I\right). \tag{23}$$

If $\alpha \notin \sigma(K'_Z)$, then the space $Z$ is the direct sum of $\operatorname{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z)$, and the operator $(K'_Z - \alpha I)$ is invertible on $Z$. Suppose that $\alpha \in \sigma(K'_Z)$. Since $K'_Z$ is compact, then $\alpha$ is a pol of the resolvent by [5, Satz 50.4]. Since

$$\operatorname{Ker}\left(K'_Z - \alpha I\right)^2 = \operatorname{Ker}\left(K'_Z - \alpha I\right), \tag{24}$$

the ascent of $(K'_Z - \alpha I)$ is equal to 1. Since $\alpha$ is a pol of the resolvent and the ascent of $(K'_Z - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space $Z$ is the direct sum of $\operatorname{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z)$ and the descent of $(K'_Z - \alpha I)$ is equal to 1. Since the descent of $(K'_Z - \alpha I)$ is equal to 1, we have

$$\left(K'_Z - \alpha I\right)^2(Z) = (K' - \alpha I)(Z). \tag{25}$$

Since the space $Z$ is the direct sum of $\operatorname{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z) = (K'_Z - \alpha I)^2(Z)$, the operator $(K'_Z - \alpha I)$ is invertible on $(K'_Z - \alpha I)(Z)$. Since $y \in Y \subset Z$, there are $y_1 \in \operatorname{Ker}(K'_Z - \alpha I)$ and $y_2 \in (K'_Z - \alpha I)(Z)$ such that $y = y_1 + y_2$. Since $X'$ is the direct sum of $\operatorname{Ker}(K' - \alpha I) = \operatorname{Ker}(K'_Z - \alpha I)$ and $(K' - \alpha I)(X') \supset (K'_Z - \alpha I)(Z)$ and $y \in (K' - \alpha I)(X')$, we obtain that $y_1 = 0$ and $y_2 = y$. Thus, $y \in (K'_Z - \alpha I)(Z)$. Since $(K'_Z - \alpha I)$ is invertible on $(K'_Z - \alpha I)(Z)$, there is $z \in (K'_Z - \alpha I)(Z)$ such that $(K'_Z - \alpha I)(z) = y$. Since $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$, we deduce that $x_2 = z \in (K'_Z - \alpha I)(Z) \subset Z$.

Now, let $w \in \operatorname{Ker}(K' - \alpha I)$. Fix a sequence $\{z_k\} \subset Y$ such that $z_k \to z = x_2$. Then

$$\begin{aligned} w(Ty) = y(Tw) &= [(K' - \alpha I)x_2](Tw) = \lim_{k\to\infty}[(K' - \alpha I)z_k](Tw) \\ &= \lim_{k\to\infty} z_k((K - \alpha I)Tw) = \lim_{k\to\infty} z_k(T(K' - \alpha I)w) = \lim_{k\to\infty} z_k(0) = 0. \end{aligned} \tag{26}$$

Since $w(Ty) = 0$ for each $w \in \operatorname{Ker}(K' - \alpha I)$, [15, Chapter 10, Theorem 3] yields $Ty \in (K - \alpha I)(X)$.

Denote by $\tilde{K}'$ the restriction of $K'$ to $(K' - \alpha I)(X)$. If we denote by $P$ the spectral projection corresponding to the spectral set $\{\alpha\}$ and the operator $K'$, then $P(X') = (K' - \alpha I)(X')$ by [5, Satz 50.2] and $\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\}$ by [14, Chapter VI, Theorem 4.1]. Therefore,

$$\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\} \subset \{\beta;\ (\beta - \alpha) \cdot \alpha > 0\} \subset \bigcup_{t>0} \{\beta;\ |\beta - \alpha - t\alpha| < |t\alpha|\}. \tag{27}$$

Since $\{\beta;\ |\beta - \alpha - t_1\alpha| < |t_1\alpha|\} \subset \{\beta;\ |\beta - \alpha - t_2\alpha| < |t_2\alpha|\}$ for $0 < t_1 < t_2$ and $\sigma(\tilde{K}')$ is a compact set (see [14, Chapter VI, Theorem 1.3, and Lemma 1.5], there is $t > 0$ such that $\sigma(\tilde{K}') \subset \{\beta;\ |\beta - \alpha - t\alpha| < |t\alpha|\}$. Therefore, $r(\tilde{K}' - \alpha I - t\alpha I) < |t\alpha|$. Since we have $r(t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)) < 1$, the series

$$V = \sum_{k=0}^{\infty} (-1)^k \left[t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)\right]^k \tag{28}$$

converges. Easy calculation yields that $V$ is the inverse operator of the operator $I + t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I) = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)$. Since $t^{-1}\alpha^{-1}y = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)x_2$, we have $x_2 = t^{-1}\alpha^{-1}Vy$. Denote $z_k = t^{-1}\alpha^{-1}[-t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)]^k y$. Then

$$x_2 = \sum_{k=0}^{\infty} z_k. \tag{29}$$

Since $K'(Y) \subset Y$, $z_k \in Y$ for each $k$. Since $KT = TK'$ on $Y$, we have $Tz_k = t^{-1}\alpha^{-1}[-t^{-1}\alpha^{-1}(K - \alpha I - t\alpha I)]^k Ty$.

