Research Article

Nonlinear Impulsive Differential Equations with Weighted Exponential or Ordinary Dichotomous Linear Part in a Banach Space

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We consider nonlinear impulsive differential equations with ψ-exponential and ψ-ordinary dichotomous linear part in a Banach space. By the help of Banach's fixed-point principle sufficient conditions are found for the existence of ψ-bounded solutions of these equations on $\mathbb{R}$ and $\mathbb{R}_+$.

1. Introduction

Impulsive differential equations are an adequate mathematical apparatus for simulation of numerous processes and phenomena in biology, physics, chemistry, control theory, and so forth which during their evolutionary development are subject to short time perturbations in the form of impulses. The qualitative investigation of these processes began with the work of Mil'man and Myshkis [1]. For the first time such equations were considered in an arbitrary Banach space in [2–5].

The problem of ψ-boundedness and ψ-stability of the solutions of differential equations in finite dimensional Euclidean spaces, introduced for the first time by Akinyele [6], has been studied since then by many authors. A beautiful explanation about the benefits of such a use of weighted stability and boundedness can be found, for example, in [7]. Inspired by the famous monographs of Coppel [8], Daleckii and Krein [9], and Massera and Schaeffer [10], where the important notion of exponential and ordinary dichotomy for ordinary differential equations is considered in detail, Diamandescu [11–13] and Boi [14, 15] introduced and studied the ψ-dichotomy for linear differential equations in a finite dimensional Euclidean space, where ψ is a nonnegative continuous diagonal matrix function. The concept of ψ-dichotomy for arbitrary Banach spaces was introduced and studied in [16, 17]. In this case $\psi(t)$ is an arbitrary bounded invertible linear operator.

A weighted dichotomy for linear differential equations with impulse effect in arbitrary Banach spaces is considered in [18] for a ψ-exponential dichotomy and in [19] for the particular case of ψ-ordinary dichotomy.

This paper considers nonlinear perturbed impulsive differential equations with a ψ-dichotomous liner part in an arbitrary Banach space. We will show that some properties of these equations will be influenced by the corresponding ψ-dichotomous impulsive homogeneous linear equation. Sufficient conditions for existence of ψ-bounded solutions of this equations on $\mathbb{R}$ and $\mathbb{R}_+$ in case of ψ-exponential or ψ-ordinary dichotomy are found.

2. Preliminaries

Let X be an arbitrary Banach space with norm $|\cdot|$ and identity Id. By J we will denote $\mathbb{R}$ or $\mathbb{R}_+ = [0, \infty)$ and by I either $\mathbb{Z}$ or $\mathbb{N} \cup \{0\}$.

We consider the nonlinear impulsive differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + F(t, x) \quad (t \neq t_n), \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) + H_n(x(t_n)) \quad (n \in I), \tag{2}$$

where $T = \{t_n\}_{n \in I}$ is a finite or infinite sequence in J. We will say that condition (H1) is satisfied if the following conditions hold:

(H 1.1) $A(t)$ $(t \in J)$ is a continuous operator-valued function with values in the Banach space $LB(X)$ of all linear bounded operators acting in X with the norm $\| \cdot \|$.

(H 1.2) $Q_n \in LB(X)$ $(n \in I)$.

(H 1.3) The function $F(t, x) : J \times X \rightarrow X$ is continuous with respect to t.

(H 1.4) $H_n : X \rightarrow X$ $(n \in I)$ are continuous operators.

(H 1.5) $t_n < t_{n+1}$ and $\lim_{n \rightarrow \pm\infty} t_n = \pm\infty$ $(n \in I)$.

Consider the corresponding linear impulsive equation

$$\frac{dx}{dt} = A(t)x \quad (t \neq t_n, \ n \in I), \tag{3}$$

$$x(t_n^+) = Q_n x(t_n) \quad (n \in I). \tag{4}$$

Definition 1. By a solution of the impulsive equation (1), (2) (or (3), (4)) we will call a function $x(t)$ which for $t \neq t_n$ satisfies (1) (or (3)) and for $t = t_n$ satisfies condition (2) (or (4)) and is continuous from the left.

As shown, for instance in [3, 4], if the operators Q_n have bounded inverse ones, then for the impulsive linear equation (3), (4) there exists a Cauchy operator $V(t)$, $(t \in J)$ by means of which each solution $x(t)$ of (3), (4) for which $x(s) = \xi \in X$ has the form

$$x(t) = V(t)V^{-1}(s)\xi \quad (t, s \in J). \tag{5}$$

Let $RL(X)$ be the subspace of all invertible operators in $LB(X)$ whose inverse operators are bounded too. Let $\psi(t) : J \rightarrow RL(X)$ be continuous with respect to $t \in J$ operator function.

