EXISTENCE OF ENTROPY SOLUTIONS FOR SOME NONLINEAR PROBLEMS IN ORLICZ SPACES

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We study in the framework of Orlicz Sobolev spaces $W_0^1 L_M(\Omega)$, the existence of entropic solutions to the nonlinear elliptic problems: $-\operatorname{div} a(x,u,\nabla u) + \operatorname{div}\phi(u) = f$ in Ω, for the case where the second member of the equation $f \in L^1(\Omega)$, and $\phi \in (C^0(\mathbb{R}))^N$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $A(u) = -\operatorname{div} a(x,u,\nabla u)$ be a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$, $1 < p < \infty$.

We consider the nonlinear elliptic problem

$$\begin{aligned} -\operatorname{div} a(x,u,\nabla u) &= f - \operatorname{div}\phi(u) \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$f \in L^1(\Omega), \qquad \phi \in \left(C^0(\mathbb{R})\right)^N. \tag{1.2}$$

Note that no growth hypothesis is assumed on the function ϕ, which implies that the term $\operatorname{div}\phi(u)$ may be meaningless, even as a distribution. The notion of entropy solution, used in [8], allows us to give a meaning to a possible solution of (1.1).

In fact Boccardo proved in [8], for p such that $2 - 1/N < p < N$, the existence and regularity of an entropy solution u of problem (1.1), that is,

$$\begin{aligned} &u \in W_0^{1,q}(\Omega), \qquad q < \tilde{p} = \frac{(p-1)N}{N-1}, \\ &T_k(u) \in W_0^{1,p}(\Omega), \quad \forall k > 0, \end{aligned}$$

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$$\int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi]\,dx \le \int_\Omega f\,T_k[u-\varphi]\,dx + \int_\Omega \phi(u)\nabla T_k[u-\varphi]\,dx \quad \forall \varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \tag{1.3}$$

where

$$T_k(s)=s \quad \text{if } |s|\le k \qquad T_k(s)=k\frac{s}{|s|} \quad \text{if } |s|>k. \tag{1.4}$$

For the case $\phi=0$ and f is a bounded measure, Bénilan et al. proved in [3] the existence and uniqueness of entropy solutions.

We mention as a parallel development, the work of Lions and Murat [14] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [10] for the study of the Boltzmann equations. They can prove existence and uniqueness of renormalized solution.

The functional setting in these works is that of the usual Sobolev space $W^{1,p}$. Accordingly, the function a is supposed to satisfy polynomial growth conditions with respect to u and its derivatives ∇u. When trying to generalize the growth condition on a, one is led to replace $W^{1,p}$ by a Sobolev space W^1L_M built from an Orlicz space L_M instead of L^p. Here the N-function M which defines L_M is related to the actual growth of the function a.

It is our purpose, in this paper, to prove the existence of entropy solution for problem (1.1) in the setting of the Orlicz Sobolev space $W_0^1L_M(\Omega)$. Our result, Theorem 3.5, generalizes [8, Theorem 2.1] and gives in particular a refinement of his result (see Remark 3.6).

For some existence results for strongly nonlinear elliptic equations in Orlicz spaces [4, 5, 6].

2. Preliminaries

2.1. Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-function, that is, M is continuous, convex, with $M(t)>0$ for $t>0$, $M(t)/t\to 0$ as $t\to 0$ and $M(t)/t\to\infty$ as $t\to\infty$.

Equivalently, M admits the representation $M(t)=\int_0^t a(\tau)\,d\tau$, where $a:\mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing, right continuous, with $a(0)=0$, $a(t)>0$ for $t>0$ and $a(t)\to\infty$ as $t\to\infty$.

In the following, we assume, for convenience, that all N-functions are twice continuously differentiable, see Benkirane and Gossez [7].

The N-function $\bar{M}$ conjugate to M is defined by $\bar{M}(t)=\int_0^t \bar{a}(\tau)\,d\tau$, where $\bar{a}:\mathbb{R}^+\to\mathbb{R}^+$ is given by $\bar{a}(t)=\sup\{s: a(s)\le t\}$, see [1, 13].

The N-function M is said to satisfy the Δ_2-condition (resp., near infinity) if for some k and for every $t\ge 0$,

$$M(2t)\le kM(t) \quad (\text{resp., for } t\ge \text{some } t_0). \tag{2.1}$$

Let M and P be two N-functions. The notation $P \ll M$ means that P grows essentially less rapidly than M, that is, for each $\epsilon > 0$, $P(t)/M(\epsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} M^{-1}(t)/P^{-1}(t) = 0$. We will extend all N-functions into even functions on all $\mathbb{R}$.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))\,dx < \infty \tag{2.2}$$

(resp., $\int_\Omega M(u(x)/\lambda)\,dx < \infty$ for some $\lambda > 0$). The space $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right) dx \le 1\right\} \tag{2.3}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinity measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\,dx$, and the dual norm on $L_{\bar{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\bar{M}}$. We say that u_n converges to u for the modular convergence in $L_M(\Omega)$ if for some $\lambda > 0$

$$\int_\Omega M\left(\frac{|u_n - u|}{\lambda}\right) dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

If M satisfies the Δ_2-condition, then the modular convergence coincide with the norm convergence.

2.3. The Orlicz Sobolev space $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order one lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M. \tag{2.5}$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\bar{M}})$ and $\sigma(\prod L_M, \prod L_{\bar{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the norm closure of $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$

$$\int_\Omega M\left(\frac{|D^\alpha u_n - D^\alpha u|}{\lambda}\right)dx \longrightarrow 0 \quad \forall |\alpha| \leq 1. \tag{2.6}$$

This implies the convergence $\sigma(\prod L_M, \prod L_{\bar{M}})$.

