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*Research Article*

# The Numbers of Positive Solutions by the Lusternik-Schnirelmann Category for a Quasilinear Elliptic System Critical with Hardy Terms

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In this paper, we study the quasilinear elliptic system with Sobolev critical exponent involving both concave-convex and Hardy terms in bounded domains. By employing the technique introduced by Benci and Cerami (1991), we obtain at least $\operatorname{cat}(\Omega) + 1$ distinct positive solutions.

## 1. Introduction and Main Result

In this paper, we are concerned with the multiplicity of positive solutions of the following critical problem:

$$\begin{gathered} -\Delta_p u - \nu \frac{|u|^{p-2}u}{|x|^p} = \frac{1}{p^*}\frac{\partial F}{\partial u}(x,u,v) + f_\lambda(x)|u|^{q-2}u \quad \text{in } \Omega, \\ -\Delta_p v - \nu \frac{|v|^{p-2}v}{|x|^p} = \frac{1}{p^*}\frac{\partial F}{\partial v}(x,u,v) + g_\mu(x)|v|^{q-2}v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1}$$

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, $0 \in \Omega$, $1 < q < p < N$, $p^* = pN/(N-p)$ is the critical Sobolev exponent, $0 < \nu < \overline{\nu}$ where $\overline{\nu} = ((N-p)/p)^p$ is the best Hardy constant, and the parameter $\lambda > 0$, $\mu > 0$, we assume that $f_\lambda(x) = \lambda f_+(x) + f_-(x)$ and $g_\mu(x) = \mu g_+(x) + g_-(x)$ where the weight functions $f$ and $g$ satisfy the following conditions:

$(H_1)$ $f, g \in C(\overline{\Omega})$ with $\|f_+\|_\infty = \|g_+\|_\infty = 1$, where $f_\pm = \max\{\pm f, 0\} \neq 0$ and $g_\pm = \max\{\pm g, 0\} \neq 0$.

And the function $F$ satisfies the following conditions:

$(f_1)$ $F \in C^1(\overline{\Omega} \times (\mathbb{R}^+)^2, \mathbb{R}^+)$, such that $\forall t > 0$

$$F(x, tu, tv) = t^{p^*} F(x,u,v) \quad \forall (x,u,v) \in \overline{\Omega} \times (\mathbb{R}^+)^2. \tag{2}$$

$(f_2)$ $F(x,u,0) = F(x,0,v) = (\partial F/\partial u)(x,u,0) = (\partial F/\partial v)(x,0,v) = 0$, where $u, v \in \mathbb{R}^+$.

$(f_3)$ $\partial F(x,u,v)/\partial u = \partial F(x,u,v)/\partial v$ are strictly increasing functions about $u$ and $v$ for all $u > 0$, $v > 0$.

$(f_4)$ $(u,v).\nabla F(x,u,v) = p^* F(x,u,v)$ with $(\partial F(x,u,v)/\partial u, \partial F(x,u,v)/\partial v) = \nabla F$.

$(f_5)$ $F(x,u,v) \leq K(|u|^p + |v|^p)^{p^*/p}$ for some constant $K > 0$.

*Remark 1.* We deduce form the conditions $(f_1)$, $(f_2)$, and $(f_3)$ that the functional $(u,v) \longrightarrow \psi(u,v) = \int_\Omega F(x,u,v)dx$ is of class $C^1(W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega), \mathbb{R}^+)$ and

$$\begin{aligned} &\left\langle \psi'(u,v), (a,b) \right\rangle \\ &= \int_\Omega \left( \frac{\partial F(x,u,v)}{\partial u} a + \frac{\partial F(x,u,v)}{\partial v} b \right) dx, \end{aligned} \tag{3}$$

where $(u,v),(a,b) \in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$, and $\partial F/\partial u, \partial F/\partial v \in C^1(\Omega\times(\mathbb{R}^+)^2,\mathbb{R}^+)$ such that $(\partial F/\partial u)(x,tu,tv) = t^{p^*-1}(\partial F/\partial u)(x,u,v)$ and $(\partial F/\partial v)(x,tu,tv) = t^{p^*-1}(\partial F/\partial v)(x,u,v)$.

Moreover, there exists $C>0$ such that

$$\left|\frac{\partial F}{\partial u}(x,u,v)\right| \le C\left(|u|^{p^*-1}+|v|^{p^*-1}\right)$$
$$\left|\frac{\partial F}{\partial v}(x,u,v)\right| \le C\left(|u|^{p^*-1}+|v|^{p^*-1}\right) \tag{4}$$
$$\forall x\in\overline{\Omega},\ u,v\in\mathbb{R}^+.$$

The proof is almost the same as that in Chu and Tang [1].

Recently, many papers have studied the multiplicity of positive solutions by way of fibering method and the notions of topological indices category for different semilinear, quasilinear, and nonlocal problems involving a critical exponent and concave and convex nonlinearities (see [2–4]). Our goal here is to give a new result for this system by linking the number of positive solutions with the topology of the domain $\Omega$. More precisely with the Category index, let us note $\operatorname{cat}_Y(X)$ is the least number of closed and contractible sets in $Y$ which cover $X$. Our main result is the following.

**Theorem 2.** *Let $N>p^2$ and $p^*-N/(N-p)\le q<p$. Suppose that $F$ satisfies $(f_1)-(f_5)$ and the functions $f$, $g$ satisfy the condition $(H_1)$. Then, there exists $\Lambda_*>0$ such that if for each $\lambda^{p/(p-q)}+\mu^{p/(p-q)}\in(0,\Lambda_*)$, problem (1) has at least $\operatorname{cat}(\Omega)+1$ distinct positive solutions.*

This paper is composed of four sections. In Section 2, we give some results for the $\mathcal{N}$ehari manifold associated of the energy functional and fibering maps. In Section 3, we will build homotopies between $\Omega$ and certain sublevel set of the energy functional associated with (1). Finally we prove the result in Section 4.

