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Research Article

Bounds on the Spectral Radius of a Nonnegative Matrix and Its Applications

Danping Huang and Lihua You

School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China

Correspondence should be addressed to Lihua You; ylhua@scnu.edu.cn

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We obtain the sharp bounds for the spectral radius of a nonnegative matrix and then obtain some known results or new results by applying these bounds to a graph or a digraph and revise and improve two known results.

1. Introduction

First we recall some basic definitions and notations that will be used in this paper. Let A be an $n \times n$ real matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Since A is not symmetric in general, the eigenvalues may be complex numbers. Without loss of generality, we assume that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, and then the spectral radius of A is defined as $\rho(A) = |\lambda_1|$; that is, it is the largest modulus of the eigenvalues of A. By the Perron-Frobenius theorem, we have the following: (1) $\rho(A)$ is an eigenvalue of A if A is a nonnegative matrix; (2) $\rho(A) = \lambda_1$ is simple if A is a nonnegative irreducible matrix.

Let $G = (V, E)$ ($\vec{G} = (V, E)$) be a graph (digraph) with vertex set $V = V(G)$ ($= V(\vec{G})$) $= \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$ (arc set $E = E(\vec{G})$). A graph G (digraph $\vec{G}$) is simple if it has no loops and multiple edges (arcs). For any pairs of vertices $v_i, v_j \in V$, if there is a (directed) path from v_i to v_j, the graph G (digraph $\vec{G}$) is called (strongly) connected. In this paper, we consider finite, simple graphs and digraphs.

Let G be a graph and $\mathrm{diag}\,(G) = \mathrm{diag}\,(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees of G, where d_i is the degree of vertex v_i.

Let $\vec{G}$ be a digraph; $N^-_{\vec{G}}(v_i) = \{v_j \in V(\vec{G}) \mid (v_j, v_i) \in E(\vec{G})\}$ and $N^+_{\vec{G}}(v_i) = \{v_j \in V(\vec{G}) \mid (v_i, v_j) \in E(\vec{G})\}$ denote the in-neighbors and out-neighbors of v_i, respectively. Let $d^-_i = |N^-_{\vec{G}}(v_i)|$ and $d^+_i = |N^+_{\vec{G}}(v_i)|$ denote the indegree and outdegree of the vertex v_i in $\vec{G}$, respectively, and $\mathrm{diag}\,(\vec{G}) = \mathrm{diag}\,(d^+_1, d^+_2, \ldots, d^+_n)$ be the diagonal matrix of the vertex outdegrees of $\vec{G}$.

Let $A(G) = (a_{ij})$ be the $(0, 1)$ adjacency matrix of G, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let $A(\vec{G}) = (a_{ij})$ denote the adjacency matrix of $\vec{G}$, where a_{ij} is equal to the number of arcs (v_i, v_j).

Then the signless Laplacian matrix of G ($\vec{G}$) is defined as

$$Q(G) = \mathrm{diag}\,(G) + A(G)$$
$$\left(Q\left(\vec{G}\right) = \mathrm{diag}\left(\vec{G}\right) + A\left(\vec{G}\right)\right). \tag{2}$$

The spectral radii of $A(G)$ and $Q(G)$ ($A(\vec{G})$ and $Q(\vec{G})$), denoted by $\rho(G)$ and $q(G)$ ($\rho(\vec{G})$ and $q(\vec{G})$), are called the (adjacency) spectral radius of G ($\vec{G}$) and the signless Laplacian spectral radius of G ($\vec{G}$), respectively.

Let $G = (V, E)$ be a connected graph and $\vec{G} = (V, E)$ be a strong connected digraph. For $u, v \in V$, the distance from u to v, denoted by $d_G(u, v)$ ($d_{\vec{G}}(u, v)$), is the length of the shortest (directed) path from u to v in G ($\vec{G}$). For $u \in V$, the transmission of vertex u in G ($\vec{G}$) is the sum of distances from u to all other vertices of G ($\vec{G}$), denoted by $\mathrm{Tr}_G(u)$ ($\mathrm{Tr}_{\vec{G}}(u)$).

The distance matrix of G ($\vec{G}$) is the $n \times n$ matrix $\mathscr{D}(G) = (d_{ij})$, where $d_{ij} = d_G(v_i, v_j)$ ($\mathscr{D}(\vec{G}) = (d_{ij})$, where $d_{ij} = d_{\vec{G}}(v_i, v_j)$). In fact, for $1 \le i \le n$, the transmission of vertex v_i, $\mathrm{Tr}_G(v_i)$ ($\mathrm{Tr}_{\vec{G}}(v_i)$), is just the ith row sum of $\mathscr{D}(G)$ ($\mathscr{D}(\vec{G})$). For convenience, we also call $\mathrm{Tr}_G(v_i)$ ($\mathrm{Tr}_{\vec{G}}(v_i)$) the distance degree (outdegree) of vertex v_i in G ($\vec{G}$), denoted by D_i (D_i^+); that is, $D_i = \sum_{j=1}^n d_{ij} = \mathrm{Tr}_G(v_i)$ ($D_i^+ = \sum_{j=1}^n d_{ij} = \mathrm{Tr}_{\vec{G}}(v_i)$). Similarly, we define $D_i^- = \sum_{j=1}^n d_{ji}$.