2.4. Let $W^{-1}L_{\bar{M}}(\Omega)$ (resp., $W^{-1}E_{\bar{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\bar{M}}(\Omega)$ (resp., $E_{\bar{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathfrak{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\bar{M}})$. Consequently, the action of a distribution in $W^{-1}L_{\bar{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

2.5. We recall the following lemmas.

Lemma 2.1 (see [5]). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P, and Q be N-functions such that $Q \ll P$, and let $f : \Omega\times\mathbb{R} \to \mathbb{R}^N$ be a Carathéodory function such that*

$$|f(x,s)| \leq c(x) + k_1P^{-1}M(k_2|s|) \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R}, \tag{2.7}$$

where $k_1, k_2 \in \mathbb{R}_+$, $c(x) \in E_Q(\Omega)$. Let N_f be the Nemytskii operator defined from $P(E_M(\Omega), 1/k_2) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < 1/k_2\}$ to $(E_Q(\Omega))^N$ by $N_f(u)(x) = f(x,u(x))$. Then N_f is strongly continuous.

Lemma 2.2 (see [5]). *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W_0^1L_M(\Omega)$ (resp., $W_0^1E_M(\Omega)$). Then $F(u) \in W_0^1L_M(\Omega)$ (resp., $W_0^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases} \tag{2.8}$$

Then $F : W_0^1L_M(\Omega) \to W_0^1L_M(\Omega)$ is sequentially continuous with respect to the weak topology $\sigma(\prod L_M, \prod E_{\bar{M}})$.*

Lemma 2.3 (see [11]). *Let Ω have the segment property. Then for each $v \in W_0^1L_M(\Omega)$, there exists a sequence $v_n \in \mathfrak{D}(\Omega)$ such that v_n converges to v for the modular convergence in $W_0^1L_M(\Omega)$. Furthermore, if $v \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$ then*

$$\|v_n\|_{L^\infty(\Omega)} \leq (N+1)\|v\|_{L^\infty(\Omega)}. \tag{2.9}$$

2.6. We introduce the following notation, see [2, 15].

Definition 2.4. Let M be an N-function, and define the following set:

$$\mathcal{A}_M = \Big\{ Q : Q \text{ is an } N\text{-function such that } \frac{Q''}{Q'} \leq \frac{M''}{M'}, \\ \int_0^1 Q \circ H^{-1}\left(\frac{1}{r^{1-1/N}}\right) dr < \infty \text{ where } H(r) = \frac{M(r)}{r} \Big\}. \tag{2.10}$$

Remark 2.5. Let $M(t) = t^p$ and $Q(t) = t^q$, then the condition $Q \in \mathcal{A}_M$ is equivalent to the following conditions:

(i) $2 - 1/N < p < N$

(ii) $q < \tilde{p} = (p-1)N/(N-1)$, see (1).

Remark 2.6. We can give some examples of N-functions M for which the set $\mathcal{A}_M$ is not empty. Here, the N-functions M are defined only at infinity.

(1) For $M(t) = t^2 \log t$ and $Q(t) = t \log t$, we have $H(t) = t \log t$ and $H^{-1}(t) = t(\log t)^{-1}$ at infinity, see [13]. Then the N-function Q belongs to $\mathcal{A}_M$.

(2) For $M(t) = t^2 \log^2 t$ at infinity and $Q(t) = t \log^2 t$, we have $H(t) = t \log^2 t$ and $H^{-1}(t) = t(\log t)^{-2}$ at infinity, see [13]. Then the N-function Q belongs to $\mathcal{A}_M$.

3. Definition and existence of entropy solutions

Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. Let M, P be two N-functions such that $P \ll M$.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\bar{M}}(\Omega)$ be a mapping (not defined everywhere) given by $A(u) = -\operatorname{div} a(x, u, \nabla u)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, $\xi, \bar{\xi}$ with $\xi \neq \bar{\xi}$,

$$|a(x,t,\xi)| \leq d(x) + k_1 \bar{P}^{-1} M(k_2 |t|) + k_3 \bar{M}^{-1} M(k_4 |\xi|), \tag{3.1}$$

$$[a(x,t,\xi) - a(x,t,\bar{\xi})][\xi - \bar{\xi}] > 0, \tag{3.2}$$

$$a(x,t,\xi)\xi \geq \alpha M\left(\frac{|\xi|}{\lambda}\right), \tag{3.3}$$

where $d(x) \in E_{\bar{M}}(\Omega)$, $d \geq 0$, $\alpha, \lambda \in \mathbb{R}_+^*$, $k_1, k_2, k_3, k_4 \in \mathbb{R}_+$.

Consider the nonlinear elliptic problem (1.1) where

$$f \in L^1(\Omega) \tag{3.4}$$

and $\phi = (\phi_1, \ldots, \phi_N)$ satisfies

$$\phi \in \left(C^0(\mathbb{R})\right)^N. \tag{3.5}$$

As in [8], we define the following notion of an entropy solution, which gives a meaning to a possible solution of (1.1).

Definition 3.1. Assume that (3.1), (3.2), (3.3), (3.4), and (3.5) hold true and $\mathcal{A}_M \neq \emptyset$. A function u is an entropy solution of problem (1.1) if

$$\begin{gathered} u \in W_0^1 L_Q(\Omega) \quad \forall Q \in \mathcal{A}_M, \\ T_k(u) \in W_0^1 L_M(\Omega) \quad \forall k > 0, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi]\,dx \le \int_\Omega f T_k[u-\varphi]\,dx + \int_\Omega \phi(u)\nabla T_k[u-\varphi]\,dx \\ \forall \varphi \in W_0^1 L_M(\Omega)\cap L^\infty(\Omega). \end{gathered} \tag{3.6}$$

We cannot use the solution u as a test function in (1.1), because u does not belong to $W_0^1 L_M(\Omega)$. An important role is played by $T_k(u)$ and the test functions

$$T_k[u-\varphi], \quad \varphi \in W_0^1 L_M(\Omega)\cap L^\infty(\Omega) \tag{3.7}$$

because both belong to $W_0^1 L_M(\Omega)$.

In Theorem 3.5, we prove the existence of solution of problem (1.1), in the framework of entropy solutions.