## 2. The $\mathcal{N}$ehari Manifold Associated with the Energy Functional and Fibering Maps

Let the Sobolev space $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ with the usual norm:

$$\|(u,v)\|_W = \left(\|u\|^p+\|v\|^p\right)^{1/p},$$
$$\|u\| = \|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p - \nu\frac{|u|^p}{|x|^p}dx\right)^{1/p}, \tag{5}$$
$$\nu\in[0,\overline{\nu}).$$

Also, the standard norm of the space $L^p(\Omega)$ is $\|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{1/p}$. Moreover, a pair of functions $(u,v)\in W$ is said be to a weak solution of problem (1) if

$$\int_\Omega\left(|\nabla u|^{p-2}\nabla u\nabla\varphi_1 - \nu\frac{|u|^{p-2}u}{|x|^p}\varphi_1\right)dx$$
$$+\int_\Omega\left(|\nabla v|^{p-2}\nabla v\nabla\varphi_2 - \nu\frac{|v|^{p-2}v}{|x|^p}\varphi_1\right)dx$$
$$-\frac{1}{p^*}\int_\Omega\left(\frac{\partial F(x,u,v)}{\partial u}\varphi_1+\frac{\partial F(x,u,v)}{\partial v}\varphi_2\right)dx$$
$$-\int_\Omega f_\lambda|u|^{q-2}u\varphi_1dx - \int_\Omega g_\mu|v|^{q-2}v\varphi_2dx = 0 \tag{6}$$

for all $(\varphi_1,\varphi_2)\in W$.

We know that looking for weak solutions of (1) is like looking for the critical points of the associated functional

$$J_{\lambda,\mu}(u,v) = \frac{1}{p}\|(u,v)\|_W^p - \frac{1}{p^*}\int_\Omega F(x,u^+,v^+)dx - \frac{1}{q}K_{f_\lambda,g_\mu}(u^+,v^+) \tag{7}$$

where $K_{f_\lambda,g_\mu}(u,v) = \int_\Omega(f_\lambda(x)|u|^q+g_\mu(x)|v|^q)dx$.

By the above Remark 1, the functional $J_{\lambda,\mu}(u,v)$ is well defined on the space $W$ and is of class $C^1(W,\mathbb{R})$.

Therefore, the solutions of (1) correspond to critical points of $J_{\lambda,\mu}$. Let us denote by $\mathcal{N}_{\lambda,\mu}$ the $\mathcal{N}$ehari manifold related to $J_{\lambda,\mu}$, given by

$$\mathcal{N}_{\lambda,\mu} := \left\{(u,v)\in W,(u,v)\ne(0,0): \left\langle J'_{\lambda,\mu}(u,v),(u,v)\right\rangle = 0\right\} \tag{8}$$

Namely,

$$\mathcal{N}_{\lambda,\mu} := \left\{u\in W,(u,v)\ne(0,0): \|(u,v)\|_W^p = \int_\Omega F(x,u^+,v^+)dx + K_{f_\lambda,g_\mu}(u^+,v^+)\right\}. \tag{9}$$

Notice that the functional $J_{\lambda,\mu}$ is not bounded below on the total space for that we consider the functional on the $\mathcal{N}$ehari manifold.

Define

$$\chi_{\lambda,\mu}(u,v) = \left\langle J'_{\lambda,\mu}(u,v),(u,v)\right\rangle = \|(u,v)\|_W^p - \int_\Omega F(x,u,v)dx - K_{f_\lambda,g_\mu}(u,v). \tag{10}$$

Let $(u,v)\in\mathcal{N}_{\lambda,\mu}$, and by easy calculation we have

$$\begin{aligned}\left\langle\chi'_{\lambda,\mu}(u,v),(u,v)\right\rangle &= p\|(u,v)\|_W^p - p^*\int_\Omega F(x,u,v)dx - qK_{\lambda,\mu}(u,v)\\ &= (p-p^*)\int_\Omega F(x,u,v)dx - (q-p)K_{f_\lambda,g_\mu}(u,v)\\ &= (p-q)\|(u,v)\|_W^p - (p^*-q)\int_\Omega F(x,u,v)dx\end{aligned}$$

$$= (p - p^*) \|(u,v)\|_W^p - (q - p^*) K_{f_\lambda, g_\mu}(u,v). \tag{11}$$

**Lemma 3.** *The functional $J_{\lambda,\mu}$ is bounded below on the $\mathcal{N}$ehari manifold $\mathcal{N}_{\lambda,\mu}$.*

*Proof.* Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$, and applying the Hölder inequality and the Sobolev embedding theorem, Young inequality, and Condition $(H_1)$ we have

$$K_{f_\lambda, g_\mu}(u,v) \le S^{-q/p} |\Omega|^{(p^*-q)/p^*} \cdot \left(\lambda^{p/(p-q)} + \mu^{p/(p-q)}\right)^{(p-q)/p} \|(u,v)\|_W^q, \tag{12}$$

and we deduce

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \left(\frac{p^*-p}{p^*p}\right)\|(u,v)\|_W^p - \left(\frac{p^*-q}{p^*q}\right) \\ &\cdot K_{f_\lambda, g_\mu}(u,v) \ge \frac{p^*-p}{p^*p}\|(u,v)\|_W^p - \frac{p^*-q}{p^*q} \\ &\cdot S^{-q/p}|\Omega|^{(p^*-q)/p^*}\left(\lambda^{p/(p-q)} + \mu^{p/(p-q)}\right)^{(p-q)/p} \\ &\cdot \|(u,v)\|_W^q \end{aligned} \tag{13}$$

Thus, $J_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. □

Now, we split the $\mathcal{N}$ehari manifold $\mathcal{N}_{\lambda,\mu}$ into three parts, namely,

$$\begin{aligned} \mathcal{N}^+_{\lambda,\mu} &:= \left\{u \in \mathcal{N}_{\lambda,\mu} : \left\langle \chi'_{\lambda,\mu}(u,v), (u,v)\right\rangle > 0\right\} \\ \mathcal{N}^-_{\lambda,\mu} &:= \left\{u \in \mathcal{N}_{\lambda,\mu} : \left\langle \chi'_{\lambda,\mu}(u,v), (u,v)\right\rangle < 0\right\} \\ \mathcal{N}^0_{\lambda,\mu} &:= \left\{u \in \mathcal{N}_{\lambda,\mu} : \left\langle \chi'_{\lambda,\mu}(u,v), (u,v)\right\rangle = 0\right\} \end{aligned} \tag{14}$$

Then, we have the following results.

