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Research Article

Convergence of One-Leg Hybrid Methods for Implicit Mixed Differential Algebraic Systems

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This paper focuses on a hybrid multistep and its twin one-leg methods and implementing them on implicit mixed differential algebraic equations. The orders of convergence for the above methods are discussed and numerical tests are solved.

1. Introduction

Consider the ordinary differential system:

$$y' = f(t, y), \quad (1)$$

where the linear multistep method (LM) [1, 2]

$$\sum_{i=0}^{k} \alpha_i y_{m+i} = h \sum_{i=0}^{k} \beta_i f(t_{m+i}, y_{m+i}) \quad (2)$$

is given, and the generating polynomials

$$\rho(\zeta) = \sum_{i=0}^{k} \alpha_i \zeta^i, \qquad \sigma(\zeta) = \sum_{i=0}^{k} \beta_i \zeta^i \quad (3)$$

have real coefficients and no common divisor. Also assume throughout the normalization that

$$\sigma(1) = 1. \quad (4)$$

Then the associated one-leg (OL) method is defined by

$$\sum_{i=0}^{k} \alpha_i y_{m+i} = hf\left(\sum_{i=0}^{k} \beta_i t_{m+i}, \sum_{i=0}^{k} \beta_i y_{m+i}\right). \quad (5)$$

The author presents hybrid multistep methods that take the form

$$y_{n+s} = h\mu f_n + \sum_{j=0}^{k-2} \gamma_{n-j} y_{n-j}, \quad \text{(6a)}$$

$$y_n + \sum_{j=1}^{k} \alpha_{n-j} y_{n-j} = h\beta_s \left(f_{n+s} - \beta^* f_{n-1}\right), \quad \text{(6b)}$$

and the one-leg twin of (6a) and (6b) takes the form

$$y_n + \sum_{j=1}^{k} \alpha_{n-j} y_{n-j} = hf\left(\beta_s t_{n+s} - \beta_s \beta^* t_{n-1}, \beta_s y_{n+s} - \beta_s \beta^* y_{n-1}\right), \quad (7)$$

where $f_{n+s} = f(t_{n+s}, y_{n+s})$, $t_{n+s} = t_n + sh$, $-1 < s < 1$, $-1 \leqslant \beta^* < 1$. β_s, and α_{n-j}, $j = 1, 2, \ldots, k$, are parameters to be determined as functions of s and β^*. Methods (6a) and (6b) with step k have order $p = k$ and y_{n+s} has order $k-1$. To evaluate the value of y_{n+s} at the off-step point, that is, t_{n+s}, consider the nodes t_n (double node), $t_{n-1}, \ldots$, and t_{n-k} (simple nodes) [3, 4].

Applying Newton's interpolation formula for this data gives the following scheme:

$$y(t_n + sh) = y_n + shf_n + s^2 (hf_n - \nabla y_n)$$

$$+\frac{s^2(s+1)}{2!}\left(hf_n-\nabla y_n-\frac{1}{2}\nabla^2 y_n\right)$$
$$+\frac{s^2(s+1)(s+2)}{3!}\left(hf_n-\nabla y_n-\frac{1}{2}\nabla^2 y_n-\frac{\nabla^3 y_n}{3}\right)$$
$$+\cdots, \tag{8}$$

where $\nabla y_n = y_n - y_{n-1}$ and f (or $f(t,y)$) is considered as a derivative of the solution $y(t)$.

The hybrid multistep method and its twin one-leg depend on two parameters, β^* and s, which control their convergence and stability; also the position of the stage point affects the stability regions of the methods. For optimal values of β^* and s, the methods have larger stability region compared to the hybrid backward differentiation formulae [3]; the corresponding one-leg twin is G-stable for $k = 2$ and $k = 3$; see [5, 6].

Differential-algebraic equations (DAEs) often take place in highly scientific technology domains, such as automatic control engineering, simulation of electrical networks, and chemical reaction kinetics [7, 8]. Some systems can be reduced to ODE systems and can be solved by numerical ODE methods. Reduction to explicit differential system (1) in some other systems can be impossible or impractical because the problem is more naturally posed in the form $F(t,y',y) = 0$ and a reduction might reduce the sparseness of Jacobian matrices. These systems are then solved directly [9, 10].

