ON THE PROJECTION CONSTANTS OF SOME TOPOLOGICAL SPACES AND SOME APPLICATIONS

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We find a lower estimation for the projection constant of the projective tensor product $X \otimes^{\wedge} Y$ and the injective tensor product $X \otimes^{\vee} Y$, we apply this estimation on some previous results, and we also introduce a new concept of the projection constants of operators rather than that defined for Banach spaces.

1. Introduction

If Y is a closed subspace of a Banach space X, then the relative projection constant of Y in X is defined by

$$\lambda(Y, X) := \inf \{ \|P\| : P \text{ is a linear projection from } X \text{ onto } Y \}. \tag{1.1}$$

And the absolute projection constant of Y is defined by

$$\lambda(Y) := \sup \{ \lambda(Y, X) : X \text{ contains } Y \text{ as a closed subspace} \}. \tag{1.2}$$

It is well known that any Banach space Y can be isometrically embedded into $l_\infty(\Gamma)$ for some index set Γ (Γ is usually taken to be U_{Y^*} where Y^* denotes the dual space of Y and U_{Y^*} denotes the set $\{f : f \in Y^*, \ \|f\| \le 1\}$) and that if Y is complemented in $l_\infty(\Gamma)$, then it is complemented in every Banach space containing it as a closed subspace, that is, Y is injective. We also know that for any such embedding the supremum in (1.2) is attained, that is, $\lambda(Y) = \lambda(Y, \ l_\infty(\Gamma))$ (see [1, 4]). For each finite-dimensional space Y_n with $\dim Y_n = n$, Kadets and Snobar [6] proved that $\lambda(Y_n) \le \sqrt{n}$. König [7] showed that for each prime number n the space $l^\infty_{n^2}$ contains an n-dimensional subspace Y_n with projection constant

$$\lambda(Y_n) = \sqrt{n} - \left(\frac{1}{\sqrt{n}} - \frac{1}{n} \right). \tag{1.3}$$

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König and Lewis [9] verified the strict inequality $\lambda(Y_n) < \sqrt{n}$ in case $n \geq 2$. Lewis [14] showed that

$$\lambda(Y_n) \leq \sqrt{n}\left[1 - n^{-2}\left(\frac{1}{5}\right)^{2n+11}\right]. \tag{1.4}$$

König and Tomczak-Jaegermann [11] also showed that there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of Banach spaces X_n with $\dim X_n = n$ such that

$$\lim_{n\to\infty} \frac{\lambda(X_n)}{\sqrt{n}} = 1. \tag{1.5}$$

In fact, it is shown in [9] that for each Banach space Y_n with dimension n, $\lambda(Y_n) \leq \sqrt{n} - c/\sqrt{n}$, where $c > 0$ is a numerical constant and the n-dimensional spaces X_n satisfy $\sqrt{n} - 2/\sqrt{n} \leq \lambda(X_n)$. The improvement of these results was given in [12], where an upper estimate for $\lambda(Y_n)$ was found in the form

$$\lambda(Y_n) \leq \begin{cases} \sqrt{n} - \dfrac{1}{\sqrt{n}} + O\left(n^{-3/4}\right), & \text{in the real field,} \\[2ex] \sqrt{n} - \dfrac{1}{2\sqrt{n}} + O\left(n^{-3/4}\right), & \text{in the complex field.} \end{cases} \tag{1.6}$$

The precise values of l_n^1, l_n^2, and l_n^p, $1 < p < \infty$, $p \neq 2$, have been calculated by Grünbaum [4], Rutovitz [15], Gordon [3], and Garling and Gordon [2]. In the case of $1 < p < 2$, the improvement of these results was given by König, Schütt, and Tomczak-Jaegermann in [10], they showed that

$$\lim_{n\to\infty} \frac{\lambda(l_n^p)}{\sqrt{n}} = \begin{cases} \sqrt{\dfrac{2}{\pi}}, & \text{in the real field,} \\[2ex] \dfrac{\sqrt{\pi}}{2}, & \text{in the complex field.} \end{cases} \tag{1.7}$$

Some other results are mentioned in [2, 3, 13, 15].