Let $\mathrm{Tr}(G) = \mathrm{diag}\,(D_1, D_2, \ldots, D_n)$ be the diagonal matrix of vertex transmissions of G, and let $\mathrm{Tr}(\vec{G}) = \mathrm{diag}\,(D_1^+, D_2^+, \ldots, D_n^+)$ be the diagonal matrix of vertex transmissions of $\vec{G}$. The distance signless Laplacian matrix of G ($\vec{G}$) is the $n \times n$ matrix defined by Aouchiche and Hansen as [1]

$$\mathscr{Q}(G) = \mathrm{Tr}(G) + \mathscr{D}(G)$$
$$\left(\mathscr{Q}\left(\vec{G}\right) = \mathrm{Tr}\left(\vec{G}\right) + \mathscr{D}\left(\vec{G}\right)\right). \tag{3}$$

The spectral radii of $\mathscr{D}(G)$ and $\mathscr{Q}(G)$ ($\mathscr{D}(\vec{G})$ and $\mathscr{Q}(\vec{G})$), denoted by $\rho^{\mathscr{D}}(G)$ and $q^{\mathscr{D}}(G)$ ($\rho^{\mathscr{D}}(\vec{G})$ and $q^{\mathscr{D}}(\vec{G})$), are called the distance spectral radius of G ($\vec{G}$) and the distance signless Laplacian spectral radius of G ($\vec{G}$), respectively.

Let G be a connected graph. The reciprocal distance matrix (also called the Harary matrix) $R(G) = (r_{ij})$ of G is the $n \times n$ matrix, where $(r_{ij}) = 1/d_{ij}$ if $i \ne j$ and $r_{ii} = 0$ for $i = 1, \ldots, n$. Clearly, the reciprocal distance matrix $R(G)$ is nonnegative and symmetric.

Let G be a graph and $\vec{G}$ be a digraph; we call G ($\vec{G}$) regular if each vertex of G ($\vec{G}$) has the same degree (outdegree). Other definitions, terminology, and notations not in the article can be found in [2–4].

In recent decades, there are many results on the bounds of the spectral radius of a nonnegative matrix and the various spectral radii of a graph or a digraph, including the spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance signless Laplacian spectral radius, and the spectral radius of the reciprocal distance matrix; see [5–16] and so on.

In this paper, we obtain the sharp bounds for the spectral radius of a nonnegative (irreducible) matrix in Section 2 and then obtain some known results or new results by applying these bounds to a graph in Section 3 or a digraph in Section 4; we revise and improve two known results.

2. Main Results

In this section, we will obtain the sharp bounds for the spectral radius of a nonnegative (irreducible) matrix and revise and improve the result of Theorem 2.9 in [9]. The techniques used in this section are motivated by [7, 9, 14] and so on.

Lemma 1 (see [2]). *If A is an $n \times n$ nonnegative matrix with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \ldots, r_n$, then $\min_{1\le i\le n} r_i \le \lambda(A) \le \max_{1\le i\le n} r_i$. Moreover, if A is irreducible, then one of the equalities holds if and only if the row sums of A are all equal.*

Theorem 2. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with row sums $r_1, r_2, \ldots, r_n$, where $r_1 \ge r_2 \ge \cdots \ge r_n$, and let S be the smallest diagonal element, T be the smallest nondiagonal element, and $\lambda(A)$ be the spectral radius of A. Take $\phi_1 = r_n$ and for $2 \le l \le n$,*

$$\phi_l = \frac{r_n + S - T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2}. \tag{4}$$

Let $\phi_t = \max_{1\le l\le n}\{\phi_l\}$ for some $1 \le t \le n$. Then $\lambda(A) \ge \phi_t$. Moreover, if A is irreducible, then

(1) *$\lambda(A) = \phi_1 = r_n$ if and only if $r_1 = r_2 = \cdots = r_n$.*

(2) *$\lambda(A) = \phi_t > r_n$ with $2 \le t \le n$ if and only if A satisfies the following conditions:*

(i) *$a_{ii} = S$ for $1 \le i \le t-1$;*

(ii) *$a_{ij} = T > 0$ for $1 \le i \le n$, $1 \le j \ne i \le t-1$;*

(iii) *$r_1 = r_2 = \cdots = r_{t-1} > r_t = r_{t+1} = \cdots = r_n$.*

Proof. If $T = 0$, then $\phi_l = \phi_1 = r_n$ for any $2 \le l \le n$ by $r_n \ge S$. Thus by Lemma 1 and $r_1 \ge r_2 \ge \cdots \ge r_n$, we have $\lambda(A) \ge r_n = \max_{1\le l\le n}\{\phi_l\} = \phi_1$, and if A is irreducible, $\lambda(A) = \phi_1 = r_n$ if and only if $r_1 = r_2 = \cdots = r_n$.

