

Research Article

Necessary and Sufficient Conditions for Set-Valued Maps with Set Optimization

Abdessamad Oussarhan  and Ikram Daidai 

Faculté des Sciences et Techniques, Université Cadi Ayyad, BP 549, Marrakech, Morocco

Correspondence should be addressed to Abdessamad Oussarhan; abdessamad.oussa@gmail.com

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Optimality conditions are studied for set-valued maps with set optimization. Necessary conditions are given in terms of S -derivative and contingent derivative. Sufficient conditions for the existence of solutions are shown for set-valued maps under generalized quasiconvexity assumptions.

1. Introduction

In recent years, a great attention has been paid to set-valued optimization problems; many authors (see, e.g., [1–7]) have concentrated on the problems with and without constraints:

$$\begin{aligned} & \text{minimize } F(x), \\ & \quad x \in M, \\ & \text{minimize } F(x), \\ & \quad G(x) \cap (-K_Z) \neq \emptyset, \\ & \quad x \in M, \end{aligned} \tag{1}$$

where F and G are set-valued maps defined between two Banach spaces X, Y and X, Z , respectively, K_Z is the pointed closed convex cone of Z , and M is a nonempty subset of X .

Studies on these problems consider two types of solutions: vector solution, given by a vector optimization, and set solution, given by a set optimization.

The vector solution cannot be often used in practice, since it depends only on special element of image set of solution and the other elements are ignored; therefore the solution concept in vector optimization is sometimes improper. In order to avoid this drawback, Kuroiwa [8] introduced in the first time the concept of set solution by using practically relevant order relations for sets. This leads to solution concepts

for set-valued optimization problems based on comparisons among values of the set-valued objective map. Hernández et al. [9] gives some links between solutions concepts in vector and set optimization.

Taa [7] gives necessary and sufficient conditions for unconstrained vector optimization in terms of S -derivatives. Jahn and Khan [3] establish optimality conditions for unconstrained vector optimization under generalized convexity assumptions. Alonso-Durán and Rodríguez-Marín [10] give optimality conditions for the considered problems in set optimization using directional derivatives under pseudoconvexity assumptions and with the notion of the contingent derivative. In this paper we study necessary conditions for both problems in terms of S -derivatives with set optimization and we derive sufficient conditions under weaker notion of pseudoconvexity assumptions that are given in [3].

This paper is divided into three sections. In the first Section we collect some of the concepts required for the paper. Section 2 is devoted to the necessary optimality conditions for the unconstrained and the constrained set optimization and Section 3 deals with the sufficient optimality conditions in set optimization.

2. Preliminaries

Let X, Y , and Z be real normed spaces, where Y and Z are partially ordered by convex pointed cones with nonempty

interiors K_Y and K_Z , respectively. Y^* and Z^* will denote the continuous duals of X and Y , respectively. The collection of nonempty subsets of Y will be denoted by $\wp(Y)$.

Let $F : X \rightrightarrows Y$ be a set-valued map. We recall that the effective domain and the graph of F are defined by

$$\begin{aligned} \text{dom}(F) &:= \{x \in X \mid F(x) \neq \emptyset\}, \\ \text{Gr}(F) &:= \{(x, y) \in X \times Y \mid y \in F(x)\}. \end{aligned} \tag{2}$$

Let $G : X \rightrightarrows Z$ be a set-valued map and let us suppose that $\text{dom } F = \text{dom } G = M$ with $M \neq \emptyset$.

Research in set-valued optimization has concentrated on the problems with and without constraints:

$$\begin{aligned} \text{minimize } & F(x) \\ & x \in M. \end{aligned} \tag{3}$$

$$\begin{aligned} \text{minimize } & F(x) \\ & G(x) \cap (-K_Z) \neq \emptyset, \\ & x \in M. \end{aligned} \tag{4}$$

A solution $\bar{x} \in M$ for these problems with the criterion of vector optimization is defined as a generalization of the notion established by Pareto. We recall this concept in the following definition. Let $F(M) = \bigcup_{x \in M} F(x)$.