For finite codimensional subspaces, Garling and Gordon [2] showed that if Y is a finite codimensional subspace of the Banach space X with codimension n, then for every $\epsilon > 0$ there exists a projection P from X onto Y with norm

$$\|P\| \leq 1 + (1+\epsilon)\sqrt{n}. \tag{1.8}$$

2. Notations and basic definitions

The sets X, Y, Z, and E denote Banach spaces, X^* denotes the conjugate space of X and U_X denotes the unit ball of the space X. Elements of X, Y, X^*, and Y^* will be denoted by $x, u, \ldots, y, v, \ldots, f, h, \ldots$, and $g, k, \ldots$, respectively. The injective tensor product $X \otimes^{\vee} Y$ between the normed spaces X and Y is defined as the completion of the smallest cross norm on the space $X \otimes Y$ and the norm on the space $X \otimes Y$ is defined by

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_{X \otimes^{\vee} Y} = \sup \left| \sum_{i=1}^{n} f\big(x_i\big) g\big(y_i\big) \right|, \tag{2.1}$$

where the supremum is taken over all functionals $f \in U_{X^*}$ and $g \in U_{Y^*}$.

The projective tensor product $X \otimes^{\wedge} Y$ between the normed spaces X and Y is defined as the completion of the largest cross norm on the space $X \otimes Y$ and the norm on $X \otimes Y$ is defined by

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_{X \otimes^{\wedge} Y} = \inf \left\{ \sum_{j=1}^{m} \| u_j \| \, \| v_j \| \right\}, \tag{2.2}$$

where the infimum is taken over all equivalent representations $\sum_{j=1}^{m} u_j \otimes v_j \in X \otimes Y$ of $\sum_{i=1}^{n} x_i \otimes y_i$ (see [5]).

If X is a Banach space on which every linear bounded operator from X into any Banach space Y is nuclear (this is the case in all finite-dimensional Banach spaces X), then for any Banach space Y the space $X \otimes^{\vee} Y$ is isomorphically isometric to $X \otimes^{\wedge} Y$ (see [16]).

The set $\Omega = \{(f,g) : f \in U_{X^*}, g \in U_{Y^*}\} = U_{X^*} \times U_{Y^*}$.

We start with the following two lemmas.

LEMMA 2.1. *For Banach spaces X and Y there is a norm one projection from $l_\infty(U_{X^*}) \otimes^{(\vee \text{ or } \wedge)} l_\infty(U_{Y^*})$ onto $l_\infty(\Omega)$.*

Proof. Since the space $l_\infty(\Omega)$ has the 1-extension property, it is sufficient to show that $l_\infty(\Omega)$ can be isometrically embedded in the space $l_\infty(U_{X^*}) \otimes^{(\vee \text{ or } \wedge)} l_\infty(U_{Y^*})$. In fact, every nonzero element $0 \neq \mathfrak{F} = \{\mathfrak{F}((f,g))\}_{f \in U_{X^*}, g \in U_{Y^*}}$ in the space $l_\infty(\Omega)$, (note that the norm in this Banach space is given by $\|\mathfrak{F}\|_{l_\infty(\Omega)} = \sup_{f \in U_{X^*}} \sup_{g \in U_{Y^*}} |\mathfrak{F}((f,g))|$) defines two scalar-valued functions $F \in l_\infty(U_{X^*})$ and $G \in l_\infty(U_{Y^*})$ by the following formulas:

$$F(f) = \sup_{g \in U_{Y^*}} \left| \mathfrak{F}\big((f,g)\big) \right|, \qquad G(g) = \sup_{f \in U_{X^*}} \left| \mathfrak{F}\big((f,g)\big) \right|. \tag{2.3}$$

Clearly the element $\acute{\mathfrak{F}} = (1/\|\mathfrak{F}\|_{l_\infty(\Omega)}) \times (F \otimes G)$ is an element of the space $l_\infty(U_{X^*}) \otimes^{(\vee \text{ or } \wedge)} l_\infty(U_{Y^*})$. Since both the injective and the projective tensor products are cross norms, $\|\acute{\mathfrak{F}}\|_{l_\infty(U_{X^*}) \otimes^{(\vee \text{ or } \wedge)} l_\infty(U_{Y^*})} = \|\mathfrak{F}\|_{l_\infty(\Omega)}$. The mapping J defined by the formula $J(\mathfrak{F}) = \acute{\mathfrak{F}}$ is the required isometric embedding. $\square$