Now we consider the case $T > 0$.

Firstly, we show $\lambda(A) \ge \phi_l$ for all $2 \le l \le n$.

Since A is a nonnegative matrix, then $a_{p,q} \ge T > 0$ for $1 \le p \ne q \le n$. Thus

$$\sum_{j=1}^{l-1} a_{ij} \ge \begin{cases} S + (l-2)T, & \text{if } 1 \le i \le l-1; \\ (l-1)T, & \text{if } l \le i \le n. \end{cases} \tag{5}$$

Let

$$x = \frac{S - r_n + (2l-3)T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2(l-1)T}. \tag{6}$$

It is easy to show that $x > 1$. Take

$$x_j = \begin{cases} x, & \text{if } 1 \le j \le l-1, \\ 1, & \text{if } l \le j \le n, \end{cases} \tag{7}$$

and let $\mathbf{U} = \mathrm{diag}\,(x_1, x_2, \ldots, x_n)$ be a diagonal matrix of order n. Let $B = \mathbf{U}^{-1}A\mathbf{U}$, and then B and A have the same eigenvalues, and $\lambda(B) = \lambda(A)$.

Now we consider the row sums of B, say, $s_1, s_2, \ldots, s_n$.

Case 1 ($1 \le i \le l-1$). Consider

$$s_i = \sum_{j=1}^{n} \frac{x_j}{x_i} a_{ij} = \sum_{j=1}^{l-1} a_{ij} + \frac{1}{x}\sum_{j=l}^{n} a_{ij}$$
$$= \frac{1}{x}\sum_{j=1}^{n} a_{ij} + \left(1 - \frac{1}{x}\right)\sum_{j=1}^{l-1} a_{ij} = \frac{1}{x} r_i + \left(1 - \frac{1}{x}\right)\sum_{j=1}^{l-1} a_{ij}$$
$$\ge \frac{1}{x} r_i + \left(1 - \frac{1}{x}\right)[S + (l-2)T]$$

$$= \frac{1}{x}(r_i - S) + S + \left(1 - \frac{1}{x}\right)(l-2)T$$
$$\geq \frac{1}{x}(r_{l-1} - S) + S + \left(1 - \frac{1}{x}\right)(l-2)T, \tag{8}$$

with equality if and only if (a) and (b) hold: (a) $a_{ii} = S$ and $a_{ij} = T$ if $1 \leq j \leq l-1$ with $j \neq i$ and (b) $r_i = r_{l-1}$.

Case 2 ($l \leq i \leq n$). Consider

$$\begin{aligned} s_i &= \sum_{j=1}^{n} \frac{x_j}{x_i} a_{ij} = x \sum_{j=1}^{l-1} a_{ij} + \sum_{j=l}^{n} a_{ij} \\ &= \sum_{j=1}^{n} a_{ij} + (x-1) \sum_{j=1}^{l-1} a_{ij} = r_i + (x-1) \sum_{j=1}^{l-1} a_{ij} \\ &\geq r_i + (x-1)(l-1)T \geq r_n + (x-1)(l-1)T, \end{aligned} \tag{9}$$

with equality if and only if (c) and (d) hold: (c) $a_{ij} = T$ if $1 \leq j \leq l-1$ and (d) $r_i = r_n$.

Noting that

$$\begin{aligned} &r_n + (x-1)(l-1)T \\ &= \frac{1}{x}(r_{l-1} - S) + S + \left(1 - \frac{1}{x}\right)(l-2)T \\ &= \frac{S + r_n - T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2} \\ &= \phi_l, \end{aligned} \tag{10}$$

then, by Lemma 1, we have $\lambda(A) = \lambda(B) \geq \min\{s_1, s_2, \ldots, s_n\} \geq \phi_l$.

Noting that $\phi_l \geq \phi_1 = r_n$ by $r_n + T \geq S$, thus $\lambda(A) \geq \phi_t$, where $\phi_t = \max_{1 \leq l \leq n}\{\phi_l\}$ for some $1 \leq t \leq n$.