Lemma 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. If $u \in (W_0^1 L_M(\Omega))^N$ then $\int_\Omega \operatorname{div} u\,dx = 0$.*

Proof of Lemma 3.2. It is sufficient to use an approximation of u. □

We recall the following lemma (see [15, Lemma 2]).

Lemma 3.3. *Let M be an N-function, $u \in W^1 L_M(\Omega)$ such that $\int_\Omega M(|\nabla u|)\,dx < \infty$, then*

$$\begin{aligned} -\mu'(t) \ge\, & N C_N^{1/N} \mu^{1-1/N}(t) \\ & \times \mathrm{C}\left(\frac{-1}{N C_N^{1/N}\mu^{1-1/N}(t)}\frac{d}{dt}\int_{\{|u|>t\}} M(|\nabla u|)\,dx\right) \quad \forall t > 0, \end{aligned} \tag{3.8}$$

where C is the function defined as

$$\mathrm{C}(t) = \frac{1}{\sup\{r \ge 0, H(r) \le t\}}, \qquad H(r) = \frac{M(r)}{r}. \tag{3.9}$$

The function C_N is the measure of the unit ball of $\mathbb{R}^N$, and $\mu(t) = \operatorname{meas}\{|u| > t\}$.

Lemma 3.4. *Let (X,τ,μ) be a measurable set such that $\mu(X) < \infty$. Let γ be a measurable function $\gamma : X \to [0,\infty)$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0, \tag{3.10}$$

then for each $\epsilon > 0$, there exists $\delta > 0$ such that $\int_A \gamma(x)\,dx < \delta$ implies

$$\mu(A) \le \epsilon. \tag{3.11}$$

THEOREM 3.5. *Under assumptions (3.1), (3.2), (3.3), (3.4), and (3.5), with $\mathscr{A}_M \neq \emptyset$, there exists an entropy solution u of problem (1.1) (in the sense of Definition 3.1).*

Remark 3.6. In the case $M(t) = t^p$, Theorem 3.5 gives a refinement of the regularity result (1) (i.e., $u \in W_0^{1,q}(\Omega)$, $q < \tilde{p} = ((p-1)N/N-1))$. In fact, by Theorem 3.5, we have $u \in W_0^1 L_Q(\Omega)$ for each $Q \in \mathscr{A}_M$ (for example for $Q(t) = t^{\tilde{p}}/\log^{\alpha}(e+t), \alpha > 1$).

Proof of Theorem 3.5
Step 1. Define, for each $n > 0$, the approximations

$$\phi_n(s) = \phi(T_n(s)), \qquad f_n(s) = T_n[f(s)]. \tag{3.12}$$

Consider the nonlinear elliptic problem

$$u_n \in W_0^1 L_M(\Omega), \quad -\operatorname{div} a(x, u_n, \nabla u_n) = f_n - \operatorname{div} \phi_n(u_n) \quad \text{in } \Omega. \tag{3.13}$$

From Gossez and Mustonen [12, Proposition 1, Remark 2], problem (3.13) has at least one solution.
Step 2. We will prove that (u_n) is bounded in $W_0^1 L_Q(\Omega)$ for each $Q \in \mathscr{A}_M$. Let φ be the truncation defined, for each $t, h > 0$, by

$$\varphi(\xi) = \begin{cases} 0 & \text{if } 0 \le \xi \le t, \\ \dfrac{1}{h}(\xi - t) & \text{if } t < \xi < t+h, \\ 1 & \text{if } \xi \ge t+h, \\ -\varphi(-\xi) & \text{if } \xi < 0. \end{cases} \tag{3.14}$$

Using the test function $v = \varphi(u_n)$ in (3.13) ($v \in W_0^1 L_M(\Omega)$ by Lemma 2.2), we have

$$\int_\Omega a(x, u_n, \nabla u_n) \varphi'(u_n) \nabla u_n \, dx = \int_\Omega f_n \varphi(u_n) \, dx + \int_\Omega \phi_n(u_n) \nabla \varphi(u_n) \, dx. \tag{3.15}$$

We claim now that

$$\int_\Omega \phi_n(u_n) \nabla \varphi(u_n) \, dx = 0. \tag{3.16}$$

Indeed,

$$\nabla \varphi(u_n) = \varphi'(u_n) \nabla u_n, \tag{3.17}$$

where

$$\varphi'(\xi) = \begin{cases} \dfrac{1}{h} & \text{if } t < |\xi| < t+h, \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

define $\theta(s) = \phi_n(s)(1/h)\chi_{\{t<|s|<t+h\}}$, and $\tilde{\theta}(s) = \int_0^s \theta(\tau)\,d\tau$, we have by Lemma 2.2, $\tilde{\theta}(u_n) \in (W_0^1 L_M(\Omega))^N$, which implies

$$\begin{aligned}\int_\Omega \phi_n(u_n)\nabla\varphi(u_n)\,dx &= \int_\Omega \phi_n(u_n)\frac{1}{h}\chi_{\{t<|u_n|<t+h\}}\nabla u_n\,dx = \int_\Omega \theta(u_n)\nabla u_n\,dx \\ &= \int_\Omega \operatorname{div}(\tilde{\theta}(u_n))\,dx = 0 \quad \text{(see Lemma 3.2)}.\end{aligned} \tag{3.19}$$

This proves (3.16). By (3.3) and (3.15), we have (where we can suppose without loss of generality that $\lambda = 1$, since one can take $u_n' = u_n/\lambda$)

$$\frac{\alpha}{h}\int_{t<|u_n|<t+h} M(|\nabla u_n|)\,dx \le \|f\|_{1,\Omega}. \tag{3.20}$$

Let $h \to 0$, then

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx \le C \quad \text{with } C = \frac{\|f\|_{1,\Omega}}{\alpha}. \tag{3.21}$$