LEMMA 2.2. *Let X and Y be two Banach spaces. Then $\lambda(X \otimes^{\vee} Y) = \lambda(X \otimes^{\vee} Y, l_\infty(\Omega))$.*

Proof. It is also sufficient to show that the space $X\otimes^{\vee}Y$ can be isometrically embedded in $l_\infty(\Omega)$. In fact, every element $\mathfrak{F} = \sum_{i=1}^{n} x_i \otimes y_i \in X\otimes^{\vee}Y$ defines a scalar-valued bounded function $\acute{\hat{\mathfrak{F}}} \in l_\infty(\Omega)$ by the formula $\acute{\hat{\mathfrak{F}}}((f,g)) = \sum_{i=1} f(x_i)g(y_i)$. Using definition (2.1) for the injective tensor product, we have $\|\mathfrak{F}\|_\vee = \|\acute{\hat{\mathfrak{F}}}\|_{l_\infty(\Omega)}$. The mapping i defined by the formula $i(\mathfrak{F}) = \acute{\hat{\mathfrak{F}}}$ is the required isometric embedding. □

We have the following theorem.

THEOREM 2.3. (1) *If Y_1 and Y_2 are complemented subspaces of Banach spaces X_1 and X_2, respectively, then the injective (resp., projective) tensor product $Y_1 \otimes^{\vee} Y_2$ (resp., $Y_1 \otimes^{\wedge} Y_2$) of the spaces Y_1 and Y_2 is complemented in the injective (resp., projective) tensor product $X_1 \otimes^{\vee} X_2$ (resp., $X_1 \otimes^{\wedge} X_2$) of the spaces X_1 and X_2 and*

$$\lambda\Big(Y_1 \otimes^{(\vee \text{ or } \wedge)} Y_2, X_1 \otimes^{(\vee \text{ or } \wedge)} X_2\Big) \le \lambda\big(Y_1, X_1\big)\lambda\big(Y_2, X_2\big). \tag{2.4}$$

(2) *If X and Y are injective spaces, then the space $X \otimes^{\vee} Y$ is injective. Moreover,*

$$\lambda\big(X \otimes^{\vee} Y\big) \le \lambda(X)\lambda(Y). \tag{2.5}$$

Proof. Let P_1 and P_2 be any projections from X_1 onto Y_1 and from X_2 onto Y_2, respectively. Then the operator P from the space $X_1 \otimes^{\vee} X_2$ onto the space $Y_1 \otimes^{\vee} Y_2$ (resp., from the space $X_1 \otimes^{\wedge} X_2$ onto the space $Y_1 \otimes^{\wedge} Y_2$) defined by

$$P\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} P_1\big(x_i\big) \otimes P_2\big(y_i\big) \tag{2.6}$$

is a projection and its norm $\|P\|$ is not exceeding $\|P_1\|\|P_2\|$. In fact, let $\sum_{i=1}^{n} x_i \otimes y_i$ be any element of the space $X_1 \otimes^{(\vee \text{ or } \wedge)} X_2$. Then, in the case of projective tensor product we have

$$\begin{aligned} \left\| P\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right\|_{Y_1\otimes^\wedge Y_2} &= \left\|\sum_{i=1}^{n} P_1\big(x_i\big) \otimes P_2\big(y_i\big)\right\|_{Y_1\otimes^\wedge Y_2} \\ &= \left\|\sum_{j=1}^{m} P_1\big(u_i\big) \otimes P_2\big(v_i\big)\right\|_{Y_1\otimes^\wedge Y_2} \\ &\le \|P_1\|\,\|P_2\| \sum_{j=1}^{m} \|u_j\|\,\|v_j\|, \end{aligned} \tag{2.7}$$

for all equivalent representations $\sum_{j=1}^{m} u_j \otimes v_j$ of $\sum_{i=1}^{n} x_i \otimes y_i$. So

$$\left\| P\left(\sum_{i=1}^{n} x_i \otimes y_i\right) \right\|_{Y_1 \otimes^\wedge Y_2} \leq \| P_1 \| \| P_2 \| \left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_{X_1 \otimes^\wedge X_2}. \tag{2.8}$$