Since $(K - \alpha I)$, $(K - \alpha I)^2$, $(K' - \alpha I)$, and $(K' - \alpha I)^2$ are Fredholm operators with index 0 (see [14, Chapter V, Theorem 3.1]), [14, Chapter VII, Theorem 3.2] yields

$$\dim \operatorname{Ker}(K - \alpha I)^2 = \dim \operatorname{Ker}(K' - \alpha I)^2 = \dim \operatorname{Ker}(K' - \alpha I) = \dim \operatorname{Ker}(K - \alpha I), \tag{30}$$

and thus $\operatorname{Ker}(K - \alpha I)^2 = \operatorname{Ker}(K - \alpha I)$. If $\alpha \notin \sigma(K)$, then the space $X$ is the direct sum of $\operatorname{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$, and the operator $(K - \alpha I)$ is invertible on $X$. Suppose that $\alpha \in \sigma(K)$. Since $K$ is compact, then $\alpha$ is a pol of the resolvent by [5, Satz 50.4]. Since

$$\operatorname{Ker}(K - \alpha I)^2 = \operatorname{Ker}(K - \alpha I), \tag{31}$$

the ascent of $(K - \alpha I)$ is equal to 1. Since $\alpha$ is a pol of the resolvent and the ascent of $(K - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space $X$ is the direct sum of $\operatorname{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$ and the descent of $(K - \alpha I)$ is equal to 1. Since the descent of $(K - \alpha I)$ is equal to 1, we have $(K - \alpha I)^2(X) = (K - \alpha I)(X)$. Since the space $X$ is the direct sum of $\operatorname{Ker}(K-\alpha I)$ and $(K-\alpha I)(X)=(K-\alpha I)^2(X)$, the operator $(K-\alpha I)$ is invertible on $(K-\alpha I)(X)$. Denote by $\hat{K}$ the restriction of $K$ to $(K-\alpha I)(X)$. If we denote by $Q$ the spectral projection corresponding to the spectral set $\{\alpha\}$ and the operator $K$, then $Q(X)=(K-\alpha I)(X)$ by [5, Satz 50.2] and $\sigma(\hat{K})=\sigma(K)\setminus\{\alpha\}$ by [14, Chapter VI, Theorem 4.1]. Since $\sigma(K)=\sigma(K')$ by [14, Chapter VI, Theorem 4.6], we obtain $\sigma(\hat{K})\subset\{\beta;\,|\beta-\alpha-t\alpha|<|t\alpha|\}$. Therefore, $r(\hat{K}-\alpha I-t\alpha I)<|t\alpha|$. Since $Ty\in(K-\alpha X)$ and $r(t^{-1}\alpha^{-1}(\hat{K}-\alpha I-t\alpha I))<1$, the series

$$\sum_{k=0}^{\infty} Tz_k=\sum_{k=0}^{\infty} t^{-1}\alpha^{-1}\left[-t^{-1}\alpha^{-1}(\hat{K}-\alpha I-t\alpha I)\right]^k Ty \tag{32}$$

converges. Since $T$ is closed, $x_2=\sum z_k$, and $\sum Tz_k$ converges, then the vector $x_2$ lies in $Y$, the domain of $T$. □

Theorem 7. *Let $f\in L_\infty(\mathcal{H})$, $f\geq 0$, and $g\in L_2(G)\cap L_p(\mathbb{R}^m)$, where $p>m/2$, $g=0$ on $\mathbb{R}^m\setminus G$. Let $L$ be a bounded linear functional on $W^{1,2}(G)$ representable by $\mu\in\mathscr{C}'(\partial G)$. If $u$ is a weak solution in $W^{1,2}(G)$ of problem (3), then $u\in L_\infty(G)$ if and only if $\mu\in\mathscr{C}_b'(\partial G)$.*

*Proof.* If $u\in L_\infty(G)$, then $\mu\in\mathscr{C}_b'(\partial G)$ by Lemma 4.

Suppose now that $\mu\in\mathscr{C}_b'(\partial G)$. Let $w=u-\mathscr{U}(g\mathcal{H}_m)$. According to Lemma 3, there is a bounded linear functional $\tilde{L}$ on $W^{1,2}(G)$ representable by $\tilde{\mu}\in\mathscr{C}_b'(\partial G)$ such that $w$ is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{aligned} \Delta w &= 0 \quad \text{on } G,\\ \frac{\partial w}{\partial n}+wf &= \tilde{L} \quad \text{on } \partial G. \end{aligned} \tag{33}$$

Define for $\varphi\in L_\infty(\mathcal{H})$ and $x\in\partial G$,

$$T\varphi(x)=\frac{1}{2}\varphi(x)+\int_{\partial G}\varphi(y)\frac{\partial}{n(y)}h_x(y)\,d\mathcal{H}(y)+\mathscr{U}(f\varphi\mathcal{H}). \tag{34}$$