Definition 2. A function $u(t) : J \rightarrow X$ is said to be ψ-bounded on J if $\psi(t)u(t)$ is bounded on J.

Let $C_\psi(X, T)$ be the space of all ψ-bounded on J functions with values in X which are continuous for $t \notin T$, have discontinuities of the first kind for $t \in T$, and are continuous from the left, which is a Banach space with the norm

$$\|\|f\|\|_{C_\psi} = \sup_{t \in J} |\psi(t)f(t)|. \tag{6}$$

Definition 3. The linear impulsive equation (3), (4) is said to have a ψ-exponential dichotomy on J if there exist a pair of mutually complementary projections P_1 and $P_2 = \text{Id} - P_1$ and positive constants N_1, N_2, ν_1, ν_2 such that

$$\left\| \psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s) \right\| \leq N_1 e^{-\nu_1(t-s)}$$
$$(s \leq t, \ s, t \in J),$$
$$\left\| \psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s) \right\| \leq N_2 e^{-\nu_2(s-t)}$$
$$(t \leq s, \ s, t \in J). \tag{7}$$

Equation (3), (4) is said to have a ψ-ordinary dichotomy on J if (7) hold with $\nu_1 = \nu_2 = 0$. In this case we denote $N = \max\{N_1, N_2\}$.

Remark 4. For $\psi(t) = \text{Id}$ for all $t \in J$ we obtain the notion exponential and ordinary dichotomy for impulsive differential equations considered in [3, 20, 21]. That is why our main results in this paper appear as a generalization of some results there.

Let us introduce the principal Green function of the nonhomogeneous equation corresponding to (3), (4) with the projections P_1 and P_2 from the definition for ψ-exponential dichotomy

$$G(t, s) = \begin{cases} V(t)P_1V^{-1}(s) & (t > s, \ t, s \in J) \\ -V(t)P_2V^{-1}(s) & (t < s, \ t, s \in J). \end{cases} \tag{8}$$

Definition 5. The nonnegative function $m(t)$ is said to be integrally bounded on J if for some l the following inequality holds:

$$B(m(t)) = \sup_{t \in J} \int_t^{t+l} m(s)\,ds < \infty. \tag{9}$$

Definition 6. The sequence of nonnegative numbers $\{m_n\}_{n \in I}$ is said to be integrally bounded if for some l the following inequality holds:

$$B(m_j) = \sup_{t \in I} \sum_{t \leq t_j < t+l} m_j < \infty. \tag{10}$$

For each integrable on J function $m(t)$ we introduce the notation

$$L(m(t)) = \int_J m(s)\,ds \tag{11}$$

and for each summable on I sequence of nonnegative numbers $\{m_n\}_{n \in I}$ the notation

$$L(m_j) = \sum_{j \in I} m_j. \tag{12}$$

Definition 7. Let $r > 0$ be an arbitrary number. We will say that the function $F(t, x)$ $(t \in J, \ x \in X)$ and the operators H_j $(j \in I)$ satisfy the condition (H2) with the operator function $\psi(t) : J \rightarrow RL(X)$, if there exist positive functions $m(t)$, $k(t)$ $(t \in J)$ and sequences of nonnegative numbers $\{m_n\}_{n \in I}$, $\{k_n\}_{n \in I}$ such that

(H 2.1) $|\psi(t)F(t, x)| \leq m(t)$ $(|\psi(t)x| \leq r, \ t \in J)$;

(H 2.2) $|\psi(t)(F(t, x_1) - F(t, x_2))| \leq k(t)|\psi(t)(x_1 - x_2)|$ $(|\psi(t)x_1|, |\psi(t)x_2| \leq r, \ t \in J)$;

(H 2.3) $|\psi(t_j)H_j(x)| \leq m_j$ $(|\psi(t_j)x| \leq r, \ j \in I)$;

(H 2.4) $|\psi(t_j)(H_j(x_1) - H_j(x_2))| \leq k_j|\psi(t_j)(x_1 - x_2)|$ $(|\psi(t_j)x_1|, |\psi(t_j)x_2| \leq r, \ j \in I)$.