Here LM (2) and OL (5) are defined for implicit mixed differential algebraic systems of the form

$$F\left(t,y',y,x\right)=0, \tag{9a}$$

$$G(t,y,x)=0, \tag{9b}$$

where F, G, x, and y are vectors of the same dimension. Rewrite (2) and (5) in the form

$$\frac{1}{h}\rho y_n-\sigma y_n'=0, \tag{10}$$

$$\frac{1}{h}\rho y_n-f\left(\sigma y_n,\tau\right)=0, \tag{11}$$

respectively, where $\tau = \sigma t_n$, substituting for y_n' in (9a) and (9b):

$$F\left(t_n,\frac{1}{\beta_k}\left(\frac{1}{h}\rho y_n-\overline{\sigma} y_n'\right),y_n,x_n\right)=0, \tag{12a}$$

$$G\left(t_n,x_n,y_n\right)=0, \tag{12b}$$

where $\overline{\sigma} y_n'$ acts only on backward data. Equations (12a) and (12b) can be solved for (y_n, x_n). In the one-leg form, the arguments are changed to τ_n, $y'(\tau_n)$, $y(\tau_n)$, and $x(\tau_n)$. The implementation of OL (5) to (9a) and (9b) gives the equations

$$F\left(\tau,\frac{1}{h}\rho y_n,\sigma y_n,\sigma x_n\right)=0, \tag{13a}$$

$$G\left(t_n,y_n,x_n\right)=0. \tag{13b}$$

As a modification technique that applies the same arguments of F on G, this implementation can be written as

$$F\left(\tau,\frac{1}{h}\rho y_n,\sigma y_n,\sigma x_n\right)=0, \tag{14a}$$

$$G\left(\tau,\sigma y_n,\sigma x_n\right)=0. \tag{14b}$$

The LMS and MOL formulations in (12a), (12b), (14a), and (14b) are easier to implement than OL method in (13a) and (13b) because both equations are evaluated on the same arguments.

In the following section, the hybrid multistep (HMS) method in (6a) and (6b) and its twin, hybrid one-leg (7) (HOL) method are defined for (9a) and (9b) and expressions for the local truncation errors of (HMS) and (HOL) are given.

2. The Hybrid Method

In the case of $k = 2$, the method in (6a) and (6b) takes the form

$$\alpha_n y_n+\alpha_{n-1}y_{n-1}+\alpha_{n-2}y_{n-2}=h\beta_s\left(f_{n+s}-\beta^* f_{n-1}\right), \tag{15a}$$

$$y_{n+s}=y_n+shf_n, \tag{15b}$$

where

$$\alpha_n=\frac{3+2s-\beta^*}{2\left(1-\beta^*\right)},$$
$$\alpha_{n-1}=\frac{-2(1+s)}{\left(1-\beta^*\right)},$$
$$\alpha_{n-2}=\frac{-\left(-1-2s-\beta^*\right)}{2\left(1-\beta^*\right)},$$
$$\beta_s=\frac{1}{\left(1-\beta^*\right)}. \tag{16}$$

Method (15a) has order 2 and its truncation error is $((2+3s(2+s)+\beta^*)/6(-1+\beta^*))h^3y'''(\eta)$, and y_{n+s} has order one and its truncation error is $T_2 = s^2h^2y''(\eta)$ due to (8), where $\eta \in (t_{n-2}, t_{n+1})$. Applying the method in (15a) and (15b) on implicit mixed differential algebraic equations (9a) and (9b) obtains the following:

$$F\left(t_{n+s},u_{n+s},y_{n+s},x_{n+s}\right)=0, \tag{17a}$$

$$G\left(t_n,x_n,y_n\right)=0, \tag{17b}$$

where

$$u_{n+s}=\frac{1}{h\beta_s}\left(\alpha_n y_n+\alpha_{n-1}y_{n-1}+\alpha_{n-2}y_{n-2}\right)+\beta^* f_{n-1}. \tag{18}$$

Let $x(t)$ and $y(t)$ be the exact solution of (9a) and (9b). The residues of (17a) and (17b) are the values of the left sides evaluated on $x(t)$ and $y(t)$. Using Taytor expansion for the second argument of F evaluated on $y(t)$,

$$\begin{aligned} u(t_{n+s}) &= \frac{1}{h\beta_s}\left(\alpha_n y(t_n) + \alpha_{n-1} y(t_{n-1}) + \alpha_{n-2} y(t_{n-2})\right) \\ &\quad + \beta^* f(t_{n-1}), \end{aligned} \tag{19}$$

leads to

$$u(t_{n+s}) \simeq y'(t_{n+s}) + T_1, \tag{20}$$

where

$$T_1 = \frac{1}{6}\left(-2 - 6s - \beta^*\right) h^2 y'''(\eta). \tag{21}$$

Theorem 1. *The order of convergence of the second-order hybrid method in (15a) and (15b) when applied to implicit mixed DAEs (9a) and (9b) is two.*