Let A be irreducible; $\phi_t = \max_{1 \leq l \leq n}\{\phi_l\}$ for some $1 \leq t \leq n$.

Case 1 ($\lambda(A) = \phi_1$). For $2 \leq l \leq n$, by $\phi_l \geq \phi_1$ and $T > 0$, we have $\phi_l = \phi_1 \Leftrightarrow r_{l-1} = r_n$. Then

$$\phi_t = \phi_1 \Longleftrightarrow \phi_l = \phi_1 \quad \forall 2 \leq l \leq n \Longleftrightarrow r_1 = r_2 = \cdots = r_n. \tag{11}$$

On the other hand, by Lemma 1 and $r_1 \geq r_2 \geq \cdots \geq r_n$, we have

$$\lambda(A) = r_n \Longleftrightarrow r_1 = r_2 = \cdots = r_n. \tag{12}$$

By (11), (12), and $\phi_1 = r_n$, (1) holds.

Case 2 ($\lambda(A) = \phi_t > \phi_1$ for some $2 \leq t \leq n$). Then $r_{t-1} > r_n$ and $T > 0$ by $\phi_t > \phi_1 = r_n$.

If $\lambda(A) = \phi_t$, then $s_1 = s_2 = \cdots = s_n = \phi_t$ by the above arguments and Lemma 1; thus (a) and (b) hold for $1 \leq i \leq t-1$ and (c) and (d) hold for $t \leq i \leq n$. Thus $a_{ii} = S$ for $1 \leq i \leq t-1$, $r_1 = r_2 = \cdots = r_{t-1} > r_t = r_{t+1} = \cdots = r_n$ and $a_{ij} = T > 0$ for $1 \leq i \leq n, 1 \leq j \neq i \leq t-1$. Now (i), (ii), and (iii) follow.

Conversely, if (i), (ii), and (iii) hold, it is easy to show that equality holds. □

Corollary 3. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with row sums $r_1, r_2, \ldots, r_n$, where $r_1 \geq r_2 \geq \cdots \geq r_n$, and let S be the smallest diagonal element, T be the smallest nondiagonal element, and $\lambda(A)$ be the spectral radius of A. Take $\phi_1 = r_n$ and, for $2 \leq l \leq n$,*

$$\phi_l = \frac{r_n + S - T + \sqrt{(r_n + T - S)^2 + 4(l-1)(r_{l-1} - r_n)T}}{2}. \tag{13}$$

Let $\phi_t = \max_{1 \leq l \leq n}\{\phi_l\}$ for some $1 \leq t \leq n$. Then $\lambda(A) \geq \phi_t$. Moreover, if A is irreducible with $T = 0$ or A is irreducible and symmetric, then

$$\lambda(A) = \phi_t \quad \textit{iff } t = 1, \; r_1 = r_2 = \cdots = r_n. \tag{14}$$

Proof. We complete the proof by the following two cases.

Case 1 ($T = 0$). It is obvious by the proof of Theorem 2.

Case 2 (A is symmetric and $T > 0$). By (i) and (ii), A is symmetric and T is the smallest nondiagonal element. We have $r_1 = r_2 = \cdots = r_{t-1} = S + (n-1)T < r_t = \cdots = r_n$. It is a contradiction by the fact $r_{t-1} \geq r_t$. □

Similar to the proof of Theorem 2 (so we omit the proof of Theorem 4), we can show Theorem 4 which revises and improves the result of Theorem 2.9 in [9].

Theorem 4. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with row sums $r_1, r_2, \ldots, r_n$, where $r_1 \geq r_2 \geq \cdots \geq r_n$, and let M be the largest diagonal element, N be the largest nondiagonal element, and $\lambda(A)$ be the spectral radius of A. Take $\phi_1 = r_1$ and, for $2 \leq l \leq n$,*

$$\phi_l = \frac{r_l + M - N + \sqrt{(r_l + N - M)^2 + 4(l-1)(r_1 - r_l)N}}{2}. \tag{15}$$

Let $\phi_t = \min_{1 \leq l \leq n}\{\phi_l\}$ for some $1 \leq t \leq n$. Then $\lambda(A) \leq \phi_t$. Moreover, if A is irreducible, then

(1) *$\lambda(A) = \phi_1 = r_1$ if and only if $r_1 = r_2 = \cdots = r_n$.*
(2) *$\lambda(A) = \phi_t < r_1$ with $2 \leq t \leq n$ if and only if A satisfies the following conditions:*
  (i) *$a_{ii} = M$ for $1 \leq i \leq t-1$;*
  (ii) *$a_{ij} = N > 0$ for $1 \leq i \leq n, 1 \leq j \neq i \leq t-1$;*
  (iii) *$r_1 = r_2 = \cdots = r_{t-1} > r_t = r_{t+1} = \cdots = r_n$.*