Since $\mathscr{U}(f\mathcal{H})\in\mathscr{C}(\mathbb{R}^m)$ by [6, Corollary 2.17 and Lemma 2.18], the operator $T$ is a bounded linear operator on $L_\infty(\mathcal{H})$ by [11, Proposition 8] and [6, Lemma 2.15]. The operator $T-(1/2)I$ is compact by [12, Theorem 20] and [6, Theorem 4.1 and Corollary 1.11]. According to [10, Theorem 1], there is $\nu\in\mathscr{C}'(\partial G)\subset(L_\infty(\mathcal{H}))'$ such that $T'\nu=\tilde{\mu}$ and

$$\int_G \nabla\mathscr{U}\nu\cdot\nabla v\,d\mathcal{H}_m+\int_{\partial G}\mathscr{U}\nu f v\,d\mathcal{H}=\int v\,d\tilde{\mu}, \tag{35}$$

for each $v\in\mathscr{D}$.

Remark that $\mathscr{C}'(\partial G)$ is a closed subspace of $(L_\infty(\mathcal{H}))'$. According to [11, Proposition 8], we have $T'(\mathscr{C}'(\partial G))\subset\mathscr{C}'(\partial G)$. Denote by $\tau$ the restriction of $T'$ to $\mathscr{C}'(\partial G)$. According to [10, Lemma 11] and [14, Chapter VI, Theorem 1.2], we have $\sigma(\tau)\subset\{\beta;\,\beta\geq 0\}$. Since $\sigma(\tau')=\sigma(\tau)$ (see [15, Chapter VIII, Section 6, Theorem 2]), each $\beta\in\sigma(T)$ is an eigenvalue (see [14, Chapter VI, Theorem 1.2]), and $T$ is the restriction of $\tau'$ to $L_\infty(\mathcal{H})$, we obtain that $\sigma(T')=\sigma(T)\subset\{\beta;\,\beta\geq 0\}$ by [15, Chapter VIII, Section 6, Theorem 2].

According to [9, Theorem 1.11], we have $\operatorname{Ker} T' \subset \mathscr{C}_b'(\partial G)$. According to [9, Lemma 1.10] and [10, Lemmas 12 and 13], $\operatorname{Ker} T' = \operatorname{Ker}(T')^2$. Denote, for $\rho \in \mathscr{C}_b'(\partial G)$, by $V\rho$ the restriction of $\mathscr{U}\rho$ to $\partial G$. Then $V$ is a closed operator from $\mathscr{C}_b'(\partial G)$ to $L_\infty(\mathscr{H})$ by [13, Lemma 5]. If $\rho \in \mathscr{C}_b'(\partial G)$, then $VT'\rho = TV\rho$ by [13, Lemma 4]. If $\rho_1, \rho_2 \in \mathscr{C}_b'(\partial G)$, then $\rho_1$ and $\rho_2$ have finite energy by [13, Proposition 23], [7, Theorem 1.20], and

$$\int \mathscr{U}\rho_1 \, d\rho_2 = \int_{\mathbb{R}^m} \nabla \mathscr{U}\rho_1 \cdot \nabla \mathscr{U}\rho_2 \, d\mathscr{H}_m = \int \mathscr{U}\rho_2 \, d\rho_1. \tag{36}$$

Since $T'\nu = \tilde{\mu} \in \mathscr{C}_b'(\partial G)$, Theorem 6 yields that $\nu \in \mathscr{C}_b'(\partial G)$. Since $\nu$ has finite energy $\int \mathscr{U}\nu \, d\nu$ and $\int \mathscr{U}\nu \, d\nu = \int |\nabla \mathscr{U}\nu|^2 d\mathscr{H}_m$ by [7, Theorem 1.20], we obtain that $\mathscr{U}\nu \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.14]). Since $\mathscr{D}$ is dense in $W^{1,2}(G)$ by [16, Theorem 2.3.2], relation (35) yields that the function $\mathscr{U}\nu$ is a weak solution in $W^{1,2}(G)$ of (33). Since $v = \mathscr{U}\nu - w$ is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{aligned} \Delta v &= 0 \quad \text{on } G, \\ \frac{\partial v}{\partial n} + vf &= 0 \quad \text{on } \partial G, \end{aligned} \tag{37}$$

and $f \geq 0$, we obtain

$$0 = \int_G \nabla v \cdot \nabla v \, d\mathscr{H}_m + \int_{\partial G} vfv \, d\mathscr{H} \geq \int_G |\nabla v|^2 d\mathscr{H}_m \geq 0. \tag{38}$$

Therefore, $\nabla v = 0$ on $G$ and there is a constant $c$ such that $v(x) = c$ for $\mathscr{H}_m$-a.a. $x \in G$ by [16, Corollary 2.1.9]. Since $\nu \in \mathscr{C}_b'(\partial G)$, the function $\mathscr{U}\nu$ is bounded in $G$. Since $u(x) = \mathscr{U}(g\mathscr{H}_m)(x) + \mathscr{U}\nu(x) - c$ for $\mathscr{H}_m$-a.a. $x \in G$ and $\mathscr{U}(g\mathscr{H}_m) \in \mathscr{C}(\mathbb{R}^m)$ by Lemma 3, we obtain $u \in L_\infty(G)$. □

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