We prove the following inequality, which allows us to obtain the boundedness of (u_n) in $W_0^1 L_Q(\Omega)$,

$$\begin{aligned}&-\frac{d}{dt}\int_{|u_n|>t} Q(|\nabla u_n|\,dx) \\ &\le -\mu_n'(t) Q\circ H^{-1}\left(-\frac{1}{NC_N^{1/N}\mu_n(t)^{1-1/N}}\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx\right).\end{aligned} \tag{3.22}$$

Indeed, let $C(s) = 1/H^{-1}(s)$, where $H(r) = M(r)/r$ and $H^{-1}(s) = \sup\{r \ge 0, H(r) \le s\}$. Then

$$C(s) = \frac{s}{M\circ H^{-1}(s)}. \tag{3.23}$$

By Lemma 3.3 we have, with $\mu_n(t) = \operatorname{meas}\{|u_n| > t\}$,

$$\begin{aligned}-\mu_n'(t) &\ge NC_N^{1/N}\mu_n(t)^{1-1/N} \\ &\quad\times C\left(-\frac{1}{NC_N^{1/N}}\mu_n(t)^{1-1/N}\frac{d}{dt}\int_{|u_n|>t} M(|\nabla u_n|)\,dx\right),\end{aligned} \tag{3.24}$$

then

$$\begin{aligned}&-\mu_n'(t)\cdot M\circ H^{-1}\left(-\frac{1}{NC_N^{1/N}\mu_n(t)^{1-1/N}}\frac{d}{dt}\int_{|u_n|>t} M(|\nabla u_n|)\,dx\right) \\ &\quad\ge NC_N^{1/N}\mu_n(t)^{1-1/N}\left(-\frac{1}{NC_N^{1/N}\mu_n(t)^{1-1/N}}\frac{d}{dt}\int_{|u_n|>t} M(|\nabla u_n|)\,dx\right),\end{aligned} \tag{3.25}$$

and also

$$\begin{aligned}&\frac{1}{\mu_n'(t)}\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx\\&\qquad\le M\circ H^{-1}\left(-\frac{1}{NC_N^{1/N}\mu_n(t)^{1-1/N}}\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx\right)\end{aligned}\tag{3.26}$$

which gives

$$\begin{aligned}&M^{-1}\left(\frac{1}{\mu_n'(t)}\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx\right)\\&\qquad\le H^{-1}\left(-\frac{1}{NC_N^{1/N}\mu_n(t)^{1-1/N}}\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx\right).\end{aligned}\tag{3.27}$$

Let $Q \in \mathcal{A}_M$ and let $D(s) = M(Q^{-1}(s))$, D is then convex, and the Jensen's inequality gives

$$D\left(\frac{\int_{\{t<|u_n|<t+h\}} Q(|\nabla u_n|)\,dx}{-\mu_n(t+h)+\mu_n(t)}\right)\le\frac{\int_{\{t<|u_n|<t+h\}} M(|\nabla u_n|)\,dx}{-\mu_n(t+h)+\mu_n(t)},\tag{3.28}$$

then

$$\begin{aligned}&Q^{-1}\left(\frac{1}{\mu_n'(t)}\frac{d}{dt}\int_{\{|u_n|>t\}} Q(|\nabla u_n|)\,dx\right)\\&\qquad\le M^{-1}\left(\frac{1}{\mu_n'(t)}\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx\right)\\&\qquad\le H^{-1}\left(-\frac{1}{NC_N^{1/N}\mu_n(t)^{1-1/N}}\frac{d}{dt}\int_{\{|u_n|>t\}} M(|\nabla u_n|)\,dx\right)\end{aligned}\tag{3.29}$$

which gives (3.22). By (3.21) and (3.22) and since the function

$$t\longrightarrow\int_{\{|u_n|>t\}} Q(|\nabla u_n|)\,dx\tag{3.30}$$

is absolutely continuous (see [15]), we have

$$\begin{aligned}\int_\Omega Q(|\nabla u_n|)\,dx&=\int_0^\infty\left(-\frac{d}{dt}\int_{\{|u_n|>t\}} Q(|\nabla u_n|)\right)dt\\&\le\int_0^\infty -\mu_n'(t)Q\circ H^{-1}\left(\frac{C}{NC_N^{1/N}\mu_n(t)^{1-1/N}}\right)dt\\&\le\frac{1}{C'}\int_0^{C'\cdot\mathrm{meas}(\Omega)} Q\circ H^{-1}\left(\frac{1}{r^{1-1/N}}\right)dr<\infty\end{aligned}\tag{3.31}$$

which implies that (∇u_n) is bounded in $L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$. Then u_n is bounded in $W_0^1L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$. Passing to a subsequence if necessary, we can assume that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^1L_Q(\Omega) \text{ for } \sigma\Big(\prod L_Q, \prod E_{\bar{Q}}\Big), \text{ a.e. in } \Omega. \tag{3.32}$$

Step 3. We prove that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^1L_M(\Omega)$ for all $k > 0$. Using the test function $T_k(u_n)$ in (3.13), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx = \int_\Omega f_n T_k(u_n)\,dx + \int_\Omega \phi_n(u_n)\nabla T_k(u_n)\,dx. \tag{3.33}$$

We claim that

$$\int_\Omega \phi_n(u_n)\nabla T_k(u_n)\,dx = 0. \tag{3.34}$$

Indeed, $\nabla T_k(u_n) = \nabla u_n \chi_{\{|u_n|\le k\}}$, define $\theta(t) = \phi_n(t)\chi_{\{|t|\le k\}}$, and $\tilde{\theta}(t) = \int_0^t \theta(\tau)\,d\tau$, we have by Lemma 2.2, $\tilde{\theta}(u_n) \in (W_0^1L_M(\Omega))^N$,

$$\begin{aligned}\int_\Omega \phi_n(u_n)\nabla T_k(u_n)\,dx &= \int_\Omega \phi_n(u_n)\chi_{\{|u_n|\le k\}}\nabla u_n\,dx\\ &= \int_\Omega \theta(u_n)\nabla u_n\,dx\\ &= \int_\Omega \operatorname{div}\big(\tilde{\theta}(u_n)\big)\,dx = 0 \qquad (\text{by Lemma 3.2})\end{aligned} \tag{3.35}$$

which proves the claim.