Definition 8. We say that the function $F(t, x)$ and the operators H_j belong to the class $ED_\psi(a_1, a_2, a_3, a_4, r)$ if the condition (H2) is fulfilled with the operator function $\psi(t)$, the functions $m(t)$, $k(t)$ are integrally bounded on J and $B(m(t)) \leq a_1$, $B(k(t)) \leq a_2$, and the sequences $\{m_j\}$ and $\{k_j\}$ are integrally bounded and $B(m_j) \leq a_3$, $B(k_j) \leq a_4$.

Definition 9. We say that the function $F(t, x)$ and the operators H_j belong to the class $D_\psi(a_1, a_2, a_3, a_4, r)$ if the condition (H2) is fulfilled with the operator function $\psi(t)$, the functions $m(t)$, $k(t)$ are integrable on J and $L(m(t)) \leq a_1$, $L(k(t)) \leq a_2$, and the sequences $\{m_j\}$ and $\{k_j\}$ are summable on I and $L(m_j) \leq a_3$, $L(k_j) \leq a_4$.

3. Main Results

Theorem 10. *Let the following conditions be fulfilled:*

(1) *The linear impulsive differential equation (3), (4) (i.e., the linear part of (1), (2)) has ψ-exponential dichotomy on $\mathbb{R}$ with projections P_1 and P_2.*

(2) *Conditions (H1) and (H2) hold.*

(3) *The function $F(t, x)$ and the operators H_j belong to the class $ED_\psi(a_1, a_2, a_3, a_4, r)$.*

(4) *The operators Q_n have bounded inverse ones.*

Then for an arbitrary $r > 0$, for sufficient small values of a_1, a_2, a_3, a_4 the impulsive equation (1), (2) has a unique solution $x(t)$, which is defined for $t \in \mathbb{R}$ and for which $|\psi(t)x(t)| \leq r$ $(t \in \mathbb{R})$.

Proof. Let $J = \mathbb{R}$. Consider in the space $C_\psi(X, T)$ the operator $Q : C_\psi(X, T) \to C_\psi(X, T)$ defined by the formula

$$Qx(t) = \int_{-\infty}^{\infty} G(t, \tau) F(\tau, x(\tau)) d\tau$$
$$+ \sum_{j=-\infty}^{\infty} G(t, t_j^+) H_j(x(t_j)), \quad (13)$$

where G is defined by (8).

Now we will show that the ball

$$S_{\psi,r} = \left\{ x : \|x\|_{C_\psi} \leq r \right\} \quad (14)$$

is invariant with respect to Q and the operator Q is contracting.

First we will prove that the operator Q maps the ball $S_{\psi,r}$ into itself. One has

$$|\psi(t) Qx(t)| \leq \left| \int_{-\infty}^{t} \psi(t) G(t, \tau) F(\tau, x(\tau)) d\tau \right|$$
$$+ \left| \int_{t}^{\infty} \psi(t) G(t, \tau) F(\tau, x(\tau)) d\tau \right|$$

$$+ \left| \sum_{t_j < t} \psi(t) G(t, t_j^+) H_j(x(t_j)) \right|$$

$$+ \left| \sum_{t \leq t_j} \psi(t) G(t, t_j^+) H_j(x(t_j)) \right|. \quad (15)$$

We will estimate the addends in (15). For $x \in S_{\psi,r}$ we obtain

$$\left| \int_{-\infty}^{t} \psi(t) G(t, \tau) F(\tau, x(\tau)) d\tau \right| = \left| \int_{-\infty}^{t} \psi(t) V(t) \right.$$
$$\left. \cdot P_1 V^{-1}(\tau) F(\tau, x(\tau)) d\tau \right| = \left| \int_{-\infty}^{t} \psi(t) V(t) \right.$$
$$\left. \cdot P_1 V^{-1}(\tau) \psi^{-1}(\tau) \psi(\tau) F(\tau, x(\tau)) d\tau \right|$$
$$\leq \int_{-\infty}^{t} \left\| \psi(t) V(t) P_1 V^{-1}(\tau) \psi^{-1}(\tau) \right\| |\psi(\tau)$$
$$\cdot F(\tau, x(\tau))| d\tau \leq N_1 \int_{-\infty}^{t} e^{-\nu_1(t-\tau)} m(\tau) d\tau$$
$$\leq N_1 \int_{s \leq 0} e^{\nu_1 s} m(t+s) ds \leq N_1 a_1 \sum_{k=0}^{\infty} e^{-\nu_1 k}$$
$$\leq \frac{N_1 a_1}{1 - e^{-\nu_1}},$$