**Lemma 4.** *Let $(u_0, v_0) \in \mathcal{N}_{\lambda,\mu}$ be a local minimizer of $J_{\lambda,\mu}$ and $(u_0, v_0) \notin \mathcal{N}^0_{\lambda,\mu}$. Then $(u_0, v_0)$ is a critical point of $J_{\lambda,\mu}$.*

*Proof.* The proof is standard; you can see [4]. □

**Lemma 5.** *There exists $\Lambda_* > 0$ such that for all $\lambda, \mu > 0$ such that $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_*$ then $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.*

*Proof.* Suppose the contrary; that is, there exist $\lambda, \mu > 0$ with $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_*$, but $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. Let $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$; we have

$$(p-q)\|(u,v)\|_W^p = (p^*-q)\int_\Omega F(x,u,v)\,dx \tag{15}$$

and

$$(p^*-p)\|(u,v)\|_W^p = (p^*-q) K_{f_\lambda, g_\mu}(u,v). \tag{16}$$

By $(f_5)$ and applying the Minkowski inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} &\int_\Omega F(x,u,v)\,dx \\ &\le K\left(\int_\Omega \left(|u|^p + |v|^p\right)^{p^*/p} dx\right)^{(p/p^*)(p^*/p)} \\ &\le K\left(\left(\int_\Omega \left(|u|^{p^*} dx\right)^{p/p^*} + \left(\int_\Omega |v|^{p^*} dx\right)^{p/p^*} dx\right)^{p^*/p} \\ &\le K S^{-p^*/p}\left(\int_\Omega \left(|\nabla u|^p dx + \int_\Omega |\nabla v|^p dx\right)^{p^*/p}, \right. \end{aligned} \tag{17}$$

so

$$\int_\Omega F(x,u,v)\,dx \le K S^{-p^*/p}\|(u,v)\|_W^{p^*}. \tag{18}$$

Combining (15) and (18), we have

$$(p^*-q) K S^{-p^*/p}\|(u,v)\|_W^{p^*} \ge (p-q)\|(u,v)\|_W^p, \tag{19}$$

then

$$\|(u,v)\|_W \ge \left(\frac{(p-q) S^{p^*/p}}{(p^*-q) K}\right)^{1/(p^*-p)}, \tag{20}$$

By (12) we have

$$\begin{aligned} (p^*-p)\|(u,v)\|_W^p &= (p^*-q) K_{f_\lambda, g_\mu}(u,v) \le (p^*-q) \\ &\cdot S^{-q/p}|\Omega|^{(p^*-q)/p^*}\left(\lambda^{p/(p-q)} + \mu^{p/(p-q)}\right)^{(p-q)/p} \\ &\cdot \|(u,v)\|_W^q, \end{aligned} \tag{21}$$

then

$$\|(u,v)\|_W \le \left(\frac{(p^*-q) S^{-q/p}|\Omega|^{(p^*-q)/p^*}}{(p^*-p)}\right)^{1/(p-q)} \cdot \left(\lambda^{p/(p-q)} + \theta^{p/(p-q)}\right)^{1/p}. \tag{22}$$

We deduct from (20) and (22) that

$$\left(\lambda^{p/(p-q)} + \mu^{p/(p-q)}\right) > \Lambda_*, \tag{23}$$

which is a contradiction. □

So, we have $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^+_{\lambda,\mu}$, and we define

$$\begin{aligned} c_{\lambda,\mu} &= \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} J(u,v), \\ c^+_{\lambda,\mu} &= \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v), \\ c^-_{\lambda,\mu} &= \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v). \end{aligned} \tag{24}$$

**Lemma 6.** ***(i)** For some $\Lambda_* > 0$ and for $\lambda^{q/(p-q)} + \mu^{q/(p-q)} \in ]0, \Lambda_*[$ so, there exists $(PS)_{c_{\lambda,\mu}}$–sequence $\{(u_n, v_n)\}$ of $\mathcal{N}_{\lambda,\mu}$ for $J_{\lambda,\mu}$.*

***(ii)** If $0 < \lambda^{q/(p-q)} + \mu^{q/(p-q)} < \Lambda_*$, then there exists a $(PS)_{c^-_{\lambda,\mu}}$–sequence $\{(u_n, v_n)\}$ of $\mathcal{N}^-_{\lambda,\mu}$ for $J_{\lambda,\mu}$.*

*Proof.* You find the same proof in the following reference [5]. □

Denote

$$S_F = \inf_{(u,v)\in W\setminus\{0\}} \left\{ \frac{\|(u,v)\|_W^p}{\left(\int_\Omega F(x,u,v)\,dx\right)^{p/p^*}} : \int_\Omega F(x,u,v)\,dx > 0 \right\}. \tag{25}$$

We define a cut-off function $\eta(x) \in C_0^\infty(\Omega)$ such that $\eta(x) = 1$ for $|x| < \rho_0$, $\eta(x) = 0$ for $|x| > 2\rho_0$, $0 \le \eta \le 1$, and $|\nabla\eta| \le C$. For $\varepsilon > 0$, let

$$u_\varepsilon(x) = \frac{\eta(x)}{\left(\varepsilon + |x|^{p/(p-1)}\right)^{(N-p)/p}}. \tag{26}$$

From Li Wang, Qiaoling Wei, and Dongsheng Kang [6], we have

$$\begin{aligned} \left(\int_\Omega |u_\varepsilon|^p\,dx\right)^{p/p^*} &= \varepsilon^{-(N-p)/p}\|U\|^p_{L^{p^*}(\mathbb{R}^N)} + O(\varepsilon), \\ \int_\Omega |\nabla u_\varepsilon|^p\,dx &= \varepsilon^{-(N-p)/p}\|\nabla U\|^p_{L^p(\mathbb{R}^N)} + O(1), \\ \frac{\int_\Omega |\nabla u_\varepsilon|^p\,dx}{\left(\int_\Omega |u_\varepsilon|^p\,dx\right)^{p/p^*}} &= S + O\left(\varepsilon^{(N-p)/p}\right), \end{aligned} \tag{27}$$

where $U(x) = (1 + |x|^{p/(p-1)})^{-(N-p)/p} \in W^{1,p}(\mathbb{R}^N)$, and verifying $S$, this

$$S = \frac{\|\nabla U\|^p_{L^p(\mathbb{R}^N)}}{\|U\|^p_{L^{p^*}(\mathbb{R}^N)}} = \inf_{u\in W_0^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\|u\|^p_{L^{p^*}(\mathbb{R}^N)}}. \tag{28}$$