Proof. Let the local truncation errors be defined by $\epsilon_n = y_n - y(t_n)$, $\phi_n = x_n - x(t_n)$, $\tilde{\epsilon}_n = y'_n - y'(t_n)$, and $\tilde{\phi}_n = x'_n - x'(t_n)$, where $\{x_{n+s}, y_{n+s}\}$ satisfies (17a) and (17b) with exact backward data:

$$\begin{aligned} 0 &= F\left(t_{n+s}, y'_{n+s}, y_{n+s}, x_{n+s}\right) \simeq F\Bigg(t_{n+s}, \\ &\quad \frac{1}{h\beta_s}\left(\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2}\right) + \beta^* f_{n-1}, y_n \\ &\quad + shy'_n, x_n + shx'_n\Bigg) \simeq F\Bigg(t_{n+s}, \\ &\quad \frac{1}{h\beta_s}\left(\alpha_n\left(y(t_n) + \epsilon_n\right) + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2}\right) \\ &\quad + \beta^* f_{n-1}, y(t_n) + \epsilon_n + sh\left(y'(t_n) + \tilde{\epsilon}_n\right), x(t_n) \\ &\quad + \phi_n + sh\left(x'(t_n) + \tilde{\phi}_n\right)\Bigg) \simeq F\Bigg(t_{n+s}, T_1 + \frac{\epsilon_n \alpha_n}{h\beta_s} \\ &\quad + y'(t_{n+s}), T_2 + \epsilon_n + sh\tilde{\epsilon}_n + y(t_{n+s}), T_3 + \phi_n \\ &\quad + sh\tilde{\phi}_n + x(t_{n+s})\Bigg), \end{aligned} \tag{22}$$

where $T_3 = s^2h^2x''(\eta)$; expanding around $(t_{n+s}, y'(t_{n+s}), y(t_{n+s}), x(t_{n+s}))$ implies that

$$\begin{aligned} &F\left(t_{n+s}, y'(t_{n+s}), y(t_{n+s}), x(t_{n+s})\right) \\ &\quad + \left(T_1 + \frac{\epsilon_n \alpha_n}{h\beta_s}\right) F_{y'} + \left(T_2 + \epsilon_n + sh\tilde{\epsilon}_n\right) F_y \\ &\quad + \left(T_3 + \phi_n + sh\tilde{\phi}_n\right) F_x \simeq 0, \end{aligned} \tag{23}$$

where the arguments of F, $F_{y'}$, F_y, and F_x are $(t_{n+s}, y'(t_{n+s}), y(t_{n+s}), x(t_{n+s}))$

$$\begin{aligned} &\epsilon_n\left(\frac{\alpha_n}{h\beta_s} F_{y'} + F_y\right) + T_1 F_{y'} + T_2 F_y + sh\tilde{\epsilon}_n F_y \\ &\quad + \left(T_3 + \phi_n + sh\tilde{\phi}_n\right) F_x \simeq 0. \end{aligned} \tag{24}$$

The errors ϵ_n and ϕ_n satisfy the following equations:

$$\begin{aligned} &\epsilon_n\left(\frac{\alpha_n}{h\beta_s} F_{y'} + F_y\right) + \phi_n F_x \\ &\quad \simeq -\left(T_1 F_{y'} + T_2 F_y + sh\tilde{\epsilon}_n F_y + \left(T_3 + sh\tilde{\phi}_n\right) F_x\right), \end{aligned} \tag{25}$$

$$0 = G(t_n, y_n, x_n) \simeq G + G_y \epsilon_n + G_x \phi_n, \tag{26}$$

where $G = F = 0$ and the arguments of G, G_y, G_x are $(t_n, y(t_n), x(t_n))$. If G_x^{-1} and $F_{y'}^{-1}$ exist, then

$$\phi_n \simeq -G_x^{-1} G_y \epsilon_n. \tag{27}$$

Therefore, ϕ_n has the same order as ϵ_n. The substitution for ϕ_n in (25) implies that

$$\epsilon_n\left(I - \frac{h\beta_s}{\alpha_n} G_x^{-1} G_y F_x + \frac{h\beta_s}{\alpha_n} F_y\right) = \gamma_1, \tag{28}$$

where

$$\begin{aligned} \gamma_1 &= -\frac{h\beta_s}{\alpha_n} \\ &\quad \cdot F_{y'}^{-1}\left(T_1 F_{y'} + T_2 F_y + sh\tilde{\epsilon}_n F_y + sh\tilde{\phi} F_x + T_3 F_x\right) \\ &= O\left(h^3\right), \\ \epsilon_n&\left(I + O(h)\right) = O\left(h^3\right), \\ \epsilon_n &= O\left(h^3\right). \end{aligned} \tag{29}$$

Then ϵ_n and ϕ_n are third order; thus the method is of second order.