$$\left| \int_{t}^{\infty} \psi(t) G(t, \tau) F(\tau, x(\tau)) d\tau \right| = \left| \int_{t}^{\infty} \psi(t) V(t) \right.$$
$$\left. \cdot P_2 V^{-1}(\tau) F(\tau, x(\tau)) d\tau \right| = \left| \int_{t}^{\infty} \psi(t) V(t) \right.$$
$$\left. \cdot P_2 V^{-1}(\tau) \psi^{-1}(\tau) \psi(\tau) F(\tau, x(\tau)) d\tau \right|$$
$$\leq \int_{t}^{\infty} \left\| \psi(t) V(t) P_2 V^{-1}(\tau) \psi^{-1}(\tau) \right\| |\psi(\tau)$$
$$\cdot F(\tau, x(\tau))| d\tau \leq N_2 \int_{t}^{\infty} e^{-\nu_2(\tau-t)} m(\tau) d\tau$$
$$\leq N_2 \int_{s \geq 0} e^{-\nu_2 s} m(t+s) ds \leq N_2 a_1 \sum_{k=0}^{\infty} e^{-\nu_2 k}$$
$$\leq \frac{N_2 a_1}{1 - e^{-\nu_2}},$$

$$\left| \sum_{t_j < t} \psi(t) G(t, t_j^+) H_j(x(t_j)) \right| = \left| \sum_{t_j < t} \psi(t) V(t) \right.$$
$$\left. \cdot P_1 V^{-1}(t_j^+) H_j(x(t_j)) \right| = \left| \sum_{t_j < t} \psi(t) V(t) \right.$$
$$\left. \cdot P_1 V^{-1}(t_j^+) \psi^{-1}(t_j^+) \psi(t_j^+) H_j(x(t_j)) \right|$$

$$\leq \sum_{t_j < t} \left\| \psi(t) V(t) P_1 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right) \right\| \left| \psi\left(t_j\right) \right.$$

$$\cdot H_j\left(x\left(t_j\right)\right) \Big| \leq N_1 \left(\sum_{t_j < t} e^{\nu_1(t_j - t)} m_j \right) \leq \frac{N_1 a_3}{1 - e^{-\nu_1 l}},$$

$$\left| \sum_{t \leq t_j} \psi(t) G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right) \right| = \left| \sum_{t \leq t_j} \psi(t) V(t) \right.$$

$$\cdot P_2 V^{-1}\left(t_j^+\right) H_j\left(x\left(t_j\right)\right) \Big| = \left| \sum_{t \leq t_j} \psi(t) V(t) \right.$$

$$\cdot P_2 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right) \psi\left(t_j^+\right) H_j\left(x\left(t_j\right)\right) \Big|$$

$$\leq \sum_{t \leq t_j} \left\| \psi(t) V(t) P_2 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right) \right\| \left| \psi\left(t_j\right) \right.$$

$$\cdot H_j\left(x\left(t_j\right)\right) \Big| \leq N_2 \left(\sum_{t \leq t_j} e^{\nu_2(t - t_j)} m_j \right) \leq \frac{N_2 a_3}{1 - e^{-\nu_2 l}}.$$

$$(16)$$

From the estimates (16) it follows that

$$\left| \psi(t) Q x(t) \right| \leq \frac{N_1 a_1}{1 - e^{-\nu_1}} + \frac{N_2 a_1}{1 - e^{-\nu_2}} + \frac{N_1 a_3}{1 - e^{-\nu_1 l}}$$

$$+ \frac{N_2 a_3}{1 - e^{-\nu_2 l}}. \tag{17}$$

Hence by $a_1 \leq r(N_2/(1 - e^{-\nu_2}) + N_1/(1 - e^{-\nu_1}))^{-1}$ and $a_3 \leq (N_2/(1 - e^{-\nu_2 l}) + N_1/(1 - e^{-\nu_1 l}))^{-1}$ we obtain

$$\left| \psi(t) Q x(t) \right| \leq r. \tag{18}$$

Thus the operator Q maps the ball $S_{\psi, r}$ into it self.