**Lemma 7.**

$$c_{0,0} = \frac{1}{N} S_F^{N/p}. \tag{29}$$

*Proof.* Set $u_0 = e_1 u_\varepsilon$ and $v_0 = e_2 u_\varepsilon$ and $(u_0, v_0) \in W$, where $e_1, e_2 \in \mathbb{R}^+$, $e_1^p + e_2^p = 1$, and $\inf_{x\in\overline{\Omega}} F(x, e_1, e_2) \ge K$. Then by $(f_5)$, the definition of $S_F$, and (27), we have

$$\begin{aligned} c_{0,0} &\le \sup_{t\ge 0} J_{0,0}(tu_0 t v_0) \\ &= \frac{1}{N}\left( \frac{\left(e_1^p + e_2^p\right)\int_\Omega |\nabla u_\varepsilon|^p\,dx}{\left(\int_\Omega F(x, e_1 u_\varepsilon, e_2 v_\varepsilon)\,dx\right)^{p/p^*}} \right)^{N/p} \\ &\le \frac{1}{N}\left( \frac{\int_\Omega |\nabla u_\varepsilon|^p\,dx}{K^{p/p^*}\left(\int_\Omega |u_\varepsilon|^{p^*}\,dx\right)^{p/p^*}} \right)^{N/p} \\ &\le \frac{1}{N}\left(\frac{1}{K^{p/p^*}}\right)^{N/p}\left(S + O\left(\varepsilon^{(N-p)/p}\right)\right)^{N/p} \\ &= \frac{1}{N}\left(\frac{1}{K^{p/p^*}}\right)^{N/p} S^{N/p} + O\left(\varepsilon^{(N-p)/p}\right) \le \frac{1}{N} S_F^{N/p} \end{aligned} \tag{30}$$

$$c_{0,0} \le \frac{1}{N} S_F^{N/p}, \tag{31}$$

We use the following relation:

$$\sup_{t\ge 0}\left(\frac{t^p}{p}A - \frac{t^{p^*}}{p^*}B\right) = \frac{1}{N}\left(\frac{A}{B^{p/p^*}}\right)^{N/p}, \quad A, B > 0. \tag{32}$$

For the reverse inequality, the application of the mountain pass theorem gives us a Palais-Smale sequence $\{(u_n, v_n)\} \subset W$ for $I_{0,0}$ at level $c_{0,0}$ and from here we can show that $\{(u_n, v_n)\}$ is bounded in $W$ using standard arguments. Since

$$\left\|(u_n^-, v_n^-)\right\|^p = \left\langle I'(u_n, v_n), (u_n^-, v_n^-)\right\rangle \longrightarrow 0. \tag{33}$$

Assuming that $u_n, v_n \ge 0$, we find

$$\begin{aligned} &\left\|(u_n, v_n)\right\|^p \longrightarrow l \\ \text{and } &\left(\int_\Omega F(x, u_n, v_n)\,dx\right) \longrightarrow l. \end{aligned} \tag{34}$$

From definition (25) of $S_F$, we get

$$\begin{aligned} S_F l^{p/p^*} &= S_F \lim_{n\to+\infty}\left(\int_\Omega F(x, u_n, v_n)\,dx\right)^{p/p^*} \\ &\le \lim_{n\to+\infty}\left\|(u_n, v_n)\right\|^p = l, \end{aligned} \tag{35}$$

then

$$l \ge S_F^{N/p}. \tag{36}$$

Since $J_{0,0}(u_n, v_n) \longrightarrow c_{0,0}$ implies $l = c_{0,0}N$, we deduce from (36) that

$$c_{0,0} \ge \frac{1}{N} S_F^{N/p}. \tag{37}$$

Then from (31) and (37) we obtain

$$c_{0,0} = \frac{1}{N} S_F^{N/p}. \tag{38}$$

□

Next we prove that $J_{\lambda,\mu}$ satisfies the Palais-Smale condition under some level. Before, we need the following lemma.

**Lemma 8.** *Let $F \in C^1(\overline{\Omega}, (\mathbb{R}^+)^2)$ with $F(x,0,0) = 0$ and $|\partial F(x,u,v)/\partial u|, |\partial F(x,u,v)/\partial v| \le C_1\left(|u|^{p-1} + |v|^{p-1}\right)$ for some $\le p < \infty$. $C_1 > 0$. Let $\{(u_k, v_k)\}$ be a bounded sequence in $L^p(\overline{\Omega}, (\mathbb{R}^+)^2)$, such that $(u_k, v_k) \rightharpoonup (u, v)$ weakly in $W$. Then*

$$\begin{aligned} \int_\Omega F(x, u_k, v_k)\,dx \longrightarrow &\int_\Omega F(x, u_k - u, v_k - v)\,dx \\ &+ \int_\Omega F(x, u, v)\,dx \end{aligned} \tag{39}$$

*as $k \longrightarrow \infty$.*

*Proof.* (The idea of this proof was borrowed from [7]) □

**Lemma 9.** *$J_{\lambda,\mu}$ satisfies the $(PS)_c$–condition for*

$$-\infty < c < c_\infty := \frac{1}{N} S_F^{N/p} - C\left(\lambda^{p/(p-q)} + \mu^{p/(p-q)}\right), \quad (40)$$

*where $C > 0$ is independent on $\lambda$ and $\mu$.*

*Proof.* The proof is similar to that of Lemma 2.1 in [8]. □

Let $(u, v) \in W$, with $\int_\Omega F(x, u, v)dx > 0$, and put

$$t_{\max} = t_{\max}(u, v, \lambda, \mu) := \left(\frac{(p-q)\|(u,v)\|_W^p}{(p^*-q)\int_\Omega F(x,u,v)\,dx}\right)^{1/(p^*-p)} > 0. \quad (41)$$

Then the following lemma holds. Its proof is similar to the lemma [4] (or see Tarantello [9]).

**Lemma 10.** *Let $(u, v) \in W$, with $\int_\Omega F(x, u, v)dx > 0$, so there are unique number positives $t^+$ and $t^-$ such that $0 < t^+ < t_{\max} < t^-$ with*

$$\begin{gathered}(t^+u, t^+v) \in \mathcal{N}_{\lambda,\mu}^+, \\ (t^-u, t^-v) \in \mathcal{N}_{\lambda,\mu}^-\end{gathered} \quad (42)$$

*and*

$$\begin{aligned}J_{\lambda,\mu}(t^+u, t^+v) &= \min_{0\le t\le t_{\max}} J_{\lambda,\mu}(tu, tv), \\ J_{\lambda,\mu}(t^-(u,v,\lambda,\mu)u, t^-(u,v,\lambda,\mu)v) &= \max_{t\ge 0} J_{\lambda,\mu}(tu, tv).\end{aligned} \quad (43)$$

**Lemma 11.** *For some $\lambda, \mu > 0$, and $\Lambda_* > 0$ such that $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_*$, we have*

$$c_{\lambda,\mu}^- < c_\infty. \quad (44)$$

*Proof.* First, we claim that there exist positive constants $C_1, C_2 > 0$ independent of $\varepsilon$ such that

$$0 < C_1 < t_\varepsilon = t^-(u_0, v_0, \lambda, \mu) < C_2 < \infty \quad (45)$$