Therefore, ϵ_n is third order small and in accordance with classical theory we conclude that the global error $\bar{\epsilon}_n$ in y_n must be second order small. Furthermore, if $\overline{\phi}_n$ denotes the global error in x_n—note that the global errors $(\bar{\epsilon}_n, \overline{\phi}_n)$ satisfy the same algebraic constraints as the local errors (ϵ_n, ϕ_n), namely, (26)—consequently, $\overline{\phi}_n$ is also second order small and thus method in (15a) and (15b) is second order small and accurate with respect to both y and x.

Modified Technique for Hybrid Method. It is noticed that the arguments of (17a) are $(t_{n+s}, y'_{n+s}, y_{n+s}, x_{n+s})$ and that of (17b) is (t_n, y_n, x_n). The arguments of (17a) and (17b) can be taken as

$$\begin{aligned} F\left(t_{n+s}, y'_{n+s}, y_{n+s}, x_{n+s}\right) &= 0, \\ G\left(t_{n+s}, y_{n+s}, x_{n+s}\right) &= 0, \end{aligned} \tag{30}$$

which is called the modified technique for hybrid method.

In this case,

$$0 = G(t_{n+s}, y_{n+s}, x_{n+s}) \simeq G\left(t_{n+s}, y_n + shy_n', x_n + shx_n'\right) \simeq G\left(t_{n+s}, T_2 + \epsilon_n + sh\widetilde{\epsilon}_n + y(t_{n+s}), T_3 + \phi_n + sh\widetilde{\phi}_n + x(t_{n+s})\right). \tag{31}$$

Expanding G around $(t_{n+s}, y(t_{n+s}), x(t_{n+s}))$ gives the following:

$$\begin{aligned} &G + (T_2 + \epsilon_n + sh\widetilde{\epsilon}_n) G_y + \left(T_3 + \phi_n + sh\widetilde{\phi}_n\right) G_x \simeq 0, \\ &T_3 + \phi_n + sh\widetilde{\phi}_n \simeq -(T_2 + \epsilon_n + sh\widetilde{\epsilon}_n) G_x^{-1} G_y. \end{aligned} \tag{32}$$

Substitute for $(T_3 + \phi_n + sh\widetilde{\phi}_n)$ in (24):

$$\begin{aligned} &\epsilon_n \left(I + \frac{h\beta_s}{\alpha_n} F_{y'}^{-1}\left(F_y - G_x^{-1} G_y F_x\right)\right) \\ &\quad \simeq -\frac{h\beta_s}{\alpha_n} F_{y'}^{-1}\left(T_1 F_{y'} + \left(F_y - G_x^{-1} G_y F_x\right)(T_2 + sh\widetilde{\epsilon})\right), \end{aligned} \tag{33}$$

where

$$\begin{aligned} \gamma_2 &= -\frac{h\beta_s}{\alpha_n} F_{y'}^{-1}\left(T_1 F_{y'} + \left(F_y - G_x^{-1} G_y F_x\right)(T_2 + sh\widetilde{\epsilon})\right) \\ &= O\left(h^3\right), \\ \epsilon_n (I + O(h)) &\simeq O\left(h^3\right), \\ \epsilon_n &= O\left(h^3\right), \end{aligned} \tag{34}$$

and the global error $\overline{\epsilon}_n$ affecting y_n is $O(h^2)$.

Since

$$\begin{aligned} 0 &= G(t_{n+s}, y_{n+s}, x_{n+s}) \simeq G\Big(t_{n+s}, y(t_n) + \overline{\epsilon}_n + sh\left(y'(t_n) + \overline{\widetilde{\epsilon}}\right), x(t_n) + \overline{\phi}_n + sh\left(x'(t_n) + \overline{\widetilde{\phi}}\right)\Big) \\ &\simeq \left(y(t_n) + \overline{\epsilon}_n + sh\left(y'(t_n) + \overline{\widetilde{\epsilon}}\right) - y(t_{n+s})\right) G_y + \left(x(t_n) + \overline{\phi}_n + sh\left(x'(t_n) + \overline{\widetilde{\phi}}\right) - x(t_{n+s})\right) G_x \\ &\simeq \left(\overline{\epsilon}_n + sh\overline{\widetilde{\epsilon}} + T_2\right) G_y + \left(\overline{\phi}_n + sh\overline{\widetilde{\phi}} + T_3\right) G_x, \\ \overline{\phi}_n &\simeq -\left(\overline{\epsilon}_n + sh\overline{\widetilde{\epsilon}} + T_2\right) G_x^{-1} G_y - \left(sh\overline{\widetilde{\phi}} + T_3\right) \\ &= O\left(h^2\right), \end{aligned} \tag{35}$$