Now we will prove that the operator Q is a contraction in the ball $S_{\psi, r}$. Let $x_1, x_2 \in S_{\psi, r}$. Using the same technique as above we obtain

$$\left| \psi(t) Q x_1(t) - \psi(t) Q x_2(t) \right| \leq \int_{-\infty}^{\infty} \left| \psi(t) G(t, \tau) \right.$$

$$\cdot \left(F\left(\tau, x_1(\tau)\right) - F\left(\tau, x_2(\tau)\right) \right) \Big| d\tau + \sum_{j=-\infty}^{\infty} \left| \psi(t) \right.$$

$$\cdot G\left(t, t_j^+\right) \left(H_j\left(x_1\left(t_j\right)\right) - H_j\left(x_2\left(t_j\right)\right) \right) \Big|$$

$$\leq \int_{-\infty}^{\infty} \left\| \psi(t) G(t, \tau) \psi^{-1}(\tau) \right\| \left| \psi(\tau) \right.$$

$$\cdot \left(F\left(\tau, x_1(\tau)\right) - F\left(\tau, x_2(\tau)\right) \right) \Big| d\tau + \sum_{j=-\infty}^{\infty} \left\| \psi(t) \right.$$

$$\cdot V(t) P_1 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right) \right\| \left| \psi\left(t_j\right) \right.$$

$$\cdot \left(H_j\left(x_1\left(t_j\right)\right) - H_j\left(x_2\left(t_j\right)\right) \right) \Big| \leq \sup_{\tau \in J} \left| \psi(\tau) \right.$$

$$\cdot \left(x_1(\tau) - x_2(\tau) \right) \Big| \int_{-\infty}^{\infty} \left\| \psi(t) G(t, \tau) \psi^{-1}(\tau) \right\|$$

$$\cdot k(\tau) d\tau + \sup_{\tau \in J} \left| \psi(\tau) \left(x_1(\tau) - x_2(\tau) \right) \right|$$

$$\cdot \sum_{j=-\infty}^{\infty} \left\| \psi(t) V(t) P_1 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right) \right\| k_j$$

$$\leq \sup_{\tau \in J} \left| \psi(\tau) \left(x_1(\tau) - x_2(\tau) \right) \right| \left(\frac{N_2 a_2}{1 - e^{-\nu_2}} + \frac{N_1 a_2}{1 - e^{-\nu_1}} \right.$$

$$+ \frac{N_2 a_4}{1 - e^{-\nu_2 l}} + \frac{N_1 a_4}{1 - e^{-\nu_1 l}} \right).$$

$$(19)$$

Hence

$$\left\| \left| Q x_1 - Q x_2 \right| \right\|_{C_\psi}$$

$$\leq \left(\frac{N_2 a_2}{1 - e^{-\nu_2}} + \frac{N_1 a_2}{1 - e^{-\nu_1}} + \frac{N_2 a_4}{1 - e^{-\nu_2 l}} + \frac{N_1 a_4}{1 - e^{-\nu_1 l}} \right) \left\| \left| x_1 - x_2 \right| \right\|_{C_\psi}. \tag{20}$$

Thus by $a_2 < (N_2/(1 - e^{-\nu_2}) + N_1/(1 - e^{-\nu_1}))^{-1}$ and $a_4 < (N_2/(1 - e^{-\nu_2 l}) + N_1/(1 - e^{-\nu_1 l}))^{-1}$ the operator Q is a contraction in the ball $S_{\psi, r}$.

From Banach's fixed point principle, the existence of a unique fixed point of the operator Q follows.

It is not hard to verify that each solution of the impulsive differential equation (1), (2) which lies in the ball $S_{\psi, r}$ is also a solution of the equation

$$x(t) = Q x(t) \tag{21}$$

and vice versa. □

Corollary 11. *If the conditions of Theorem 10 are fulfilled and if, moreover, $F(t, 0) = 0$ ($t \in \mathbb{R}$), then $x = 0$ is a unique solution of (1), (2) in $C_\psi(X, T)$.*

Theorem 12. *Let the following conditions be fulfilled:*

(1) *The linear impulsive differential equation (3), (4) (i.e., the linear part of (1), (2)) has ψ-exponential dichotomy on $\mathbb{R}$ with projections P_1 and P_2.*

(2) *Conditions (H1) and (H2) hold.*

(3) *The function $F(t, x)$ and the operators H_j belong to the class $D_\psi(a_1, a_2, a_3, a_4, r)$.*

(4) *The operators Q_n have bounded inverse ones.*

Then for an arbitrary $r > 0$, for sufficient small values of a_1, a_2, a_3, a_4 the impulsive equation (1), (2) has a unique solution $x(t)$, which is defined for $t \in \mathbb{R}$ and for which $|\psi(t)x(t)| \leq r$ ($t \in \mathbb{R}$).