Let $u_0 = e_1 u_\varepsilon$ and $v_0 = e_2 v_\varepsilon$. We obtain

$$\begin{aligned}&\|(u_0, v_0)\|^p - t_\varepsilon^{p^*-p}\int_\Omega F(x, u_0, v_0)\,dx \\ &= t_\varepsilon^{q-p} K_{f_\lambda, g_\mu}(u_0, v_0).\end{aligned} \quad (46)$$

Then, by $(f_5)$ and (27) we deduct that

$$t_\varepsilon^{p^*-p} \le \frac{\int_\Omega |\nabla U|^p\,dx}{\left(K\int_\Omega |U|^{p^*}dx\right)^{p/p^*}} + O\left(\varepsilon^{(N-p)/p}\right), \quad (47)$$

then $t_\varepsilon$ is bounded above as $\varepsilon \longrightarrow 0$. Using Lemma 10, we have

$$t_\varepsilon \ge t_{\max}(u_\varepsilon, v_\varepsilon, \lambda, \mu) > 0, \quad (48)$$

then we can also suppose that $t_\varepsilon$ is bounded below. By a direct calculation we have

$$\int_\Omega |u_\varepsilon|^q\,dx \ge \begin{cases} C\varepsilon^{(-(N-p)/p)q+N((p-1)/p)} & \text{if } p^* - \dfrac{N}{N-p} < q, \\ C\varepsilon^{(-(N-p)/p)q+N((p-1)/p)}|\ln\varepsilon|, & \text{if } q = p^* - \dfrac{N}{N-p}, \end{cases} \quad (49)$$

and the constant $C$ is a positive. So

$$\begin{aligned}J_{\lambda,\mu}(t_\varepsilon u_0, t_\varepsilon u_0) \le{}& \frac{1}{N}S_F^{N/p} + O\left(\varepsilon^{(N-p)/p}\right) - (\lambda+\mu) \\ &\cdot \begin{cases} C\varepsilon^{((p-1)/p)(N-q((N-p)/p))}, & \text{if } p^* - \dfrac{N}{N-p} < q, \\ C\varepsilon^{((p-1)/p)(N-q((N-p)/p))}|\ln\varepsilon|, & \text{if } q = p^* - \dfrac{N}{N-p} \end{cases}\end{aligned} \quad (50)$$

We have $p^* - \dfrac{N}{N-p} < q < p$, and there exist $\tau > 0$ such that

$$\begin{aligned}&\frac{p-q}{q}\frac{p-1}{p}\left(N - q\frac{N-p}{p}\right) < \tau \\ &< \frac{N-p}{p} - \frac{p-1}{p}\left(N - q\frac{N-p}{p}\right).\end{aligned} \quad (51)$$

Let

$$\lambda + \mu = \varepsilon^\tau \quad (52)$$

By the following relation, for $x, y > 0$ and $s \in [0, 1]$, we have $(x + y)^s < x^s + y^s$, and we obtain

$$\lambda^{p/(p-q)} + \mu^{p/(p-q)} < \varepsilon^{\tau(p/(p-q))}. \quad (53)$$

By (51) we have

$$\tau + \frac{p-1}{p}\left(N - q\frac{N-p}{p}\right) < \min\left\{\tau\frac{p}{p-q}, \frac{N-p}{p}\right\}. \quad (54)$$

Then, there exists $\Lambda_* > 0$ such that $\lambda^{p/(p-q)} + \mu^{p/(p-q)} \in (0, \Lambda_*)$, and we have

$$J_{\lambda,\mu}(t_\varepsilon u_0, t_\varepsilon u_0) \le c_\infty \quad (55)$$

so by definition $c_{\lambda,\mu}^-$ we deduct that

$$c_{\lambda,\mu}^- < c_\infty. \quad (56)$$

For the case $q = p^* - N/(N-p)$, so we get the same result. □

For the existence of the first solution of our problem (1)

**Lemma 12.** *There exists $\Lambda_* > 0$ such that if $\lambda, \mu \in (0, \Lambda_*)$, then $J_{\lambda,\mu}$ has a minimizer $(u_\lambda^+, u_{\lambda,\mu}^+) \in \mathcal{N}_{\lambda,\mu}^+$ and its satisfies*

**(i)** *$J_{\lambda,\mu}(u_{\lambda,\mu}^+, v_{\lambda,\mu}^+) = c_{\lambda,\mu}^+$*

**(ii)** *$(u_{\lambda,\mu}^+, v_{\lambda,\mu}^+)$ is a positive solution of (1).*

*Proof.* Taking into account the fact that $\mathcal{N}^-_{\lambda,\mu} \subset \mathcal{N}_{\lambda,\mu}$ and Lemma 11 we have

$$c_{\lambda,\mu} \le c^-_{\lambda,\mu} < c_\infty. \tag{57}$$

Hence, for the proof of (i) just use the following Lemmas 11 and 9. Now let $(u^+_{\lambda,\mu}, v^+_{\lambda,\mu})$ be solution of problem (1) such that $J_{\lambda,\mu}(u^+_{\lambda,\mu}, v^+_{\lambda,\mu}) = c_{\lambda,\mu}$. Moreover, we have $(u^+_{\lambda,\mu}, v^+_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. In fact, if $(u^+_{\lambda,\mu}, v^+_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$, by Lemma 10, there are unique $t^+_0, t^-_0$ such that $(t^+_0 u^+_{\lambda,\mu}, t^+_0 v^+_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t^-_0 u^-_{\lambda,\mu}, t^+_0 v^-_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t^+_0 < t^-_0 = 1$. Since

$$\begin{gathered}\frac{d}{dt} J_{\lambda,\mu}\left(t^+_0 u^+_{\lambda,\mu}, t^+_0 v^+_{\lambda,\mu}\right) = 0 \\ \text{and } \frac{d^2}{dt^2} J_{\lambda,\mu}\left(t^+_0 u^+_{\lambda,\mu}, t^+_0 v^+_{\lambda,\mu}\right) > 0,\end{gathered} \tag{58}$$

there exists $t^+_0 < \bar{t} \le t^-_0$ such that $J_{\lambda,\mu}(t^+_0 u^+_{\lambda,\mu}, t^+_0 v^+_{\lambda,\mu}) < J_{\lambda,\mu}(\bar{t} u^+_{\lambda,\mu}, \bar{t} v^+_{\lambda,\mu})$. By Lemma 10

$$\begin{aligned} J_{\lambda,\mu}\left(t^+_0 u^+_{\lambda,\mu}, t^+_0 v^+_{\lambda,\mu}\right) &< J_{\lambda,\mu}\left(\bar{t} u^+_{\lambda,\mu}, \bar{t} v^+_{\lambda,\mu}\right) \\ &\le J_{\lambda,\mu}\left(t^-_0 u^+_{\lambda,\mu}, t^-_0 v^+_{\lambda,\mu}\right) \\ &= J_{\lambda,\mu}\left(u^+_{\lambda,\mu}, v^+_{\lambda,\mu}\right), \end{aligned} \tag{59}$$

which is impossible and by the maximum principle, we deduct that $(u^+_{\lambda,\mu}, v^+_{\lambda,\mu})$ is a positive solution of problem (1). □