On the other hand, (3.33) can be written as

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx &= \int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)\,dx\\ &= \int_\Omega f_n T_k(u_n)\,dx,\end{aligned} \tag{3.36}$$

which implies, with (3.3), that $\nabla T_k(u_n)$ is bounded in $(L_M(\Omega))^N$, and $T_k(u_n)$ is bounded in $(W_0^1L_M(\Omega))^N$. Since $u_n \to u$ a.e. in Ω then $T_k(u_n) \to T_k(u)$ a.e. in Ω. Then

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^1L_M(\Omega) \text{ for } \sigma\Big(\prod L_M, \prod E_{\bar{M}}\Big). \tag{3.37}$$

Step 4. We will prove that $\nabla u_n \to \nabla u$ a.e. in Ω. Let $\lambda > 0$, $\epsilon > 0$. For $B > 1$, $k > 0$, we consider as in [9] for $n, m \in \mathbb{N}$,

$$\begin{aligned} E_1 &= \{|\nabla u_n| > B\} \cup \{|\nabla u_m| > B\} \cup \{|u_n| > B\} \cup \{|u_m| > B\},\\ E_2 &= \{|u_n - u_m| > k\},\\ E_3 &= \{|u_n - u_m| \le k, |u_n| \le B, |u_m| \le B, |\nabla u_n| \le B,\\ &\qquad |\nabla u_m| \le B, |\nabla u_n - \nabla u_m| \ge \lambda\},\end{aligned} \tag{3.38}$$

we have $\{|\nabla u_n - \nabla u_m| \ge \lambda\} \subset E_1 \cup E_2 \cup E_3$.

Since (u_n) and (∇u_n) are bounded in $L^1(\Omega)$ (since u_n is bounded in $W_0^1 L_Q(\Omega)$), we have

$$2B\mu(E_1) < \int_{E_1} |\nabla u_n| + |u_n|\, dx < \int_{\Omega} |\nabla u_n| + |u_n|\, dx < C. \tag{3.39}$$

Then meas $E_1 \le \epsilon$ for B sufficiently large enough, independently of n, m. Thus we fix B in order to have

$$\operatorname{meas} E_1 \le \epsilon. \tag{3.40}$$

Now we claim that meas $E_3 \le \epsilon$ for n and m large. Let C_1 be such that $\|u_n\|_1 \le C_1$ and $\|\nabla u_n\|_1 \le C_1$. As in [9], the assumption (3.2) gives the existence of a measurable function $\gamma(x)$ such that

$$\begin{gathered} \operatorname{meas}\left(\{x \in \Omega : \gamma(x) = 0\}\right) = 0, \\ [a(x,t,\xi) - a(x,t,\bar{\xi})]\,[\xi - \bar{\xi}] \ge \gamma(x) > 0, \end{gathered} \tag{3.41}$$

for all $t \in \mathbb{R}$, $\xi, \bar{\xi} \in \mathbb{R}^N$, $|t|, |\xi|, |\bar{\xi}| \le B$, $|\xi - \bar{\xi}| \ge \lambda$ a.e. in Ω. We have

$$\begin{aligned} \int_{E_3} \gamma(x)\, dx &\le \int_{E_3} [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u_m)]\,[\nabla u_n - \nabla u_m]\, dx \\ &\le \int_{E_3} [a(x,u_m,\nabla u_m) - a(x,u_n,\nabla u_m)]\,[\nabla u_n - \nabla u_m]\, dx \\ &\quad + \int_{E_3} [a(x,u_n,\nabla u_n) - a(x,u_m,\nabla u_m)]\,[\nabla u_n - \nabla u_m]\, dx. \end{aligned} \tag{3.42}$$

Using the test function $T_k(u_n - u_m)$ in (3.13) and integrating on E_3, we obtain

$$\begin{aligned} &\int_{E_3} [a(x,u_n,\nabla u_n) - a(x,u_m,\nabla u_m)]\,\nabla T_k(u_n - u_m)\, dx \\ &\quad = \int_{E_3} (f_n - f_m)\, T_k(u_n - u_m)\, dx \\ &\qquad + \int_{E_3} [\phi_n(u_n) - \phi_m(u_m)]\,\nabla T_k(u_n - u_m)\, dx, \end{aligned} \tag{3.43}$$

with

$$\begin{aligned} &\int_{E_3} [\phi_n(u_n) - \phi_m(u_m)]\,\nabla T_k(u_n - u_m)\, dx \\ &\quad \le 2B \int_{E_3} |\phi_n(u_n) - \phi_m(u_m)|\, dx \\ &\quad \le 2B \int_{E_3} [|\phi(T_n(u_n)) - \phi(u_n)| + |\phi(u_n) - \phi(u_m)| \\ &\qquad\qquad + |\phi(u_m) - \phi(T_m(u_m))|]\, dx. \end{aligned} \tag{3.44}$$

Let $n_0 \geq B$, then for $n, m \geq n_0$ we have $T_n(u_n) = u_n$ and $T_m(u_m) = u_m$ on E_3, which implies that the first and the third integral of the last inequality vanish. The second integral of (3.42) is bounded for $n, m \geq n_0$ by

$$2k\|f\|_{1,\Omega} + 2B\int_{E_3} \left|\phi(u_n) - \phi(u_m)\right| dx. \tag{3.45}$$