Proof. Let $J = \mathbb{R}$. In the proof of Theorem 10 it was mentioned that each solution $x(t)$ of the impulsive differential equation

(1), (2) that remains for $t \in J$ in the ball $S_{\psi,r}$ satisfies the equation

$$x(t) = \int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau)) \, d\tau$$
$$+ \sum_{j=-\infty}^{\infty} G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right) \tag{22}$$

and vice versa.

We consider again in the space $C_\psi(X,T)$ the operator $Q : C_\psi(X,T) \to C_\psi(X,T)$ defined in (13). For $|\psi(t)Qx(t)|$ we obtain the following estimate:

$$\left|\psi(t) Qx(t)\right| \le \int_{-\infty}^{\infty} \left|\psi(t) G(t,\tau) F(\tau, x(\tau))\right| d\tau$$

$$+ \sum_{j=-\infty}^{\infty} \left|\psi(t) G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right)\right| \le \int_{-\infty}^{\infty} \left|\psi(t)\right.$$

$$\left. \cdot G(t,\tau) \psi^{-1}(\tau) \psi(\tau) F(\tau, x(\tau))\right| d\tau + \sum_{j=-\infty}^{\infty} \left|\psi(t)\right.$$

$$\left. \cdot V(t) P_1 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right) \psi\left(t_j^+\right) H_j\left(x\left(t_j\right)\right)\right|$$

$$\le \int_{-\infty}^{\infty} \left\|\psi(t) G(t,\tau) \psi^{-1}(\tau)\right\| \left|\psi(\tau) F(\tau, x(\tau))\right| d\tau$$

$$+ \sum_{j=-\infty}^{\infty} \left\|\psi(t) V(t) P_1 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right)\right\| \left|\psi\left(t_j\right)\right. \tag{23}$$

$$\left. \cdot H_j\left(x\left(t_j\right)\right)\right| \le N \int_{-\infty}^{\infty} \left|\psi(\tau) F(\tau, x(\tau))\right| d\tau$$

$$+ N \sum_{j=-\infty}^{\infty} \left|\psi\left(t_j\right) H_j\left(x\left(t_j\right)\right)\right| = N \int_{-\infty}^{\infty} m(\tau) \, d\tau$$

$$+ N \sum_{j=-\infty}^{\infty} m_j = N\left(L\left(m(t)\right) + L\left(m_j\right)\right) \le N\left(a_1\right.$$

$$\left. + a_3\right).$$

Thus by sufficiently small a_1 and a_3 the operator Q maps the ball $S_{\psi,r}$ into it self.

Now we will prove that the operator Q is a contraction in the ball $S_{\psi,r}$. Let $x_1, x_2 \in S_{\psi,r}$. We obtain

$$\left|\psi(t) Qx_1(t) - \psi(t) Qx_2(t)\right| \le \int_{-\infty}^{\infty} \left|\psi(t) G(t,\tau)\right.$$

$$\left. \cdot \left(F(\tau, x_1(\tau)) - F(\tau, x_2(\tau))\right)\right| d\tau + \sum_{j=-\infty}^{\infty} \left|\psi(t)\right.$$

$$\left. \cdot G\left(t, t_j^+\right)\left(H_j\left(x_1\left(t_j\right)\right) - H_j\left(x_2\left(t_j\right)\right)\right)\right|$$

$$\le \int_{-\infty}^{\infty} \left\|\psi(t) G(t,\tau) \psi^{-1}(\tau)\right\| \left|\psi(\tau)\right.$$

$$\left. \cdot \left(F(\tau, x_1(\tau)) - F(\tau, x_2(\tau))\right)\right| d\tau + \sum_{j=-\infty}^{\infty} \left\|\psi(t)\right.$$

$$\left. \cdot V(t) P_1 V^{-1}\left(t_j^+\right) \psi^{-1}\left(t_j^+\right)\right\| \left|\psi\left(t_j\right)\right.$$

$$\left. \cdot \left(H_j\left(x_1\left(t_j\right)\right) - H_j\left(x_2\left(t_j\right)\right)\right)\right| \le N \sup_{\tau \in J} \left|\psi(\tau)\right.$$

$$\left. \cdot \left(x_1(\tau) - x_2(\tau)\right)\right| \left(\int_{-\infty}^{\infty} k(\tau) \, d\tau + \sum_{j=-\infty}^{\infty} k_j\right) \le N$$

$$\cdot \sup_{\tau \in J} \left|\psi(\tau)\left(x_1(\tau) - x_2(\tau)\right)\right| \left(a_2 + a_4\right).$$

$$\tag{24}$$

Hence

$$\left\|Qx_1 - Qx_2\right\|_{C_\psi} \le N\left(a_2 + a_4\right) \left\|\left|x_1 - x_2\right|\right\|_{C_\psi}. \tag{25}$$