## 3. Some Technical Results

**Lemma 13.** *Let $(\lambda_n)$ and $(\mu_n)$ decreasing sequences in $(0, \Lambda_*)$ for some $\Lambda_* > 0$ and converging to 0, so $\lim_{n\to+\infty} c^-_{\lambda_n,\mu_n} = c_{0,0}$.*

*Proof.* By Lemma 6 there exists a sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$, $u_n, v_n \ge 0$ such that

$$\begin{gathered} J_{\lambda_n,\mu_n}(u_n, v_n) = c^-_{\lambda_n,\mu_n} \\ \text{and } J'_{\lambda_n,\mu_n}(u_n, v_n) = 0. \end{gathered} \tag{60}$$

There exists a real number sequence $t_n$ satisfying $(t_n u_n, t_n v_n) \in \mathcal{N}_{0,0}$. So

$$\begin{aligned} c_{0,0} &\le J_{0,0}(t_n u_n, t_n v_n) \\ &= J_{\lambda_n,\mu_n}(t_n u_n, t_n v_n) + \frac{t_n^q}{q} K_{f_{\lambda_n}, g_{\mu_n}}\left(u^+_n, v^+_n\right) \end{aligned} \tag{61}$$

$$\le c^-_{\lambda_n,\mu_n} + \frac{t_n^q}{q} K_{f_{\lambda_n}, g_{\mu_n}}\left(u^+_n, v^+_n\right) \tag{62}$$

Since, by Lemma 10 for all $n$ we have

$$0 < c^-_{\lambda_1,\mu_1} \le c^-_{\lambda_n,\mu_n} \le c_{0,0}. \tag{63}$$

Moreover, $(t_n u_n, t_n v_n) \in \mathcal{N}_{0,0}$ implies that

$$\left\|(u_n, v_n)\right\|_W = t_n^{q-p} K_{f_{\lambda_n}, g_{\mu_n}}\left(u^+_n, v^+_n\right) \tag{64}$$

and we deduct that $J_{\lambda_n,\mu_n}(u_n, v_n) = c^-_{\lambda_n,\mu_n} \le c_{0,0}$ and $J'_{\lambda_n,\mu_n}(u_n, v_n) = 0$. We get that $\{(u_n, v_n)\}$ is bounded in $W$.

We can now say that $t_n$ is a bounded sequence, if we assume by contradiction that $\lim_{n\to+\infty} t_n = \infty$. We find $\lim_{n\to+\infty} K_{f_{\lambda_n}, g_{\mu_n}}(u^+_n, v^+_n) = 0$, so $\lim_{n\to+\infty}\|(u_n, v_n)\|_W = 0$ which implies by (60) that

$$\lim_{n\to+\infty} c^-_{\lambda_n,\mu_n} = 0 \tag{65}$$

which is a contradiction with (63). □

We consider the following lemma. See section 5.3 in [10].

**Lemma 14.** *Suppose that $X$ is Banach space and $F \in \mathscr{C}^1(X, \mathbb{R})$. Assume that, for $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$,*

(1) *$F$ satisfies the $(PS)_c$ condition for $c \le c_0$,*
(2) *$cat(\{x \in X, F(x) \le c_0\}) \ge k$.*

*Then $F$ has at least $k$ critical points in $\{x \in X, F(x) \le c_0\}$.*

Let us consider tow subset of $\mathbb{R}^N$

$$\begin{aligned} \Omega^+_r &:= \left\{x \in \mathbb{R}^N : dist(x, \Omega) < r\right\}, \\ \Omega^-_r &:= \{x \in \Omega : dist(x, \partial\Omega) > r\} \end{aligned} \tag{66}$$

Note that $\Omega^+_r$ and $\Omega^-_r$ are homotopically equivalent to $\Omega$ for some $r > 0$. We may assume $B_r := B_r(0) \subset \Omega$. We consider

$$\begin{aligned} &W_r \\ &:= \left\{(u, v) \in W^{1,p}_0(B_r) \times W^{1,p}_0(B_r) : u, v \text{ are radial}\right\}. \end{aligned} \tag{67}$$

Recall that $u \in W^{1,p}_0(B_r)$ implies that $u$ is extension in $\Omega$ with $u = 0$ outside of $B_r$. Let $J_{\lambda,\mu,B_r} : W_r \longrightarrow \mathbb{R}$ as

$$\begin{aligned} J_{\lambda,\mu,B_r}(u, v) := \frac{1}{p}\|(u, v)\|^p_{W_r} - \frac{1}{p^*}\int_\Omega F\left(x, u^+, v^+\right) dx \\ - \frac{1}{q} K_{f_\lambda, g_\mu}\left(u^+, v^+\right). \end{aligned} \tag{68}$$

We denote by

$$\tilde{c}_{\lambda,\mu} := \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu,B_r}} J_{\lambda,\mu,B_r}(u, v). \tag{69}$$

Similar to $J_{\lambda,\mu}$, $J_{\lambda,\mu,B_r}$ can be shown to satisfy restricted versions of the same three Lemmas 7, 9, and 11. We consider

$$\mathcal{N}^-_{\lambda,\mu,\tilde{c}_{\lambda,\mu}} := \left\{(u, v) \in \mathcal{N}^-_{\lambda,\mu} : J_{\lambda,\mu}(u, v) \le \tilde{c}_{\lambda,\mu}\right\} \tag{70}$$