Definition 1. Let $\bar{x} \in M$. It is said that \bar{x} is

- (i) a minimum solution for (3) and we denote $\bar{x} \in \min(F, K_Y)$ (or $\bar{x} \in \min(F)$), if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(M) - \bar{y}) \cap (-K_Y) = \{0\}; \tag{5}$$

- (ii) a weak minimum solution for (3) and we denote $\bar{x} \in W \min(F, K_Y)$ (or $\bar{x} \in W \min(F)$), if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(M) - \bar{y}) \cap (-\text{int}(K_Y)) = \emptyset. \tag{6}$$

Let \leq^l ($<^l$, resp.) be the following relation defined between two nonempty subsets A, B of Y :

$$\begin{aligned} A \leq^l B &\iff B \subset A + K_Y, \\ (\text{resp. } A <^l B &\iff B \subset A + \text{int}(K_Y)). \end{aligned} \tag{7}$$

Using the above relations, Kuroiwa [8], in a natural way, introduced the following notion of l-minimal set (weakly l-minimal set, resp.).

Definition 2. Let $\mathcal{S} \subset \wp(Y)$. It is said that $A \in \mathcal{S}$ is

- (i) a lower minimal (or l-minimal) set of \mathcal{S} , if $B \in \mathcal{S}$ and $B \leq^l A$ imply $A \leq^l B$. The family of l-minimal sets of \mathcal{S} is denoted by $\text{l-min}(\mathcal{S}, K_Y)$ (or $\text{l-min}(\mathcal{S})$);

- (ii) a lower weak minimal (or l-w minimal) set of \mathcal{S} , if $B \in \mathcal{S}$ and $B <^l A$ imply $A <^l B$. The family of weakly l-minimal sets of \mathcal{S} is denoted by $\text{l-W min}(\mathcal{S}, K_Y)$ (or $\text{l-W min}(\mathcal{S})$).

In this way, the problems (3) and (4) can be written in set optimization with the following forms:

$$\begin{aligned} \text{l-minimize } & F(x) \\ & x \in M. \end{aligned} \tag{SP_1}$$

$$\begin{aligned} \text{l-minimize } & F(x) \\ & G(x) \cap (-K_Z) \neq \emptyset, \\ & x \in M. \end{aligned} \tag{SP_2}$$

In these cases, \bar{x} is a l-minimum (l-w minimum, resp.) solution of F , if $\bar{x} \in M$ (with $G(\bar{x}) \cap -K_Z \neq \emptyset$ in the problem (SP₂)) and $F(\bar{x})$ is a l-minimal (l-w minimal, resp.) set of the family of images of F , that is, the family

$$\mathcal{F} = \{F(x) \mid x \in M\}. \tag{8}$$

The next proposition supplies a characterization of l-w minimum (see [10, Proposition 18]).

Proposition 3. $\bar{x} \in M$ is an l-w minimal solution of (SP₁) if and only if for each $x \in M$ one of the following conditions is satisfied:

- (i) $F(x) <^l F(\bar{x})$ and $F(\bar{x}) <^l F(x)$.
- (ii) There exists $\bar{y} \in F(\bar{x})$ such that $(F(x) - \bar{y}) \cap (-\text{int}(K_Y)) = \emptyset$.

Let us recall the following definition.

Definition 4. Let $(\bar{x}, \bar{y}) \in \text{Gr}(F)$. The contingent derivative $DF(\bar{x}, \bar{y})$ is the set-valued map from X into Y defined by $y \in DF(\bar{x}, \bar{y})(x)$ if there exist sequences $(t_n) \rightarrow 0^+$, $(x_n, y_n) \rightarrow (x, y)$ such that

$$\bar{y} + t_n y_n \in F(\bar{x} + t_n x_n), \quad \forall n \in \mathbb{N}. \tag{9}$$

The following definition has been introduced by Shi [6]. It is an extension of the set-valued derivative in Definition 4.