3. Various Spectral Radii of a Graph

Let G be a graph. In Section 1, the (adjacency) matrix $A(G)$, the signless Laplacian matrix $Q(G)$, the distance matrix $\mathcal{D}(G)$ (if G is connected), the distance signless Laplacian matrix $\mathcal{Q}(G)$ (if G is connected), the reciprocal distance matrix $R(G)$ (if G is connected), the (adjacency) spectral radius $\rho(G)$, the signless Laplacian spectral radius $q(G)$, the distance spectral radius $\rho^{\mathscr{D}}(G)$, the distance signless Laplacian spectral radius $q^{\mathscr{D}}(G)$, and the spectral radius of the reciprocal distance matrix $\lambda(R(G))$ are defined. Now, in this section, we will apply Theorem 2, Corollary 3, and Theorem 4 to $A(G)$, $Q(G)$, $\mathscr{D}(G)$, $\mathscr{Q}(G)$, and $R(G)$ and obtain some new results or known results.

3.1. Adjacency Spectral Radius of a Graph. Let G be a graph. By applying Corollary 3 and Theorem 4 to the (adjacency) matrix $A(G)$ with $S = 0$, $T = 0$, $M = 0$, $N = 1$, and $r_i = d_i$ for any $1 \le i \le n$, we have the following.

Corollary 5. *Let G be a graph on n vertices with degree sequence $d_1, d_2, \ldots, d_n$, where $d_1 \ge d_2 \ge \cdots \ge d_n$. Then one has*

$$d_n \le \rho(G) \le \min_{1\le i\le n} \left\{ \frac{d_i - 1 + \sqrt{(d_i+1)^2 + 4(i-1)(d_1 - d_i)}}{2} \right\}. \tag{16}$$

Moreover, if G is connected, then the left equality holds if and only if G is a regular graph, the right equality holds if and only if G is a regular graph, or there exists some t with $2 \le t \le n$ such that G is a bidegreed graph with $d_1 = \cdots = d_{t-1} = n-1 > d_t = \cdots = d_n$.

Remark 6. The left inequality in Corollary 5 can be obtained by Lemma 1 immediately, and the right inequality in Corollary 5 is the result of Theorem 2.2 in [13].

3.2. Signless Laplacian Spectral Radius of a Graph. Let G be a graph. By applying Corollary 3 and Theorem 4 to the signless Laplacian matrix $Q(G)$ with $S = d_n$, $T = 0$, $M = d_1$, $N = 1$, and $r_i = 2d_i$ for any $1 \le i \le n$, we have the following.

Corollary 7. *Let G be a graph on n vertices with degree sequence $d_1, d_2, \ldots, d_n$, where $d_1 \ge d_2 \ge \cdots \ge d_n$. Then one has*

$$2d_n \le q(G) \le \min_{1\le i\le n} \left\{ \frac{d_1 + 2d_i - 1 + \sqrt{(2d_i - d_1 + 1)^2 + 8(i-1)(d_1 - d_i)}}{2} \right\}. \tag{17}$$

Moreover, if G is connected, then the left equality holds if and only if G is a regular graph, the right equality holds if and only if G is a regular graph, or there exists some t with $2 \le t \le n$ such that G is a bidegreed graph in which $d_1 = \cdots = d_{t-1} = n-1 > d_t = \cdots = d_n$.

Remark 8. The left inequality in Corollary 7 can be obtained by Lemma 1 immediately, and the right inequality in Corollary 7 is the result of Theorem 3.2 in [15].

3.3. Distance Spectral Radius of a Graph. Let G be a connected graph and d be the diameter of G. Then the distance matrix $\mathscr{D}(G) = (d_{ij})$ is nonnegative and symmetric. By applying Corollary 3 and Theorem 4 to the distance matrix $\mathscr{D}(G)$ with $S = 0$, $T = 1$, $M = 0$, $N = d$, and $r_i = D_i$ for any $1 \le i \le n$, we note that $d_{21} = \cdots = d_{n1} = d$ implies a contradiction. Then we have the following.

Corollary 9. *Let G be a connected graph on n vertices and d be the diameter of G, with distance degree sequence $D_1, D_2, \ldots, D_n$ such that $D_1 \ge D_2 \ge \cdots \ge D_n$. Let*

$$f(i) = \frac{D_n - 1 + \sqrt{(D_n+1)^2 + 4(i-1)(D_{i-1} - D_n)}}{2}. \tag{18}$$

Then one has

$$\max_{2\le i\le n} \{D_n, f(i)\} \le \rho^{\mathscr{D}}(G) \le \min_{1\le i\le n} \left\{ \frac{D_i - d + \sqrt{(D_i+d)^2 + 4d(i-1)(D_1 - D_i)}}{2} \right\}. \tag{19}$$

Moreover, one of the equalities holds if and only if $D_1 = D_2 = \cdots = D_n$.