For a.e. $x \in \Omega$ and $\epsilon_1 > 0$ there exist $\eta(x) \geq 0$ (meas$\{x \in \Omega : \eta(x) = 0\} = 0$) such that $|s - s'| \leq \eta(x)$, $|s|, |s'|, |\xi| \leq B$ implies

$$\left|a(x,s,\xi) - a(x,s',\xi)\right| \leq \epsilon_1. \tag{3.46}$$

We use now the continuity of ϕ, to obtain for a.e. $x \in \Omega$ and $\epsilon_2 > 0$, $\eta_1(x) \geq 0$ (meas$\{x \in \Omega : \eta_1(x) = 0\} = 0$) such that $|s - s'| \leq \eta_1(x)$, $|s|, |s'| \leq B$ implies

$$\left|\phi(s) - \phi(s')\right| \leq \epsilon_2. \tag{3.47}$$

Then

$$\begin{aligned}
\int_{E_3} \gamma(x)\, dx \leq \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} & \left[a(x,u_m,\nabla u_m) - a(x,u_n,\nabla u_m)\right] \\
& \times \left[\nabla u_n - \nabla u_m\right] dx \\
+ \int_{E_3 \cap \{x \in \Omega : \eta(x) \geq k\}} & \left[a(x,u_m,\nabla u_m) - a(x,u_n,\nabla u_m)\right] \\
& \times \left[\nabla u_n - \nabla u_m\right] dx \\
+ 2k\|f\|_{1,\Omega} + 2B\int_{E_3 \cap \{x \in \Omega : \eta_1(x) < k\}} & \left|\phi(u_n) - \phi(u_m)\right| dx \\
+ \int_{E_3 \cap \{x \in \Omega : \eta_1(x) \geq k\}} & \left|\phi(u_n) - \phi(u_m)\right| dx.
\end{aligned} \tag{3.48}$$

By using for the first integral the definition of E_3 and condition (3.1), for the second integral the definition of E_3 and (3.46), for the fourth integral the definition of E_3 and $|\phi(u_n)| \leq C(B)$ (since $|u_n| \leq B$ and ϕ continuous), and for the last integral the definition of E_3 and (3.47), we obtain

$$\begin{aligned}
\int_{E_3} \gamma(x)\, dx \leq C'(B) \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} & \left[1 + d(x)\right] dx + 2C_1(B)\epsilon_1 \\
& + 2k\|f\|_{1,\Omega} + 2C(B)\,\text{meas}\left\{x \in \Omega : \eta_1(x) < k\right\} + C_2\epsilon_2.
\end{aligned} \tag{3.49}$$

We have meas$\{x \in \Omega : \eta(x) < k\} \to 0$ when $k \to 0$, and meas$\{x \in \Omega : \eta_1(x) < k\} \to 0$ when $k \to 0$. Let $\epsilon > 0$ and let δ be the real, in Lemma 3.4, corresponding to ϵ, we choose ϵ_1, ϵ_2 such that

$$2C_1(B)\epsilon_1 \leq \frac{\delta}{5}, \qquad C_2\epsilon_2 \leq \frac{\delta}{5}, \tag{3.50}$$

and k such that

$$C'(B)\int_{E_3\cap\{x\in\Omega:\eta(x)<k\}}\left[1+d(x)\right]dx<\frac{\delta}{5},\qquad 2k\|f\|_{1,\Omega}\leq\frac{\delta}{5},$$
$$2C(B)\operatorname{meas}\{x\in\Omega:\eta_1(x)<k\}<\frac{\delta}{5}. \tag{3.51}$$

Then $\int_{E_3}\gamma(x)\,dx<\delta$ and Lemma 3.4 implies that

$$\operatorname{meas}E_3<\epsilon\quad\forall n,m\geq n_0. \tag{3.52}$$

This completes the proof of the claim.

Let the last k be fixed, u_n a Cauchy sequence in measure, we choose n_1 such that

$$\operatorname{meas}E_2\leq\epsilon\quad\forall n,m\geq n_1. \tag{3.53}$$

Then

$$\operatorname{meas}\{x\in\Omega:|\nabla u_n-\nabla u_m|\geq\lambda\}\leq\epsilon\quad\forall n,m\geq\max(n_1,n_0) \tag{3.54}$$

and $\nabla u_n\to\nabla u$ in measure, consequently

$$\nabla u_n\longrightarrow\nabla u\quad\text{a.e. in }\Omega. \tag{3.55}$$

Step 5. Let $\varphi\in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$. From Lemma 2.3, there exists a sequence $(\varphi_j)\in\mathfrak{D}(\Omega)$ such that φ_j converges to φ for the modular convergence in $W_0^1L_M(\Omega)$ with

$$\|\varphi_j\|_{L^\infty(\Omega)}\leq(N+1)\|\varphi\|_{L^\infty(\Omega)}. \tag{3.56}$$

Using $T_k[u_n-\varphi_j]$ as a test function in (3.13) we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k[u_n-\varphi_j]\,dx\\&\quad=\int_\Omega f_nT_k[u_n-\varphi_j]\,dx+\int_\Omega\phi_n(u_n)\nabla T_k[u_n-\varphi_j]\,dx\end{aligned} \tag{3.57}$$

which gives, if $n\to\infty$,

$$\begin{aligned}&\liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k[u_n-\varphi_j]\,dx\\&\quad\geq\liminf_{n\to\infty}\int_\Omega\left[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla\varphi_j)\right]\nabla T_k[u_n-\varphi_j]\,dx\\&\qquad+\lim_{n\to\infty}\int_\Omega a(x,T_{k+\|\varphi_j\|_{L^\infty(\Omega)}}(u_n),\nabla\varphi_j)\nabla T_k[u_n-\varphi_j]\,dx\\&\quad\geq\int_\Omega\left[a(x,u,\nabla u)-a(x,u,\nabla\varphi_j)\right]\nabla T_k[u-\varphi_j]\,dx\\&\qquad+\int_\Omega a(x,u,\nabla\varphi_j)\nabla T_k[u-\varphi_j]\,dx,\end{aligned} \tag{3.58}$$

where we have used Fatou lemma for the first integral, and for the second the convergences $\nabla T_k[u_n-\varphi_j] \to \nabla T_k[u-\varphi_j]$ by (3.37) in $(L_M(\Omega))^N$ for $\sigma(\prod L_M, \prod E_{\bar{M}})$ and $a(x, T_{k+\|\varphi_j\|_{L^\infty(\Omega)}}(u_n), \nabla\varphi_j) \to a(x, T_{k+\|\varphi_j\|_{L^\infty(\Omega)}}(u), \nabla\varphi_j)$ strongly in $(E_{\bar{M}}(\Omega))^N$ by (3.1), which implies that

$$\liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k[u_n-\varphi_j]\,dx \geq \int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi_j]\,dx. \quad (3.59)$$