Definition 5. Let $(\bar{x}, \bar{y}) \in \text{Gr}(F)$. The S-derivative $SF(\bar{x}, \bar{y})$ is the set-valued map from X into Y defined by $y \in SF(\bar{x}, \bar{y})(x)$ if there exist sequences $(t_n) \subset]0, +\infty[$, $(x_n, y_n) \rightarrow (x, y)$ such that

$$\begin{aligned} t_n x_n &\longrightarrow 0, \\ \bar{y} + t_n y_n &\in F(\bar{x} + t_n x_n), \\ &\forall n \in \mathbb{N}. \end{aligned} \tag{10}$$

Remark 6. Let $(\bar{x}, \bar{y}) \in \text{Gr}(F)$. It is easy to see the following:

- (i) The set-valued derivatives $DF(\bar{x}, \bar{y})$ and $SF(\bar{x}, \bar{y})$ are positively homogeneous with closed graphs.

- (ii) $(0, 0) \in \text{Gr}(DF(\bar{x}, \bar{y})) \subset \text{Gr}(SF(\bar{x}, \bar{y}))$.
- (iii) $DF(\bar{x}, \bar{y}) = SF(\bar{x}, \bar{y})$ whenever the graph of F is convex in $X \times Y$.

For sufficient condition for l-w minimal solution of problems (SP_1) and (SP_2) , we need certain convexity assumptions which are taken from [3, 4].

Definition 7. Let $\Gamma \subset Y$ and $(\bar{x}, \bar{y}) \in \text{Gr}(F)$. F is called Γ -contingently quasiconvex at (\bar{x}, \bar{y}) , if, for every $x \in M$, the condition $(F(x) - \bar{y}) \cap \Gamma \neq \emptyset$ ensures that $DF(\bar{x}, \bar{y})(x - \bar{x}) \cap \Gamma \neq \emptyset$.

Definition 8. We say that F is K_Y -pseudoconvex at $(\bar{x}, \bar{y}) \in \text{Gr}(F)$ if and only if

$$F(x) - \bar{y} \subset DF(\bar{x}, \bar{y})(x - \bar{x}) + K_Y. \tag{11}$$

Remark 9. Let $\Gamma \subset Y$. The Γ -contingent quasiconvexity reduces to the K_Y -pseudoconvexity.

3. Necessary Optimality Conditions

According to derived necessary condition, we recall the following notion of the strict l-w minimum and the concept of the K_Y -w minimal property given in [10].

Definition 10. Let \bar{x} be an l-w minimum solution of (SP_1) . \bar{x} is called strict l-w minimum of (SP_1) , if there exists a neighbourhood U of \bar{x} such that $F(x) \not\prec^l F(\bar{x})$ for all $x \in U \cap M$.

Definition 11 (domination property). A subset $A \subset Y$ has the K_Y -w minimal property if for all $y \in A$ there exists $a \in W \min(A)$ such that $a - y \in (-\text{int}(K_Y)) \cup \{0\}$.

The following Lemma has been established in [10] without proof; we give a simple proof for reader's convenience.

Lemma 12. Let $x, \bar{x} \in M$ and $\bar{y} \in F(\bar{x})$. If $F(x) \not\prec^l F(\bar{x})$, $W \min(F(\bar{x})) = \{\bar{y}\}$, and $F(\bar{x})$ has the K_Y -w minimal property, one has

$$(F(x) - \bar{y}) \cap (-\text{int}(K_Y)) = \emptyset. \tag{12}$$

Proof. Suppose the contrary; then there exist $y \in F(x)$ and $k \in \text{int}(K_Y)$ such that

$$\bar{y} = y + k \in F(x) + \text{int}(K_Y). \tag{13}$$

Since $W \min(F(\bar{x})) = \{\bar{y}\}$ and $F(\bar{x})$ have the K_Y -w minimal property we get

$$y_1 - \bar{y} \in \text{int}(K_Y) \cup \{0\} \quad \forall y_1 \in F(\bar{x}), \tag{14}$$

that is,

$$y_1 \in \bar{y} + \text{int}(K_Y) \cup \{0\} \quad \forall y_1 \in F(\bar{x}). \tag{15}$$

From (13) we have

$$y_1 \in F(x) + \text{int}(K_Y), \quad \forall y_1 \in F(\bar{x}), \tag{16}$$

and hence

$$F(\bar{x}) \subset F(x) + \text{int}(K_Y), \quad (\text{i.e., } F(x) \prec^l F(\bar{x})). \tag{17}$$

This contradicts $F(x) \not\prec^l F(\bar{x})$. □

Necessary conditions for the problem (SP_1) are given in the following.