Remark 10. The right inequality in Corollary 9 is the result of Corollary 1.8 in [6].

By applying Theorem 2 and Corollary 3 to the distance matrix $\mathscr{D}(G)$ with $S = 0$, $T = 1$, and $r_i = D_i$ for $i = 1, 2, \ldots, n$, we have the following.

Corollary 11 (see [16, Theorem 2]). *Let G be a connected graph on n vertices with distance degree sequence $D_1, D_2, \ldots, D_n$ such that $D_1 \ge D_2 \ge D_{i-1} > D_i \ge \cdots \ge D_n$ for some $2 \le i \le n$. Then*

$$\rho^{\mathscr{D}}(G) > \frac{D_n - 1 + \sqrt{(D_n+1)^2 + 4(i-1)(D_{i-1} - D_n)}}{2}. \tag{20}$$

3.4. Distance Signless Laplacian Spectral Radius of a Graph. Let G be a connected graph and d be the diameter of G. Then the distance matrix $\mathscr{Q}(G)$ is nonnegative and symmetric. By applying Corollary 3 and Theorem 4 to the distance matrix $\mathscr{Q}(G)$ with $S = D_n$, $T = 1$, $M = D_1$, $N = d$, and $r_i = 2D_i$ for $i = 1, 2, \ldots, n$, we note that $d_{21} = \cdots = d_{n1} = d$ implies a contradiction. Then we have the following.

Corollary 12. *Let G be a connected graph on n vertices with distance degree sequence $D_1, D_2, \ldots, D_n$ such that $D_1 \geq D_2 \geq \cdots \geq D_n$ and d be the diameter of G. Let*

$$f(i) = \frac{3D_n - 1 + \sqrt{(D_n+1)^2 + 8(i-1)(D_{i-1} - D_n)}}{2},$$
$$g(i) = \frac{D_1 + 2D_i - d + \sqrt{(2D_i - D_1 + d)^2 + 8d(i-1)(D_1 - D_i)}}{2}. \tag{21}$$

Then one has

$$\max_{2\leq i\leq n}\{2D_n, f(i)\} \leq q^{\mathcal{D}}(G) \leq \min_{1\leq i\leq n}\{g(i)\}. \tag{22}$$

Moreover, one of the equalities holds if and only if $D_1 = D_2 = \cdots = D_n$.

Remark 13. The right inequality in Corollary 12 is the result of Theorem 3.8 in [9].

By applying Theorem 2 and Corollary 3 to the distance matrix $\mathcal{Q}(G)$ with $S = D_n$, $T = 1$, and $r_i = 2D_i$ for $i = 1, 2, \ldots, n$, we have the following.

Corollary 14. *Let G be a connected graph on n vertices with distance degree sequence $D_1, D_2, \ldots, D_n$ such that $D_1 \geq D_2 \geq D_{i-1} > D_i \geq \cdots \geq D_n$ for some $2 \leq i \leq n$. Then $q^{\mathcal{D}}(G) > f(i)$.*

3.5. Spectral Radius of the Reciprocal Distance Matrix. By applying Corollary 3 and Theorem 4 to the reciprocal distance matrix $R(G)$ with $S = 0$, $T = 1/d$, $M = 0$, $N = 1$, and $r_i = R_i$ for $i = 1, \ldots, n$, we have the following.

Corollary 15. *Let G be a connected graph on n vertices, d be the diameter of G, $R_i = \sum_{j=1}^{n} r_{ij}$, and the row sum sequence be $R_1, R_2, \ldots, R_n$ of $R(G)$ satisfying $R_1 \geq R_2 \geq \cdots \geq R_n$. Let*

$$f(i) = \frac{R_n - 1/d + \sqrt{(R_n + 1/d)^2 + (4/d)(i-1)(R_{i-1} - R_n)}}{2}, \tag{23}$$
$$g(i) = \frac{R_i - 1 + \sqrt{(R_i+1)^2 + 4(i-1)(R_1 - R_i)}}{2}.$$

Then

$$\max_{2\leq i\leq n}\{R_n, f(i)\} \leq \lambda(R(G)) \leq \min_{1\leq i\leq n}\{g(i)\}. \tag{24}$$

Moreover, the left equality holds if and only if $R_1 = R_2 = \cdots = R_n$, and the right equality holds if and only if either $R_1 = R_2 = \cdots = R_n$ or there exists some t with $2 \leq t \leq n$ such that G is a graph with $t-1$ vertices of degree $n-1$ and the remaining $n-t+1$ vertices have equal degree less than $n-1$.