Let

$$\zeta(u, v) := \frac{N}{S_F^{N/p}} \int_\Omega x F(x, u, v)\, dx, \tag{71}$$

for all $(u, v) \in \mathcal{N}^-_{\lambda,\mu}$, that is, $\zeta(u, v) \in \mathbb{R}^N$

and the map $\varpi : \Omega_r^- \longrightarrow \mathcal{N}^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}$ given by

$$\varpi(y)(x) := \begin{cases} \left(u_{\lambda,\mu}(x-y), v_{\lambda,\mu}(x-y)\right) & \text{if } x \in B_r(y), \\ o & \text{if } x \notin B_r(y), \end{cases} \tag{72}$$

with $u_{\lambda,\mu}, v_{\lambda,\mu}$ radial. For all $y \in \Omega_r^-$

$$\begin{aligned} \frac{S_F^{N/p}}{N}(\zeta \circ \varpi)(y) &= \int_\Omega xF\left(x, u_{\lambda,\mu}(x-y), v_{\lambda,\mu}(x-y)\right) dx \\ &= \int_\Omega (y+z) F\left(x, u_{\lambda,\mu}(z), v_{\lambda,\mu}(z)\right) dx \\ &= \int_\Omega yF\left(x, u_{\lambda,\mu}(z), v_{\lambda,\mu}(z)\right) dx. \end{aligned} \tag{73}$$

Then $\zeta \circ \varpi$ can be rewritten

$$\begin{aligned} \zeta \circ \varpi(y) &= \frac{N}{S_F^{N/p}} \int_\Omega F\left(x, u_{\lambda,\mu}(z), v_{\lambda,\mu}(z)\right) dx \\ &=: \chi(\lambda,\mu)\, y. \end{aligned} \tag{74}$$

*Remark 15.*

$$\lim_{\lambda,\mu \longrightarrow 0} \widetilde{c}_{\lambda,\mu} = \widetilde{c}_{0,0}, \tag{75}$$

and

$$\lim_{\lambda,\mu \longrightarrow 0} \chi(\lambda,\mu) = 1. \tag{76}$$

Next, we define the map $H_{\lambda,\mu} : [0,1] \times \mathcal{N}^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}} \longrightarrow \mathbb{R}^N$ by

$$H_{\lambda,\mu}(t,u,v) := \left(t + \frac{1-t}{\chi(\lambda,\mu)}\right)\zeta(u,v). \tag{77}$$

**Lemma 16.** *For some $\Lambda_*$ such that $\lambda^{p/(p-q)} + \mu^{p/(p-q)} \in (0,\Lambda_*)$ we have*

$$H_{\lambda,\mu}\left([0,1] \times \mathcal{N}^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}\right) \subset \Omega_r^+. \tag{78}$$

*Proof.* We show by the absurd that there exist $(t_n)$ sequence of $[0,1]$, $\lambda_n, \mu_n \longrightarrow 0$, and $\{(u_n,v_n)\} \subset \mathcal{N}^-_{\lambda,\mu,\widetilde{c}_{\lambda,\mu}}$ such that

$$H_{\lambda_n,\mu_n}(t_n,u_n,v_n) \notin \Omega_r^+, \tag{79}$$

and let $t_n \longrightarrow t_0 \in [0,1]$ (up to a subsequence of $(t_n)$). By Remark 15, we have

$$\chi(\lambda_n,\mu_n) \longrightarrow 1 \tag{80}$$

$$\begin{aligned} c^-_{\lambda_n,\mu_n} &\le \frac{1}{p}\|(u_n,v_n)\|_W^p - \frac{1}{p^*}\int_\Omega F(x,u_n,v_n)\,dx \\ &\quad - \frac{1}{q}K_{f_{\lambda_n},g_{\mu_n}}(u_n^+,v_n^+) \le \widetilde{c}_{\lambda_n,\mu_n}. \end{aligned} \tag{81}$$

and

$$\begin{aligned} &\|(u_n,v_n)\|_W^p - \int_\Omega F(x,u_n,v_n)\,dx - K_{f_{\lambda_n},g_{\mu_n}}(u_n^+,v_n^+) \\ &\quad = 0. \end{aligned} \tag{82}$$

Standard calculations show that $(u_n,v_n)$ is bounded in $W$ and by this we obtain

$$\begin{aligned} c^-_{\lambda_n,\mu_n} + o(1) &\le \frac{1}{p}\|(u_n,v_n)\|_W^p \\ &\quad - \frac{1}{p^*}\int_\Omega F(x,u_n,v_n)\,dx \\ &\le \widetilde{c}_{\lambda_n,\mu_n} + o(1), \end{aligned} \tag{83}$$

and

$$\|(u_n,v_n)\|_W^p - \int_\Omega F(x,u_n,v_n)\,dx = o(1), \tag{84}$$

as $n \longrightarrow +\infty$. We have by Lemmas 13 and 7 and its restricted version for $J_{\lambda,\mu,B_r}$ that

$$c^-_{\lambda_n,\mu_n} \text{ and } \widetilde{c}_{\lambda_n,\mu_n} \text{ both converge to } \frac{1}{N}S_F^{N/p}, \tag{85}$$

then by (83), (84) and (85)

$$\begin{aligned} &\|(u_n,v_n)\|_W^p \longrightarrow S_F^{N/p} \\ &\text{and } \int_\Omega F(x,u_n,v_n)\,dx \longrightarrow S_F^{N/p}. \end{aligned} \tag{86}$$

Now, it is easy to see that the sequence $(\widetilde{u}_n,\widetilde{v}_n)$ given by

$$(\widetilde{u}_n,\widetilde{v}_n) = \left(\frac{u_n}{\left(\int_\Omega F(x,u_n,v_n)\,dx\right)^{1/p^*}}, \frac{v_n}{\left(\int_\Omega F(x,u_n,v_n)\,dx\right)^{1/p^*}}\right) \tag{87}$$

verifies

$$\begin{aligned} &\int_\Omega F(x,\widetilde{u}_n,\widetilde{v}_n)\,dx = 1 \\ &\text{and } \|(\widetilde{u}_n,\widetilde{v}_n)\|_W^p \longrightarrow S_F \end{aligned} \tag{88}$$

For a subsequence of $\{(\widetilde{u}_n,\widetilde{v}_n)\}$ we have

$$\begin{aligned} &(\widetilde{u}_n,\widetilde{v}_n) \longrightarrow (\widetilde{u},\widetilde{v}) \quad \textit{a.e.} \text{ on } \mathbb{R}^N \\ &|\nabla\widetilde{u}_n - \widetilde{u}|^p + |\nabla\widetilde{v}_n - \widetilde{v}|^p\,dx \longrightarrow \omega \quad \text{in } \mathcal{M}\left(\mathbb{R}^N\right), \\ &\int_\Omega F(x,\widetilde{u}_n,\widetilde{v}_n)\,dx \longrightarrow \tau \quad \text{in } \mathcal{M}\left(\mathbb{R}^N\right). \end{aligned} \tag{89}$$