Theorem 13. Let \bar{x} be a strict l-w minimum of (SP_1) . If $W \min(F(\bar{x})) = \{\bar{y}\}$ and $F(\bar{x})$ has the K_Y -w minimal property, then

$$SF(\bar{x}, \bar{y})(x) \cap (-\text{int}(K_Y)) = \emptyset, \quad \forall x \in M. \tag{18}$$

Proof. Suppose the contrary; then there exist $x \in M$ and $y \in Y$ such that

$$y \in SF(\bar{x}, \bar{y})(x) \cap (-\text{int}(K_Y)), \tag{19}$$

and hence there exist $(t_n)_{n \in \mathbb{N}} > 0$ and $(x_n, y_n) \rightarrow (x, y)$ such that

$$\begin{aligned} t_n x_n &\rightarrow 0, \\ t_n y_n &\in F(\bar{x} + t_n x_n) - \bar{y}, \\ &\forall n \in \mathbb{N}, \end{aligned} \tag{20}$$

and from the hypothesis we have that \bar{x} is a strict l-w minimum of (SP_1) ; then there exists a neighbourhood U of \bar{x} such that $F(x) \not\prec^l F(\bar{x})$ for all $x \in U \cap M$. Since $\bar{x} + t_n x_n \rightarrow \bar{x}$ then there exists $n_0 \in \mathbb{N}$ such that $\bar{x} + t_n x_n \in U \cap M$ for all $n \geq n_0$ and then

$$F(\bar{x} + t_n x_n) \not\prec^l F(\bar{x}), \quad \forall n \geq n_0, \tag{21}$$

by Lemma 12 and hypothesis we get

$$[F(\bar{x} + t_n x_n) - \bar{y}] \cap (-\text{int}(K_Y)) = \emptyset, \quad \forall n \geq n_0. \tag{22}$$

On the other hand, $y \in -\text{int}(K_Y)$; then there exists $n_1 \in \mathbb{N}$ such that

$$t_n y_n \in -\text{int}(K_Y), \quad \forall n \geq n_1, \tag{23}$$

and for (20) we have

$$[F(\bar{x} + t_n x_n) - \bar{y}] \cap (-\text{int}(K_Y)) \neq \emptyset \quad \forall n \geq n_1. \tag{24}$$

This contradicts (22) for all $n \geq \max(n_0, n_1)$. □

As an immediate consequence we have the following corollary.

Corollary 14. Let \bar{x} be an l-w minimum of (SP_1) . Let $\bar{y} \in F(\bar{x})$. Let us suppose that there exists a neighbourhood U of \bar{x} such that for each $x \in U \cap M$ one of the following conditions is satisfied:

- (a) $\bar{y} \notin F(x) + \text{int}(K_Y)$ or
- (b) $\bar{y} \in W \min(F(x), K_Y)$.

Then,

- (i) \bar{x} is a strict l-w minimum of (SP_1) ;
- (ii) $SF(\bar{x}, \bar{y})(x) \cap (-\text{int}(K_Y)) = \emptyset$, for all $x \in M$.

Proof. It is obvious that (b) \Rightarrow (a) $\Rightarrow F(x) \not\prec^l F(\bar{x})$. Then if there exists a neighbourhood U of \bar{x} such that for each $x \in U \cap M$ the condition (a) holds, we deduce from Definition 10 that (i) holds. On the other hand if (a) holds we have

$$(F(x) - \bar{y}) \cap (-\text{int}(K_Y)) = \emptyset, \quad \forall x \in U \cap M. \quad (25)$$

By using similar arguments as in Theorem 13, we establish (ii). \square

Another consequences of Theorem 13 and Corollary 14 are given in the following corollaries.