Remark 16. The right inequality in Corollary 15 is the result (i) of Theorem 4 in [16].

4. Various Spectral Radii of a Digraph

Let $\vec{G}$ be a strong connected digraph. In Section 1, the adjacency matrix $A(\vec{G})$, the signless Laplacian matrix $Q(\vec{G})$, the distance matrix $\mathcal{D}(\vec{G})$ (if $\vec{G}$ is connected), the distance signless Laplacian matrix $\mathcal{Q}(\vec{G})$ (if $\vec{G}$ is connected), the adjacency spectral radius $\rho(\vec{G})$, the signless Laplacian spectral radius $q(\vec{G})$, the distance spectral radius $\rho^{\mathcal{D}}(\vec{G})$, and the distance signless Laplacian spectral radius $q^{\mathcal{D}}(\vec{G})$ are defined. Now, in this section, we will apply Theorem 2, Corollary 3, and Theorem 4 to $A(\vec{G})$, $Q(\vec{G})$, $\mathcal{D}(\vec{G})$, and $\mathcal{Q}(\vec{G})$, obtain some new results or known results, and revise and improve the result of Theorem 2.5 in [11].

4.1. Adjacency Spectral Radius of a Digraph. Let $\vec{G}$ be a digraph. By applying Corollary 3 and Theorem 4 to the (adjacency) matrix $A(\vec{G})$ with $S = 0, T = 0$, $M = 0$, $N = 1$, and $r_i = d_i^+$ for $i = 1, \ldots, n$, we have the following.

Corollary 17. *Let $\vec{G}$ be a digraph on n vertices with outdegree sequence $d_1^+, d_2^+, \ldots, d_n^+$ such that $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Then one has*

$$d_n^+ \leq \rho(\vec{G}) \leq \min_{1\leq i\leq n}\left\{\frac{d_i^+ - 1 + \sqrt{(d_i^+ + 1)^2 + 4(i-1)(d_1^+ - d_i^+)}}{2}\right\}. \tag{25}$$

Moreover, if $\vec{G}$ is a strong connected digraph, then the left equality holds if and only if $\vec{G}$ is a regular digraph, the right equality holds if and only if $\vec{G}$ is a regular digraph, or there exists some t with $2 \leq t \leq n$ such that $\vec{G}$ is a bidegreed digraph with $d_1^+ = \cdots = d_{t-1}^+ > d_t^+ = \cdots = d_n^+$ and the indegrees $d_1^- = \cdots = d_{t-1}^- = n-1$.

4.2. Signless Laplacian Spectral Radius of a Digraph. Let $\vec{G}$ be a digraph. By applying Corollary 3 and Theorem 4 to the signless Laplacian matrix $Q(\vec{G})$ with $S = d_n^+, T = 0$, $M = d_1^+, N = 1$, and $r_i = 2d_i^+$ for $i = 1, \ldots, n$, we have the following.

Corollary 18. *Let $\vec{G}$ be a digraph on n vertices with outdegree sequence $d_1^+, d_2^+, \ldots, d_n^+$ such that $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Then one has*

$$2d_n^+ \leq q(\vec{G}) \leq \min_{1\leq i\leq n}\left\{\frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8(i-1)(d_1^+ - d_i^+)}}{2}\right\}. \quad (26)$$

Moreover, if $\vec{G}$ is a strong connected digraph, then the left equality holds if and only if $\vec{G}$ is a regular digraph, the right equality holds if and only if $\vec{G}$ is a regular digraph, or there exists some t with $2 \leq t \leq n$ such that $\vec{G}$ is a bidegreed digraph with $d_1^+ = \cdots = d_{t-1}^+ > d_t^+ = \cdots = d_n^+$ and the indegrees $d_1^- = \cdots = d_{t-1}^- = n-1$.

Remark 19. The left inequality in Corollary 18 can be obtained by Lemma 1 immediately, and the right inequality in Corollary 18 revises and improves Proposition 20.

Proposition 20 (see [11, Theorem 2.5]). *Let $\vec{G}$ be a strong connected digraph on n vertices with outdegree sequence $d_1^+, d_2^+, \ldots, d_n^+$ such that $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Then one has*

$$q(\vec{G}) \leq \min_{1\leq i\leq n}\left\{\frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8(i-1)(d_1^+ - d_i^+)}}{2}\right\}. \quad (27)$$

Moreover, if $i = 1$, the equality holds if and only if $\vec{G}$ is a regular digraph. If $2 \leq i \leq n$, the equality holds if and only if $\vec{G}$ is a regular digraph or a bidegreed digraph in which $d_1^+ = d_2^+ = \cdots = d_{i-1}^+ = n-1$ and $d_i^+ = \cdots = d_n^+ = \delta^+$.

The following example shows that the result of Proposition 20 is incorrect.