By the same way used in [[10],Lemma 1.40] (see also [7] ), we obtain

$$\begin{aligned} S_F &= \|(\widetilde{u},\widetilde{v})\|^p + \|\omega\|, \\ 1 &= \int_\Omega F(x,\widetilde{u},\widetilde{v})\,dx + \|\tau\| \end{aligned} \tag{90}$$

and

$$\|\tau\|^{p/p^*} \leq S_F \|\omega\|. \quad (91)$$

Since

$$\left(\int_\Omega F(x,\tilde{u},\tilde{v})\,dx\right)^{p/p^*} \leq S_F^{-1} \|(\tilde{u},\tilde{v})\|_W^p. \quad (92)$$

It is easy to confirm that $\int_\Omega F(x,\tilde{u},\tilde{v})dx$ and $\|\omega\|$ are equal either to 0 or to 1. Since $S_F$ is independent of $\Omega$ and is never achieved except when $\Omega = \mathbb{R}^N$ (see also [11]), so necessarily $\int_\Omega F(x,\tilde{u},\tilde{v})dx = 0$. Then the measure $\omega$ is concentrated at a single point $y$ of $\overline{\Omega}$,

and we have

$$\zeta(u_n,v_n) \longrightarrow \int_\Omega x\,d\omega(x) = y \in \overline{\Omega} \subset \Omega_r^+. \quad (93)$$

Therefore

$$H_{\lambda_n,\mu_n}(t_n,u_n,v_n) := \left(t_n + \frac{t_n-1}{\chi(\lambda,\mu)}\right)\zeta(u_n,v_n) \longrightarrow y \in \overline{\Omega} \subset \Omega_r^+, \quad (94)$$

and this is impossible. □

**Lemma 17.** *For some $\Lambda_* > 0$ such that $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda_*$, we have*

$$cat\left(\mathcal{N}^-_{\lambda,\mu,\tilde{c}_{\lambda,\mu}}\right) \geq cat\ (\Omega). \quad (95)$$

*Proof.* This classic proof is omitted for brevity. An identical proof can be found in [12], Lemma 14. □

## 4. The Proof of Theorem 2

Denote by $J_{\mathcal{N}^-_{\lambda,\mu}}$ the restriction of $J_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$.

**Lemma 18.** *If $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda^*$, for some $\Lambda^* > 0$, so the functional $J_{\mathcal{N}^-_{\lambda,\mu}}$ verifies the Palais-Smale condition for $c < c_\infty$.*

*Proof.* By [[10], Proposition 5.12], there exists a sequence $\{\sigma_n\} \subset \mathbb{R}$. If $(u_n,v_n)$ is a $(PS)_c$ for $I_{\mathcal{N}^-_{\lambda,\mu}}$ at level $c$, there exists a sequence $\{\sigma_n\} \subset \mathbb{R}$ such that

$$J'_{\lambda,\mu}(u_n) = \sigma_n \chi'_{\lambda,\mu}(u_n,v_n) + o(1), \quad (96)$$

where

$$\begin{aligned}\chi_{\lambda,\mu}(u_n,v_n) &= \left\langle J'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\right\rangle \\ &= \|(u_n,v_n)\|_W^p - \int_\Omega F(x,u_n,v_n)\,dx \\ &\quad - K_{f_\lambda,g_\mu}(u_n,v_n)\end{aligned} \quad (97)$$

Recall that $(u_n,v_n) \in \mathcal{N}^-_{\lambda,\mu}$, so $\langle \chi'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle < 0$.

If $\langle \chi'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle \longrightarrow 0$,

$$(p-q)\|(u_n,v_n)\|_W^p = (p^*-q)\int_\Omega F(x,u_n,v_n)\,dx + o(1) \quad (98)$$

and

$$(p^*-p)\|(u_n,v_n)\|_W^p = (p^*-q)K_{f_\lambda,g_\mu}(u_n,v_n) + o(1). \quad (99)$$

By the same argument employed in Lemma 5, we get

$$\|(u_n,v_n)\|_W \geq \left(\frac{(p-q)S^{p^*/p}}{(p^*-q)K}\right)^{1/(p^*-p)} + o(1), \quad (100)$$

and

$$\|(u_n,v_n)\|_W \leq \left(\frac{(p^*-q)S^{-q/p}|\Omega|^{(p^*-q)/p^*}}{(p^*-p)}\right)^{1/(p-q)} \cdot \left(\lambda^{p/(p-q)} + \mu^{p/(p-q)}\right)^{1/p} \quad (101)$$

and we deduct that $\lambda^{p/(p-q)} + \mu^{p/(p-q)} > \Lambda^*$. This is contradiction.

Moreover we assume that $\langle \chi'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle \longrightarrow l$, as $n \longrightarrow +\infty$. Since $\langle J'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle = 0$, so $\sigma_n \longrightarrow 0$ as $n \longrightarrow +\infty$ then $J'_{\lambda,\mu}(u_n,v_n) \longrightarrow 0$. Thus,

$$\begin{aligned}&J_{\lambda,\mu}(u_n,v_n) \longrightarrow c \in \left(0,c_{\lambda_\mu}\right), \\ &\text{and } J'_{\lambda,\mu}(u_n,v_n) \longrightarrow 0,\end{aligned} \quad (102)$$

then by Lemma 9 the proof is finished. □

**Lemma 19.** *For some $\Lambda^* > 0$ such that if $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \Lambda^*$, then every critical point $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$ of $J_{\mathcal{N}^-_{\lambda,\mu}}$ is a critical point of $J_{\lambda,\mu}$ in $W$.*

*Proof.* For the proof of this lemma, it is similar to Lemma 18. □

*Proof of Theorem 2.* Applying Lemmas 9 and 12, $J_{\mathcal{N}^-_{\lambda,\mu}}$ satisfies $(PS)_c$ condition for all $c \in (0,c_{\lambda,\mu})$. Then, by Lemmas 17 and 14, $J_{\mathcal{N}^-_{\lambda,\mu}}$ admits at least cat $(\Omega)$ critical points in $\mathcal{N}^-_{\lambda,\mu,\tilde{c}_{\lambda,\mu}}$. Hence, we deduce from Lemma 19 that $J_{\lambda,\mu}$ has at least cat $(\Omega)$ critical points in $\mathcal{N}^-_{\lambda,\mu}$. Moreover, $\mathcal{N}^-_{\lambda,\mu} \cap \mathcal{N}^+_{\lambda,\mu} = \emptyset$, $J_{\lambda,\mu}$ at least cat $(\Omega)$ + 1 critical points in $W$. □

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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