Corollary 15. *Let \bar{x} be a strict l-w minimum of (SP_1) . If $F(x) \not\prec^l F(\bar{x})$, $W \min(F(\bar{x})) = \{\bar{y}\}$, and $F(\bar{x})$ has the K_Y -w minimal property, then*

$$DF(\bar{x}, \bar{y})(x) \cap (-\text{int}(K_Y)) = \emptyset, \quad \forall x \in M. \quad (26)$$

Corollary 16. *Let \bar{x} be an l-w minimum of (SP_1) . Let $\bar{y} \in F(\bar{x})$. Let us suppose that there exists a neighbourhood U of \bar{x} such that for each $x \in U \cap M$ one of the following conditions is satisfied:*

- (a) $\bar{y} \notin F(x) + \text{int}(K_Y)$ or
- (b) $\bar{y} \in W \min(F(x), K_Y)$.

Then

- (i) \bar{x} is a strict l-w minimum of (SP_1) ;
- (ii) $DF(\bar{x}, \bar{y})(x) \cap (-\text{int}(K_Y)) = \emptyset$, for all $x \in M$.

Remark 17. If \bar{x} is a strict l-w minimum solution of (SP_1) and $\bar{y} \in W \min(F(\bar{x}))$, Theorem 13 and Corollaries 14 and 16 are not guaranteed if the other conditions are not satisfied. Indeed, let us recall the example considered in Alonso-Durán and Rodríguez-Marín [10]: let $F : (0, 2) \Rightarrow \mathbb{R}^2$ be defined by

$$F(x) = \begin{cases} \{(y, z) \in \mathbb{R}^2 \mid (y - 2x + 2)^2 + z^2 \leq x^2\}, & \text{if } x \leq 1, \\ \{(y, z) \in \mathbb{R}^2 \mid (y - 2x + 2)^2 + z^2 \leq (2 - x)^2\}, & \text{if } x > 1. \end{cases} \quad (27)$$

Let $K_Y = \mathbb{R}_+^2$. Then $\bar{x} = 1$ is a strict l-w minimum of F and $\bar{y} = (-1, 0) \in W \min(F(1))$. But observe that $W \min(F(\bar{x})) \neq \{\bar{y}\}$ and for all neighbourhoods U of \bar{x} there exists $x \in U \cap M$ such $\bar{y} \in F(x) + \text{int}(K_Y)$ and $\bar{y} \notin W \min(F(x))$. On the other hand for all $k \in -\text{int}(K_Y)$ and $x \in M$ we take $t_n = 0$ for each $n \in \mathbb{N}$, then for every sequence $(x_n, y_n) \rightarrow (x, k)$ we have

$$(-1, 0) + t_n y_n \in F(1 + t_n x_n), \quad \forall n \in \mathbb{N}, \quad (28)$$

that is

$$DF(1, (-1, 0))(x) \cap (-\text{int}(K_Y)) \neq \emptyset, \quad \forall x \in M. \quad (29)$$

Hence

$$SF(1, (-1, 0))(x) \cap (-\text{int}(K_Y)) \neq \emptyset, \quad \forall x \in M. \quad (30)$$

In the following we are going to prove necessary optimality conditions for (SP_2) in terms of contingent derivative.

In the sequel the couple (F, G) is a set-valued map from X into $Y \times Z$ defined by

$$(F, G)(x) = (F(x), G(x)). \quad (31)$$

Let $\bar{z} \in G(\bar{x}) \cap (-K_Z)$ and we consider the following problem (SP_3) with respect to $K_Y \times (K_Z + \bar{z})$:

$$\begin{aligned} & \text{l-minimize} && (F, G)(x), \\ & \text{subject to} && x \in M. \end{aligned} \quad (SP_3)$$

The following result compares the set of strict l-w minimum solution of (SP_2) to the set of strict l-w minimum solution of (SP_3) .