And in the case of injective tensor product we have

$$\begin{aligned}
&\left\| P\left(\sum_{i=1}^{n} x_i \otimes y_i\right) \right\|_{Y_1 \otimes^\vee Y_2} \\
&\quad = \left\| \sum_{i=1}^{n} P_1(x_i) \otimes P_2(y_i) \right\|_{Y_1 \otimes^\vee Y_2} \\
&\quad = \sup \left\{ \left| \sum_{i=1}^{n} f\big(P_1(x_i)\big) g\big(P_2(y_i)\big) \right| : f \in U_{Y_1^*},\ g \in U_{Y_2^*} \right\} \\
&\quad = \sup \left\{ \left| f\left(P_1 \left(\sum_{i=1}^{n} g\big(P_2(y_i)\big) x_i \right)\right) \right| : f \in U_{Y_1^*},\ g \in U_{Y_2^*} \right\} \\
&\quad \leq \sup \left\{ \| P_1 \| \left\| \sum_{i=1}^{n} g\big(P_2(y_i)\big) x_i \right\|_{X_1} : g \in U_{Y_2^*} \right\} \\
&\quad = \| P_1 \| \sup \left\{ \sup \left\{ \left| \sum_{i=1}^{n} f(x_i) g\big(P_2(y_i)\big) \right| : f \in U_{X_1^*} \right\},\ g \in U_{Y_2^*} \right\} \\
&\quad \leq \| P_1 \| \| P_2 \| \sup \left\{ \left| \sum_{i=1}^{n} f(x_i) g(y_i) \right| : f \in U_{X_1^*},\ g \in U_{X_2^*} \right\} \\
&\quad \leq \| P_1 \| \| P_2 \| \left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_{X_1 \otimes^\vee X_2}.
\end{aligned} \tag{2.9}$$

Thus in both cases, $\|P\| \leq \|P_1\| \|P_2\|$. Taking the infimum of each side with respect to all such P_1 and P_2, we get inequality (2.4). To prove inequality (2.5), we apply inequality (2.4) and get in particular

$$\begin{aligned}
\lambda\big(X \otimes^\vee Y, l_\infty(U_{X^*}) \otimes^\vee l_\infty(U_{Y^*})\big) &\leq \lambda\big(X, l_\infty(U_{X^*})\big)\lambda\big(Y, l_\infty(U_{Y^*})\big) \\
&= \lambda(X)\lambda(Y).
\end{aligned} \tag{2.10}$$

Using Lemma 2.2 and definition (1.2), we get $\lambda(X \otimes^\vee Y, l_\infty(\Omega)) \geq \lambda(X \otimes^\vee Y, l_\infty(U_{X^*}) \otimes^\vee l_\infty(U_{Y^*}))$. We claim that the sign $\geq$ is an equal sign. In fact, if P is any projection from $l_\infty(U_{X^*}) \otimes^\vee l_\infty(U_{Y^*})$ onto $X \otimes^\vee Y$ and J is the embedding given in Lemma 2.1, then $\acute{P} = PJ$ is a projection from $l_\infty(\Omega)$ onto $X \otimes^\vee Y$ with $\|\acute{P}\| \leq \|P\|$. This is the sufficient condition for the two infimum

$\lambda(X\otimes^{\vee}Y, l_\infty(\Omega))$ and $\lambda(X\otimes^{\vee}Y, l_\infty(U_{X^*})\otimes^{\vee} l_\infty(U_{Y^*}))$ to be equal. Therefore

$$\lambda\big(X\otimes^{\vee} Y\big) = \lambda\big(X\otimes^{\vee} Y, l_\infty\big(U_{X^*}\otimes^{\vee} U_{Y^*}\big)\big). \tag{2.11}$$

Using inequality (2.10), we get (2.5). □

Remark 2.4. Since $\lambda(l_\infty(\Gamma)) = 1$ for any index set Γ, we conclude that $\lambda(l_\infty(\Gamma) \otimes^{(\vee \text{ or } \wedge)} l_\infty(\Lambda), X\otimes^{(\vee \text{ or } \wedge)} Y) = 1$ for every $X \supset l_\infty(\Gamma)$ and $Y \supset l_\infty(\Lambda)$.

We have the following two corollaries.