For $n \geq k+(N+1)\|\varphi\|_{L^\infty(\Omega)}$,

$$\begin{aligned}\int_\Omega \phi_n(u_n)\nabla T_k[u_n-\varphi_j]\,dx &= \int_\Omega \phi\big(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u_n)\big)\nabla T_k[u_n-\varphi_j]\,dx \\ &\underset{n\to\infty}{\longrightarrow} \int_\Omega \phi\big(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)\big)\nabla T_k[u-\varphi_j]\,dx,\end{aligned} \quad (3.60)$$

we have used the convergences $\nabla T_k[u_n-\varphi_j] \rightharpoonup \nabla T_k[u-\varphi_j]$ by (3.37) in $(L_M(\Omega))^N$ and $\phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u_n)) \to \phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u))$ strongly in $(E_{\bar{M}}(\Omega))^N$ since ϕ is continuous. On the other hand, since $f_n \to f$ strongly in $L^1(\Omega)$ and $T_k[u_n-\varphi_j] \rightharpoonup T_k[u-\varphi_j]$ weakly* in $L^\infty(\Omega)$, we have

$$\int_\Omega f_n T_k[u_n-\varphi_j]\,dx \longrightarrow \int_\Omega f\,T_k[u-\varphi_j]\,dx. \quad (3.61)$$

Then

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi_j]\,dx \geq{}& \int_\Omega \phi\big(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)\big)\nabla T_k[u-\varphi_j]\,dx \\ &+\int_\Omega f\,T_k[u-\varphi_j]\,dx.\end{aligned} \quad (3.62)$$

Now, if $j\to\infty$ in (3.62), we get

$$\begin{aligned}&\liminf_{j\to\infty}\int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi_j]\,dx \\ &\quad\geq \liminf_{j\to\infty}\int_\Omega \big[a(x,u,\nabla u)-a(x,u,\nabla\varphi_j)\big]\nabla T_k[u-\varphi_j]\,dx \\ &\qquad+\lim_{j\to\infty}\int_\Omega a(x,u,\nabla\varphi_j)\nabla T_k[u-\varphi_j]\,dx \\ &\quad\geq \int_\Omega \big[a(x,u,\nabla u)-a(x,u,\nabla\varphi)\big]\nabla T_k[u-\varphi]\,dx \\ &\qquad+\int_\Omega a(x,u,\nabla\varphi)\nabla T_k[u-\varphi]\,dx,\end{aligned} \quad (3.63)$$

where we have used Fatou lemma for the first integral, and for the second the convergences $\nabla T_k[u-\varphi_j] \rightharpoonup \nabla T_k[u-\varphi]$ in $(L_M(\Omega))^N$ for the modular convergence and $a(x,u,\nabla\varphi_j) \to a(x,u,\nabla\varphi)$ in $(L_{\bar{M}}(\Omega))^N$ for the modular convergence, which implies that

$$\liminf_{j\to\infty}\int_\Omega a(x,u,\nabla u)\nabla T_k\left[u-\varphi_j\right]dx \geq \int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi]\,dx. \tag{3.64}$$

On the other hand, since $\nabla T_k[u-\varphi_j] \to \nabla T_k[u-\varphi]$ in $(L_M(\Omega))^N$ for the modular convergence, then weakly for $\sigma(\prod L_M, \prod L_{\bar{M}})$ and $\phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)) \in (L_{\bar{M}}(\Omega))^N$ we have

$$\begin{aligned}
&\int_\Omega \phi\left(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)\right)\nabla T_k\left[u-\varphi_j\right]dx \\
&\quad \underset{j\to\infty}{\longrightarrow} \int_\Omega \phi\left(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)\right)\nabla T_k[u-\varphi]\,dx \\
&\quad = \int_\Omega \phi(u)\nabla T_k[u-\varphi]\,dx.
\end{aligned} \tag{3.65}$$

Since $f \in L^1(\Omega)$ and $T_k[u-\varphi_j] \rightharpoonup T_k[u-\varphi]$ weakly* in $L^\infty(\Omega)$, we have

$$\int_\Omega f\,T_k\left[u-\varphi_j\right]dx \longrightarrow \int_\Omega f\,T_k[u-\varphi]\,dx. \tag{3.66}$$

Then

$$\int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi]\,dx \geq \int_\Omega \phi(u)\nabla T_k[u-\varphi]\,dx + \int_\Omega f\,T_k[u-\varphi]\,dx \tag{3.67}$$

and u is an entropy solution of problem (1.1). □

Theorem 3.7. *Suppose, in Theorem 3.5, that the N-function M satisfies, furthermore, the Δ_2-condition and $f \geq 0$, then the entropy solution u of problem (1.1) satisfies $u \geq 0$.*

Proof of Theorem 3.7. Using $\varphi = T_l(u^+)$ as test function in the definition of entropy solution, we obtain

$$\begin{aligned}
&\int_\Omega a(x,u,\nabla u)\nabla T_k\left[u-T_l\left(u^+\right)\right]dx \\
&\quad \leq \int_\Omega f\,T_k\left[u-T_l\left(u^+\right)\right]dx + \int_\Omega \phi(u)\nabla T_k\left[u-T_l\left(u^+\right)\right]dx.
\end{aligned} \tag{3.68}$$

We have

$$\int_\Omega f\,T_k\left[u-T_l\left(u^+\right)\right]dx \leq \int_{\{u\geq l\}} f\,T_k\left[u-T_l(u)\right]dx. \tag{3.69}$$

Indeed,

$$\begin{aligned}\int_\Omega f T_k\left[u-T_l(u^+)\right] dx = \int_{u\geq l} f T_k\left[u-T_l(u^+)\right] dx \\ + \int_{0<u<l} f T_k\left[u-T_l(u^+)\right] dx \\ + \int_{u\leq 0} f T_k\left[u-T_l(u^+)\right] dx.\end{aligned} \tag{3.70}$$

If $0 < u < l$ then $u - T_l(u^+) = 0$ and $\int_{0<u<l} f T_k[u - T_l(u^+)]\, dx = 0$. If $u \leq 0$ then $u - T_l(u^+) = u$ and $\int_{u\leq 0} f T_k[u - T_l(u^+)]\, dx \leq 0$ since f is positive. If $u \geq l$ then $u^+ = u$ and $\int_{u\geq l} f T_k[u - T_l(u^+)]\, dx \leq \int_{u\geq l} f T_k[u - T_l(u)]\, dx$.