Proposition 18. *If \bar{x} is a strict l-w minimal solution of (SP_2) then for all $\bar{z} \in G(\bar{x}) \cap (-K_Z)$, \bar{x} is a strict l-w minimal solution of (SP_3) with respect to $K_Y \times (K_Z + \bar{z})$,*

Proof. Suppose the contrary; then, for every neighbourhood U of \bar{x} , there exists $x \in U \cap M$ such that

$$(F, G)(\bar{x}) \subset (F, G)(x) + \text{int}(K_Y \times K_Z + \bar{z}), \quad (32)$$

then,

$$\begin{aligned} F(\bar{x}) & \subset F(x) + \text{int}(K_Y), \\ G(\bar{x}) & \subset G(x) + \text{int}(K_Z + \bar{z}), \end{aligned} \quad (33)$$

and since $\bar{z} \in G(\bar{x})$, we get

$$\begin{aligned} F(\bar{x}) & \subset F(x) + \text{int}(K_Y), \\ 0 & \in G(x) + K_Z. \end{aligned} \quad (34)$$

Thus for every neighbourhood U of \bar{x} there exists $x \in U \cap M$ such that

$$\begin{aligned} F(x) & <^l F(\bar{x}), \\ G(x) \cap (-K_Z) & \neq \emptyset. \end{aligned} \quad (35)$$

This contradicts \bar{x} is a strict l-w minimal solution of (SP_2) . \square

Let us formulate necessary conditions for the problem (SP_2) . In the sequel we consider the following subset of M :

$$S = \{x \in M \mid G(x) \cap (-K_Z) \neq \emptyset\}. \quad (36)$$

Theorem 19. *Let \bar{x} be a strict l-w minimum solution of (SP_2) and $\bar{z} \in G(\bar{x}) \cap (-K_Z)$. If $F(\bar{x})$ has the K_Y -w minimal property and $W \min(F(\bar{x}), K_Y) = \{\bar{y}\}$, then*

$$D(F, G)(\bar{x}, \bar{y}, \bar{z})(x) \cap (-\text{int}(K_Y \times (K_Z + \bar{z}))) = \emptyset, \quad \forall x \in M. \quad (37)$$

Proof. Suppose the contrary; then there exist $x \in M$ and $(y, z) \in Y \times Z$ such that

$$(y, z) \in D(F, G)(\bar{x}, \bar{y}, \bar{z})(x) \cap (-\text{int}(K_Y \times (K_Z + \bar{z}))), \quad (38)$$

and hence there exist $(t_n)_{n \in \mathbb{N}} \rightarrow 0^+$ and $(x_n, y_n, z_n) \rightarrow (x, y, z)$ such that

$$t_n(y_n, z_n) \in (F, G)(\bar{x} + t_n x_n) - (\bar{y}, \bar{z}), \quad \forall n \in \mathbb{N}. \quad (39)$$

On the other hand, $(y, z) \in -\text{int}(K_Y \times (K_Z + \bar{z}))$, then there exists $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} t_n y_n &\in -\text{int}(K_Y), \\ z_n &\in -\text{int}(K_Z + \bar{z}), \\ \forall n &\geq n_1, \end{aligned} \quad (40)$$

and hence

$$\begin{aligned} t_n y_n &\in -\text{int}(K_Y), \\ \bar{z} + t_n z_n &\in -K_Z + (1 - t_n)\bar{z}, \\ \forall n &\geq n_1, \end{aligned} \quad (41)$$

as $t_n \rightarrow 0^+$ there exists $n_2 \in \mathbb{N}$ such that $1 - t_n > 0$ for every $n \geq n_2$, then $(1 - t_n)\bar{z} \in -K_Z$. Let $N = \max(n_1, n_2)$, by (39) we get

$$\begin{aligned} [F(\bar{x} + t_n x_n) - \bar{y}] \cap (-\text{int}(K_Y)) &\neq \emptyset, \\ G(\bar{x} + t_n x_n) \cap (-K_Z) &\neq \emptyset \\ \forall n &\geq N. \end{aligned} \quad (42)$$

From hypothesis we have \bar{x} is a strict l-w minimum solution of (SP_2) ; then there exists a neighbourhood U of \bar{x} such that $F(x) \not\prec^l F(\bar{x})$ for all $x \in U \cap S$. Since $\bar{x} + t_n x_n \rightarrow \bar{x}$ then there exists $n_0 \in \mathbb{N}$ such that $\bar{x} + t_n x_n \in U \cap M$ for all $n \geq n_0$; thus $\bar{x} + t_n x_n \in U \cap S$ for all $n \geq \max(n_0, N)$; hence,

$$F(\bar{x} + t_n x_n) \not\prec^l F(\bar{x}) \quad \forall n \geq \max(n_0, N), \quad (43)$$

and, by Lemma 12 and hypothesis, we get

$$\begin{aligned} [F(\bar{x} + t_n x_n) - \bar{y}] \cap (-\text{int}(K_Y)) &= \emptyset \\ \forall n &\geq \max(n_0, N). \end{aligned} \quad (44)$$

This contradicts (42). \square

As an immediate consequence we have the following corollary.