Corollary 2.5. *For any finite sequence $\{X_i\}_{i=1}^n$ of Banach spaces with complemented subspaces $\{Y_i\}_{i=1}^n$, the relative projection constant of the injective (resp., projective) tensor product $\bigotimes_{i=1}^n Y_i$ of the spaces Y_i in the space $\bigotimes_{i=1}^n X_i$ satisfies*

$$\lambda\left(\bigotimes_{i=1}^n Y_i, \bigotimes_{i=1}^n X_i\right) \le \prod_{i=1}^n \lambda\big(Y_i, X_i\big). \tag{2.12}$$

Corollary 2.6. *Let $\{Y_i\}_{i=1}^n$ be a finite sequence of finite-dimensional Banach spaces. Then the relation between the absolute projection constant of the projective (or injective) tensor product $\bigotimes_{i=1}^n Y_i$ and the direct sum $\sum_{i=1}^n \bigoplus Y_i$ (with the supremum norm) is as follows:*

$$\lambda\left(\bigotimes_{i=1}^n Y_i\right) \le \left(\lambda\left(\sum_{i=1}^n \bigoplus Y_i\right)\right)^n. \tag{2.13}$$

Proof. In fact, the proof is a combination of Corollary 2.5 and the results of [3, Theorem 4]. □

3. Applications

In this section, using Theorem 2.3, we obtain new results.

(1) For finite-dimensional Banach spaces X and Y with dimensions n and m, respectively, we have

$$\begin{aligned}\lambda(X\otimes Y) \le \sqrt{nm} - \frac{1}{\sqrt{nm}} + O\big(nm^{-3/4}\big) \\ -\left\{\left(\sqrt{m}-\frac{1}{\sqrt{m}}\right)\left(\frac{1}{\sqrt{n}} - O\big(n^{-3/4}\big)\right)\right. \\ \left. + \left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)\left(\frac{1}{\sqrt{m}} - O\big(m^{-3/4}\big)\right)\right\},\end{aligned} \tag{3.1}$$

in the real field and

$$\lambda(X \otimes Y) \leq \sqrt{nm} - \frac{1}{2\sqrt{nm}} + O\left(nm^{-3/4}\right)$$
$$- \left\{ \left(\sqrt{m} - \frac{1}{2\sqrt{m}}\right)\left(\frac{1}{2\sqrt{n}} - O\left(n^{-3/4}\right)\right) \right. \tag{3.2}$$
$$\left. + \left(\sqrt{n} - \frac{1}{2\sqrt{n}}\right)\left(\frac{1}{2\sqrt{m}} - O\left(m^{-3/4}\right)\right) \right\},$$

in the complex field. Compare this result with the result in (1.6).

(2) For any positive integer m (not necessarily prime) with a prime factorization $m = \prod_{i=1}^{n} q_i$ where the numbers q_i are distinct prime numbers, the space $\bigotimes_{i=1}^{n} l_{q_i^2}^{\infty}$ contains a subspace Y of dimension m with

$$\lambda(Y) \leq \sqrt{\prod_{i=1}^{n} q_i} - \left(\frac{1}{\sqrt{\prod_{i=1}^{n} q_i}} - \frac{1}{\prod_{i=1}^{n} q_i}\right) - C(m), \tag{3.3}$$

where $C(m)$ is a positive number depending on m (in case of $m = q_1 q_2$, $C(m) = [(1/\sqrt{q_1} - 1/q_1)(\sqrt{q_2} - 1/\sqrt{q_2}) + (1/\sqrt{q_2} - 1/q_2)(\sqrt{q_1} - 1/\sqrt{q_1})]$). Comparing this result with (1.3), we mention that the m^2-dimension of the space $\bigotimes_{i=1}^{n} l_{q_i^2}^{\infty}$ is not a square of a prime number, so it gives a new subspace Y with a new projection constant.

(3) For numbers p, q with $1 \leq p, q \leq 2$, we have

$$\lim_{n,m\to\infty} \frac{\lambda\left(l_p^n \otimes l_q^m\right)}{\sqrt{nm}} \leq \begin{cases} \dfrac{2}{\pi}, & \text{in the real field,} \\[2ex] \dfrac{\pi}{4}, & \text{in the complex field.} \end{cases} \tag{3.4}$$

4. The projection constants of operators

Now we start with our basic definitions of the projection constants of operators.