Thus by sufficiently small a_2 and a_4 the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle follows the existence of a unique fixed point of the operator Q. $\quad\square$

Theorem 13. *Let the following conditions be fulfilled:*

(1) *The linear impulsive differential equation (3), (4) (i.e., the linear part of (1), (2)) has ψ-exponential dichotomy on $\mathbb{R}$ with projections P_1 and P_2.*

(2) *Conditions (H1) and (H2) hold.*

(3) *The function $F(t,x)$ and the operators H_j belong to the class $ED_\psi(a_1, a_2, a_3, a_4, r)$.*

(4) *The operators Q_n have bounded inverse ones.*

Then for any $r > 0$ by sufficient small a_1, a_2, a_3, a_4 there exists $\rho < r$ such that the impulsive equation (1), (2) has for each $\xi \in X_1 = P_1 X$ with $|\psi(0)\xi| \le \rho$ a unique solution $x(t)$ on R_+ for which $P_1 x(0) = \xi$ and $|\psi(t)x(t)| \le r$ $(t \in \mathbb{R}_+)$.

Proof. Let $J = \mathbb{R}_+$ and $x(t)$ be a solution of (1), (2) that remains for $t \in J$ in the ball $S_{\psi,r} = \{x : \|x\|_{C_\psi} \le r\}$. From the results obtained in [18, Theorem 1 and Remark 2] it follows that such $x(t)$ satisfies the integral equation

$$x(t) = V(t) \xi + \int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau)) \, d\tau$$
$$+ \sum_{j=-\infty}^{\infty} G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right), \tag{26}$$

where $\xi = P_1 x(0)$. The converse is also true: a solution of the (26) satisfies the differential equation (1), (2) for $t \in J$.

Let $\xi \in X_1$ and $|\psi(0)\xi| \le \rho < r$. We consider in the space $C_\psi(X,T)$ the operator $Q : C_\psi(X,T) \to C_\psi(X,T)$ defined by the formula

$$Qx(t) = V(t) \xi + \int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau)) \, d\tau$$
$$+ \sum_{j=-\infty}^{\infty} G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right). \tag{27}$$

First we will prove that the operator Q maps the ball $S_{\psi,r}$ into it self. Indeed one has

$$\left|\psi(t) Qx(t)\right| \leq \left|\psi(t) V(t) \xi\right| + \left|\psi(t)\right.$$

$$\left. \cdot \left(\int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau)) d\tau \right.\right. \tag{28}$$

$$\left.\left. + \sum_{j=-\infty}^{\infty} G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right)\right)\right|.$$

For the first addend with $\rho \leq r/2N_1$ we obtain

$$\left|\psi(t) V(t) \xi\right| \leq N_1 e^{-\nu_1 t} \left|\psi(0) \xi\right| \leq N_1 e^{-\nu_1 t} \rho \leq \frac{r}{2}. \tag{29}$$

Using the same technique as in the proof of Theorem 10 we obtain for the second addend the estimate

$$\left|\psi(t) \left(\int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau)) d\tau \right.\right.$$

$$\left.\left. + \sum_{j=-\infty}^{\infty} G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right)\right)\right| \leq \frac{N_1 a_1}{1 - e^{-\nu_1}} \tag{30}$$

$$+ \frac{N_2 a_1}{1 - e^{-\nu_2}} + \frac{N_1 a_3}{1 - e^{-\nu_1 l}} + \frac{N_2 a_3}{1 - e^{-\nu_2 l}}.$$

Hence by $a_1 \leq (r/2)(N_2/(1 - e^{-\nu_2}) + N_1/(1 - e^{-\nu_1}))^{-1}$ and $a_3 \leq (r/2)(N_2/(1 - e^{-\nu_2 l}) + N_1/(1 - e^{-\nu_1 l}))^{-1}$ we obtain

$$\left|\psi(t) Qx(t)\right| \leq r. \tag{31}$$

Thus the operator Q maps the ball $S_{\psi,r}$ into it self.