Corollary 20. *Let \bar{x} be an l-w minimum of (SP_2) and $\bar{z} \in G(\bar{x}) \cap (-K_Z)$. Let $\bar{y} \in F(\bar{x})$. Let us suppose that there exists a neighbourhood U of \bar{x} such that for each $x \in U \cap S$ one of the following conditions is satisfied:*

$$(a) \bar{y} \notin F(x) + \text{int}(K_Y) \text{ or}$$

$$(b) \bar{y} \in W \min(F(x), K_Y).$$

Then

- (i) \bar{x} is a strict l-w minimum of (SP_2) ;
- (ii) $D(F, G)(\bar{x}, \bar{y}, \bar{z})(x) \cap (-\text{int}(K_Y \times (K_Z + \bar{z}))) = \emptyset$, for all $x \in M$.

4. Sufficient Optimality Conditions

It is well known from vector optimization that we can derive sufficient condition under Γ -contingently quasiconvex assumptions. Next, we establish sufficient condition with similar assumptions for set optimization. The following terminology is used. Let Y^* denote the dual space of Y , and let

$$K_Y^+ = \{u \in Y^* \mid u(y) \geq 0, \forall y \in K_Y\} \quad (45)$$

denote the nonnegative dual cone of K_Y .

The next theorem provides a sufficient condition for the l-w minimum solution of (SP_2) .

Theorem 21. *Let $(\bar{x}, \bar{y}, \bar{z}) \in \text{Gr}(F, G)$ and $M - \bar{x} \subset \text{dom}[D(F, G)(\bar{x}, \bar{y}, \bar{z})]$. Assume that there are $u \in K_Y^+ \setminus \{0_{Y^*}\}$ and $v \in K_Z^+$ such that $v(\bar{z}) = 0$ and*

$$\begin{aligned} u(y) + v(z) &\geq 0, \\ \text{for every } (y, z) &\in D(F + K_Y, G + K_Z)(\bar{x}, \bar{y}, \bar{z})(x - \bar{x}) \quad \forall x \in M. \end{aligned} \quad (46)$$

If $(F + K_Y, G + K_Z) : \widehat{M} \Rightarrow Y \times Z$ is Γ -contingently quasiconvex at $(\bar{x}, \bar{y}, \bar{z})$ with

$$\begin{aligned} \widehat{M} &= \{x \in M \mid G(x) \cap (-K_Z + \text{cone}(\bar{z}) - \text{cone}(\bar{z})) \\ &\neq \emptyset\}, \end{aligned} \quad (47)$$

$$\Gamma = (-\text{int}(K_Y)) \times (-K_Z + \text{cone}(\bar{z}) - \text{cone}(\bar{z})),$$

then \bar{x} is an l-w minimal solution of (SP_2) on \widehat{M} .

Proof. Let us show that, for every $x \in \widehat{M}$,

$$D(F + K_Y, G + K_Z)(\bar{x}, \bar{y}, \bar{z})(x - \bar{x}) \cap \Gamma = \emptyset. \quad (48)$$

Assume the contrary; then there exist $x' \in \widehat{M}$ and $(y', z') \in Y \times Z$ such that

$$\begin{aligned} (y', z') &\in D(F + K_Y, G + K_Z)(\bar{x}, \bar{y}, \bar{z})(x' - \bar{x}), \\ y' &\in -\text{int}(K_Y), \\ z' &\in -K_Z + \text{cone}(\bar{z}) - \text{cone}(\bar{z}). \end{aligned} \quad (49)$$

Since $u \in K_Y^+ \setminus \{0_{Y^*}\}$, $v \in K_Z^+$, and $v(\bar{z}) = 0$, we have

$$u(y') + v(z') < 0. \quad (50)$$

This contradicts (46).