On the other hand, we claim that

$$\int_\Omega \phi(u)\nabla T_k\left[u-T_l(u^+)\right] dx = 0. \tag{3.71}$$

Indeed, if $0 < u < l$, then $u - T_l(u^+) = 0$, $\int_{0<u<l} \phi(u)\nabla T_k[u - T_l(u^+)]\, dx = 0$. If $u \leq 0$, then $u - T_l(u^+) = u$,

$$\begin{aligned}\int_{u\leq 0} \phi(u)\nabla T_k\left[u-T_l(u^+)\right] dx &= \int_{-k\leq u\leq 0} \phi(u)\nabla u\, dx \\ &= \int_\Omega \phi(u)\nabla u \chi_{\{-k\leq u\leq 0\}}\, dx.\end{aligned} \tag{3.72}$$

We verify that the third integral of the last inequality vanishes. For this, define $\theta(t) = \phi(t)\chi_{\{-k\leq t\leq 0\}}$, and $\tilde{\theta}(t) = \int_0^t \theta(\tau)\, d\tau$ we have, by Lemma 2.2, $\tilde{\theta}(u) \in (W_0^1 L_M(\Omega))^N$ which implies

$$\begin{aligned}\int_\Omega \phi(u)\nabla u \chi_{\{-k\leq u\leq 0\}}\, dx &= \int_\Omega \theta(u)\nabla u\, dx \\ &= \int_\Omega \operatorname{div}\left(\tilde{\theta}(u)\right) dx = 0 \quad (\text{by Lemma 3.2}).\end{aligned} \tag{3.73}$$

If $u \geq l$ then $u^+ = u$ and

$$\begin{aligned}\int_{\{u\geq l\}} \phi(u)\nabla T_k\left[u-T_l(u^+)\right] dx &= \int_{l\leq u\leq l+k} \phi(u)\nabla u\, dx \\ &= \int_\Omega \phi(u)\nabla u \chi_{\{l\leq u\leq l+k\}}\, dx.\end{aligned} \tag{3.74}$$

Similarly, we verify that

$$\int_\Omega \phi(u)\nabla u \chi_{\{l\leq u\leq l+k\}}\, dx = 0. \tag{3.75}$$

This completes the proof of the claim which implies that

$$\int_{\Omega} a(x,u,\nabla u)\nabla T_k\big[u - T_l(u^+)\big]\,dx \le \int_{u\ge l} f\,T_k\big[u - T_l(u)\big]\,dx \tag{3.76}$$

or

$$\begin{aligned}
&\int_{\Omega} a(x,u,\nabla u)\nabla T_k\big[u - T_l(u^+)\big]\,dx\\
&\qquad = \int_{l\le u\le l+k} a(x,u,\nabla u)\nabla u\,dx + \int_{-k\le u\le 0} a(x,u,\nabla u)\,\nabla u\,dx\\
&\qquad \ge \int_{l\le u\le l+k} M\left(\frac{|\nabla u|}{\lambda}\right)dx + \int_{-k\le u\le 0} M\left(\frac{|\nabla u|}{\lambda}\right)dx,
\end{aligned} \tag{3.77}$$

which gives

$$\int_{l\le u\le l+k} M\left(\frac{|\nabla u|}{\lambda}\right)dx + \int_{-k\le u\le 0} M\left(\frac{|\nabla u|}{\lambda}\right)dx \le \int_{u\ge l} f\,T_k\big[u - T_l(u)\big]\,dx. \tag{3.78}$$

Letting $l \longrightarrow \infty$ in (3.78) we have

$$\begin{aligned}
&\int_{u\ge l} f\,T_k\big[u - T_l(u)\big]\,dx \longrightarrow 0 \quad \text{since } f\,T_k[2u] \in L^1(\Omega),\\
&\int_{l\le u\le l+k} M\left(\frac{|\nabla u|}{\lambda}\right)dx \ge \int_{l\le u\le k} M\left(\frac{|\nabla u|}{\lambda}\right)dx\\
&\qquad\qquad = \int_{l\le u} M\left(\frac{|\nabla T_k(u)|}{\lambda}\right)dx\\
&\qquad\qquad \longrightarrow 0, \quad \text{when } l \longrightarrow \infty,
\end{aligned} \tag{3.79}$$

since $M(|\nabla T_k(u)|/\lambda) \in L^1(\Omega)$ and M satisfies the Δ_2-condition. Then

$$\int_{-k\le u\le 0} M\left(\frac{|\nabla u|}{\lambda}\right)dx = 0 \quad \forall k, \tag{3.80}$$

which implies that,

$$\begin{aligned}
&\int_{u\le 0} M\left(\frac{|\nabla u|}{\lambda}\right)dx = \int_{\Omega} M\left(\frac{|\nabla u^-|}{\lambda}\right)dx = 0,\\
&\nabla u^- = 0, \quad u^- = c \quad \text{a.e. in } \Omega.
\end{aligned} \tag{3.81}$$

Or $u^- \in W_0^1 L_Q(\Omega)$ then $u^- = 0$ a.e. in Ω which proves that

$$u \ge 0 \quad \text{a.e. in } \Omega. \tag{3.82}$$

□

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