Now we will prove that the operator Q is a contraction in the ball $S_{\psi,r}$. Let $x_1, x_2 \in S_{\psi,r}$. We obtain as in the proof of Theorem 10 the estimate

$$\left\|\left|Qx_1 - Qx_2\right|\right\|_{C_\psi}$$
$$\leq \left(\frac{N_2 a_2}{1 - e^{-\nu_2}} + \frac{N_1 a_2}{1 - e^{-\nu_1}} + \frac{N_2 a_4}{1 - e^{-\nu_2 l}} + \frac{N_1 a_4}{1 - e^{-\nu_1 l}}\right) \left\|\left|x_1 - x_2\right|\right\|_{C_\psi}. \tag{32}$$

By $a_2 < (N_2/(1 - e^{-\nu_2}) + N_1/(1 - e^{-\nu_1}))^{-1}$ and $a_4 < (N_2/(1 - e^{-\nu_2 l}) + N_1/(1 - e^{-\nu_1 l}))^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows. $\square$

Theorem 14. *Let the following conditions be fulfilled:*

(1) *The linear impulsive differential equation (3), (4) (i.e., the linear part of (1), (2)) has ψ-exponential dichotomy on $\mathbb{R}$ with projections P_1 and P_2.*

(2) *Conditions (H1) and (H2) hold.*

(3) *The function $F(t, x)$ and the operators H_j belong to the class $D_\psi(a_1, a_2, a_3, a_4, r)$.*

(4) *The operators Q_n have bounded inverse ones.*

Then for any $r > 0$ by sufficient small a_1, a_2, a_3, a_4 there exists $\rho < r$ such that the impulsive equation (1), (2) has for each $\xi \in X_1 = P_1 X$ with $|\psi(0)\xi| \leq \rho$ a unique solution $x(t)$ on R_+ for which $P_1 x(0) = \xi$ and $|\psi(t)x(t)| \leq r$ $(t \in \mathbb{R}_+)$.

Proof. Let $J = \mathbb{R}_+$, $\xi \in X_1$, and $|\psi(0)\xi| \leq \rho < r$. We consider again in the space $C_\psi(X, T)$ the operator $Q : C_\psi(X, T) \to C_\psi(X, T)$ defined by formula (27).

First we will prove that the operator Q maps the ball $S_{\psi,r}$ into it self. One has

$$\left|\psi(t) Qx(t)\right| \leq \left|\psi(t) V(t) \xi\right| + \left|\psi(t)\right.$$

$$\left. \cdot \left(\int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau)) d\tau \right.\right. \tag{33}$$

$$\left.\left. + \sum_{j=-\infty}^{\infty} G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right)\right)\right|.$$

For the first addend with $\rho \leq r/2N_1$ we obtain

$$\left|\psi(t) V(t) \xi\right| \leq N_1 e^{-\nu_1 t} \left|\psi(0) \xi\right| \leq N_1 e^{-\nu_1 t} \rho \leq \frac{r}{2}. \tag{34}$$

For the second addend with $a_1 + a_3 \leq r/2N$ as in the proof of Theorem 12 one has

$$\left|\psi(t) \left(\int_{-\infty}^{\infty} G(t,\tau) F(\tau, x(\tau)) d\tau \right.\right.$$

$$\left.\left. + \sum_{j=-\infty}^{\infty} G\left(t, t_j^+\right) H_j\left(x\left(t_j\right)\right)\right)\right| \leq N\left(a_1 + a_3\right) \leq \frac{r}{2}. \tag{35}$$

Thus the operator Q maps the ball $S_{\psi,r}$ into it self.

Let $x_1, x_2 \in S_{\psi,r}$. As in the proof of Theorem 12 we obtain the estimate

$$\left\|\left|Qx_1 - Qx_2\right|\right\|_{C_\psi} \leq N\left(a_2 + a_4\right) \left\|\left|x_1 - x_2\right|\right\|_{C_\psi}. \tag{36}$$

Hence by $a_2 + a_4 < N^{-1}$ the operator Q is a contraction in the ball $S_{\psi,r}$.

From Banach's fixed point principle the existence of a unique fixed point of the operator Q follows.

In the proof of Theorem 13 it was already mentioned that every solution of the impulsive differential equation (1), (2) which lies in the ball $S_{\psi,r}$ fulfills the equality

$$x(t) = Qx(t) \tag{37}$$

and vice versa. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

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