On the other hand, we have that $(F + K_Y, G + K_Z)$ is Γ -contingently quasiconvex at $(\bar{x}, \bar{y}, \bar{z})$; thus (48) ensures that there is no $x \in \widehat{M}$ such that

$$(F(x) + K_Y, G(x) + K_Z) - (\bar{y}, \bar{z}) \cap \Gamma \neq \emptyset, \quad (51)$$

that is

$$\begin{aligned} (F(x) + K_Y - \bar{y}) \cap (-\text{int}(K_Y)) &\neq \emptyset, \\ (G(x) + K_Z - \bar{z}) \cap (-K_Z + \text{cone}(\bar{z}) - \text{cone}(\bar{z})) &\neq \emptyset. \end{aligned} \quad (52)$$

Hence for every $x \in \widehat{M}$ there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(x) - \bar{y}) \cap (-\text{int}(K_Y)) = \emptyset, \quad (53)$$

by Proposition 3, and we deduce that \bar{x} is a l-w minimal solution of (SP_2) on \widehat{M} . \square

As an immediate consequence we have the following corollary.

Corollary 22. *Under the setting of Theorem 21, if the map $(F, G) : \widehat{M} \rightrightarrows Y \times Z$ is $K_Y \times K_Z$ -pseudoconvex at $(\bar{x}, \bar{y}, \bar{z})$ then \bar{x} is a l-w minimal solution of (SP_2) on \widehat{M} .*

From Theorem 21, we obtain the following sufficient optimality condition for (SP_1) .

Theorem 23. *Let $(\bar{x}, \bar{y}) \in \text{Gr}(F)$ and $M - \bar{x} \subset \text{dom}[DF(\bar{x}, \bar{y})]$. Assume that there exists $u \in K_Y^+ \setminus \{0_{Y^*}\}$ such that*

$$\begin{aligned} u(y) &\geq 0, \\ \text{for every } y \in D(F + K_Y)(\bar{x}, \bar{y})(x - \bar{x}) &\quad \forall x \in M. \end{aligned} \quad (54)$$

If $F + K_Y : M \rightrightarrows Y$ is Γ -contingently quasiconvex at (\bar{x}, \bar{y}) with, $\Gamma = -\text{int}(K_Y)$, then \bar{x} is an l-w minimal solution of (SP_1) .

The following two corollaries of the above result are immediate.

Corollary 24. *Let $(\bar{x}, \bar{y}) \in \text{Gr}(F)$ and $M - \bar{x} \subset \text{dom}[DF(\bar{x}, \bar{y})]$. Assume that*

$$\begin{aligned} D(F + K_Y)(\bar{x}, \bar{y})(x - \bar{x}) \cap (-\text{int}(K_Y)) &= \emptyset, \\ \forall x \in M. \end{aligned} \quad (55)$$

If $F + K_Y : M \rightrightarrows Y$ is Γ -contingently quasiconvex at (\bar{x}, \bar{y}) with, $\Gamma = -\text{int}(K_Y)$, then \bar{x} is an l-w minimal solution of (SP_1) .

Corollary 25. *Let $(\bar{x}, \bar{y}) \in \text{Gr}(F)$ and $M - \bar{x} \subset \text{dom}[DF(\bar{x}, \bar{y})]$. Assume that*

$$\begin{aligned} D(F + K_Y)(\bar{x}, \bar{y})(x - \bar{x}) \cap (-\text{int}(K_Y)) &= \emptyset, \\ \forall x \in M. \end{aligned} \quad (56)$$

If $F : M \rightrightarrows Y$ is K_Y -pseudoconvex at (\bar{x}, \bar{y}) , then \bar{x} is an l-w minimal solution of (SP_1) .

5. Conclusions

This paper deals with a set-valued optimization problem which involves a set-valued objective and set-valued constraints. Since such problems involve set-valued maps, optimality conditions are often given using various notions of set-valued derivatives. In this paper, we use the notion of the so-called S -derivative (and also the contingent derivative) to give necessary optimality conditions for the considered problems. For the sufficient optimality conditions, certain generalized notion of convexity is employed.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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