Definition 4.1. (1) A linear bounded operator A from a Banach space X into a Banach space Y is said to be left complemented with respect to a Banach space Z (Z contains Y as a closed subspace) if and only if there exists a linear bounded operator B from Z into X such that the composition AB is a projection from Z onto Y. In this case Z is said to be a left complementation of A.

If $P_Z(A)$ denotes the convex set of all operators B from Z into X such that the composition AB is a projection, then

(2) the left relative projection constant of the operator A with respect to the space Z is defined as

$$\lambda_l(A, Z) := \inf\left\{\|AB\| : B \in P_Z(A)\right\}. \tag{4.1}$$

(3) And the left absolute projection constant of A is defined as

$$\lambda_l(A) := \sup\left\{\lambda_l(A, Z) : Z \text{ is a left complementation of the operator } A\right\}. \tag{4.2}$$

We define the same analogy from the right.

Remark 4.2. We notice the following.

(1) From the definition of $\lambda_l(A, Z)$, the infimum in (4.1) is taken only with respect to the projections that are factored (through X) into two operators one of them is A and the other is an operator from Z into X, so $1 \leq \lambda(Y, Z) \leq \lambda_l(A, Z)$ for every left complementation Z of A.

(2) If A is a projection from X onto Y, then A is left complemented with respect to Y. In fact AJ is a projection for any embedding J from Y into X.

(3) If I_Y is the identity operator on Y and X contains Y as a complemented subspace, then $I_Y P = P$ for every projection P from X onto Y and hence I_Y is left complemented with respect to X. Moreover, $\lambda_l(I_Y, X) = \lambda(Y, X)$, that is, the relative projection constant of the identity operator on the space Y with respect to the space X is the relative projection constant of the space Y in the space X.

(4) If Z is a left complementation of the linear bounded operator $A : X \to Y$, then Y is complemented in Z and the operator A is onto.

(5) If Z is a separable or reflexive Banach space and X is a Banach space, then for any index set Γ the space Z is not a right complementation of any linear bounded operator from $l_\infty(\Gamma)$ into X. In particular, if X is a Banach space, then for any index set Γ, the space $l_\infty(\Gamma)$ is not a left complementation of any linear bounded operator from X into the space c_0.

The following lemma is parallel to that lemma mentioned in [8] for Banach spaces and we omit the proof since the proof is nearly similar.

Lemma 4.3. *Let Γ be an index set such that Y is isometrically embedded into $l_\infty(\Gamma)$ and let A be a linear bounded operator from X onto Y such that $l_\infty(\Gamma)$ is one of its left complementation. Then for a given $B \in P_{l_\infty(\Gamma)}(A)$,*

(1) *For all Banach spaces $E, Z, E \subseteq Z$ and every linear bounded operator T from E into Y there is an operator $\hat{T}$ from Z into Y extending the operator T with $\|\hat{T}\| \leq \|AB\|\|T\|$, that is, the space Y has $\|AB\|$-extension property, and in particular, if $Z \supseteq X$, the operator A has a linear extension $\hat{A}$ from Z into Y with $\|\hat{A}\| \leq \|AB\|\|A\|$. That is, the extension constant $c(A)$ of the operator A defined by ($c(A) := \sup_{X \subset Z} \inf\{\|\hat{A}\| : \hat{A}$ is an extension of A and $\hat{A} : Z \to Y\}$) satisfies $c(A) \leq \|AB\|\|A\|$.*

(2) *For every Banach space $Z \supseteq Y$, there exists a projection P from Z onto Y such that $\|P\| \leq \|AB\|$.*

The following theorem is also parallel to that given in (1.3) for Banach spaces.

THEOREM 4.4. *Let Y be isometrically embedded in $l_\infty(\Gamma)$ and let A be a linear bounded operator from X onto Y such that $l_\infty(\Gamma)$ is a left complementation of A. Then A is left complemented with respect to any other Banach space Z containing Y as a closed subspace. Moreover,*

$$\lambda_l(A,Z) \le \lambda_l\big(A, l_\infty(\Gamma)\big) \tag{4.3}$$

for every Banach space Z containing Y as a closed subspace, that is, $\lambda_l(A)$ attains its supremum at $l_\infty(\Gamma)$. Therefore,

$$\lambda_l(A) = \lambda_l\big(A, l_\infty(\Gamma)\big), \qquad c(A) \le \|A\|\lambda_l(A). \tag{4.4}$$

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