Example 21. Let $n \geq 5$ and D_1 is shown in Figure 1. For D_1, the outdegree sequence is $3 = d_1^+ > d_2^+ = d_3^+ = \cdots = d_n^+ = 2$ and the indegree $d_1^- = n-1$. We have $q(D_1) = 3 + \sqrt{3}$ by direct computation. It is clear that

$$q(D_1) = 3 + \sqrt{3} = \min_{1\leq i\leq n}\left\{\frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8(i-1)(d_1^+ - d_i^+)}}{2}\right\}. \quad (28)$$

4.3. Distance Spectral Radius of a Digraph. Let $\vec{G}$ be a strong connected digraph and d be the diameter of $\vec{G}$. By applying Theorems 2 and 4 to the distance matrix $\mathscr{D}(\vec{G})$ with $S = 0, T = 1, M = 0, N = d$, and $r_i = D_i^+$ for $i = 1, \ldots, n$, we note that $d_{21} = \cdots = d_{n1} = d$ implies a contradiction. Then we have the following.

Corollary 22. *Let $\vec{G}$ be a strong connected digraph on n vertices with distance outdegree sequence $D_1^+, D_2^+, \ldots, D_n^+$ such that $D_1^+ \geq D_2^+ \geq \cdots \geq D_n^+$, and let d be the diameter of $\vec{G}$. Let*

$$f(i) = \frac{D_n^+ - 1 + \sqrt{(D_n^+ + 1)^2 + 4(i-1)(D_{i-1}^+ - D_n^+)}}{2},$$
$$g(i) = \frac{D_i^+ - d + \sqrt{(D_i^+ + d)^2 + 4d(i-1)(D_1^+ - D_i^+)}}{2}. \quad (29)$$

Then one has

$$\max_{2\leq i\leq n}\{D_n^+, f(i)\} \leq \rho^{\mathscr{D}}(\vec{G}) \leq \min_{1\leq i\leq n}\{g(i)\}. \quad (30)$$

Moreover, the left equality holds if and only if $D_1^+ = \cdots = D_n^+$ or there exists some t with $2 \leq t \leq n$ such that $D_1^+ = \cdots = D_{t-1}^+ > D_t^+ = \cdots = D_n^+$ and $D_1^- = \cdots = D_{t-1}^- = n-1$ and the right equality holds if and only if $D_1^+ = \cdots = D_n^+$.

4.4. Distance Signless Laplacian Spectral Radius of a Digraph. Let $\vec{G}$ be a strong connected digraph and d be the diameter of $\vec{G}$. By applying Theorems 2 and 4 to the distance signless Laplacian matrix $\mathscr{Q}(\vec{G})$ with $S = D_n^+, T = 1, M = D_1^+, N = d$, and $r_i = 2D_i^+$ for $i = 1, \ldots, n$, we note two facts: the first fact is that (i) and (iii) of (2) in Theorem 2 cannot hold at the same time by $a_{ii} = D_i^+ = \sum_{1\leq j\leq n} d_{ij}$ and $r_i = 2D_i^+$, and the second fact is that $d_{21} = \cdots = d_{n1} = d$ implies a contradiction. Then we have the following.

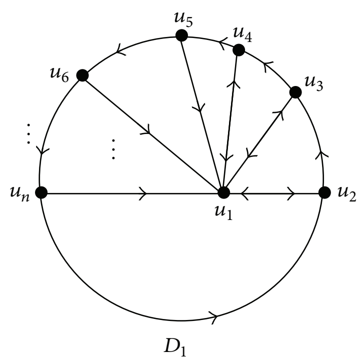

FIGURE 1: The digraphs D_1.

Corollary 23. *Let $\vec{G}$ be a strong connected digraph on n vertices with distance outdegree sequence $D_1^+, D_2^+, \ldots, D_n^+$ such that $D_1^+ \geq D_2^+ \geq \cdots \geq D_n^+$, and let d be the diameter of $\vec{G}$. Let*

$$f(i) = \frac{3D_n^+ - 1 + \sqrt{(D_n^+ + 1)^2 + 8(i-1)(D_{i-1}^+ - D_n^+)}}{2},$$
$$g(i) = \frac{D_1^+ + 2D_i^+ - d + \sqrt{(2D_i^+ - D_1^+ + d)^2 + 8d(i-1)(D_1^+ - D_i^+)}}{2}. \quad (31)$$

Then one has

$$\max_{2\leq i\leq n} \{D_n^+, f(i)\} \leq q^{\mathscr{D}}(\vec{G}) \leq \min_{1\leq i\leq n} \{g(i)\}. \quad (32)$$

Moreover, one of the equalities holds if and only if $D_1^+ = \cdots = D_n^+$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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