thus the method is of second order. □

3. The One-Leg Twin

In the case of $k = 2$, method (7) takes the form

$$\begin{aligned} &\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} \\ &\quad = hf\left(\beta_s t_{n+s} - \beta_s \beta^* t_{n-1}, \beta_s y_{n+s} - \beta_s \beta^* y_{n-1}\right), \end{aligned} \tag{36a}$$

$$y_{n+s} = y_n + shf_n. \tag{36b}$$

Method (36a) has order 2 and its truncation error is $(1/6 - (1+s)^2/2(-1+\beta^*)^2) h^3 y'''(\eta)$ and y_{n+s} has order one and its truncation error is $T_2 = s^2 h^2 y''(\eta)$ due to (8).

Theorem 2. *The order of convergence of the second-order one-leg twin in (36a) and (36b) when applied to implicit mixed DAE is two.*

Proof. Applying the method in (36a) and (36b) on the implicit mixed differential algebraic equations (9a) and (9b) obtains the following:

$$F\left(\tau_n, \frac{1}{h}\rho y_n, \sigma y_n, \sigma x_n\right) = 0, \tag{37a}$$

$$G(t_n, y_n, x_n) = 0. \tag{37b}$$

The residues of (37a) and (37b) are the values of the left sides evaluated on $x(t)$ and $y(t)$:

$$\begin{aligned} 0 &\simeq F\Big(\tau_n, \frac{1}{h}(\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2}), \beta_s y_{n+s} - \beta_s \beta^* y_{n-1}, \beta_s x_{n+s} - \beta_s \beta^* x_{n-1}\Big), \\ 0 &\simeq F\Big(\tau_n, \\ &\quad \frac{1}{h}(\alpha_n (y(t_n) + \epsilon_n) + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2}), \\ &\quad \beta_s \left(y(t_n) + \epsilon_n + sh\left(y'(t_n) + \widetilde{\epsilon}_n\right)\right) - \beta_s \beta^* y_{n-1}, \\ &\quad \beta_s \left(x(t_n) + \phi_n + sh\left(x'(t_n) + \widetilde{\phi}_n\right)\right) - \beta_s \beta^* x_{n-1}\Big), \\ 0 &\simeq F\Big(\tau_n, \overline{T}_1 + \frac{\alpha_n \epsilon_n}{h} + y'(\tau_n), T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n + y(\tau_n), T_5 + \beta_s \phi_n + \beta_s sh\widetilde{\phi}_n + x(\tau_n)\Big), \end{aligned} \tag{38}$$

where

$$\begin{aligned} \overline{T}_1 &= \left(\frac{1}{6} - \frac{(1+s)^2}{2(-1+\beta^*)^2}\right) h^2 y'''(\eta), \\ T_4 &= \sigma y(t_n) - y(\tau_n) = \frac{(1+s)^2 \beta^*}{2(-1+\beta^*)^2} h^2 y''(\eta), \\ T_5 &= \sigma x(t_n) - x(\tau_n) = \frac{(1+s)^2 \beta^*}{2(-1+\beta^*)^2} h^2 x''(\eta), \end{aligned} \tag{39}$$

and expanding around $\tau_n, y'(\tau_n), y(\tau_n), x(\tau_n)$ implies that

$$\begin{aligned} 0 &\simeq F\left(\tau_n, y'(\tau_n), y(\tau_n), x(\tau_n)\right) + \left(\overline{T}_1 + \frac{\alpha_n \epsilon_n}{h}\right) F_{y'} \\ &\quad + \left(T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n\right) F_y \\ &\quad + \left(T_5 + \beta_s \phi_n + \beta_s sh\widetilde{\phi}_n\right) F_x, \end{aligned} \tag{40}$$

$$0 \simeq \left(\overline{T}_1 + \frac{\alpha_n \epsilon_n}{h}\right) F_{y'} + \left(T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n\right) F_y + \left(T_5 + \beta_s \phi_n + \beta_s sh\widetilde{\phi}_n\right) F_x, \tag{41}$$

where the arguments of $F, F_{y'}, F_y, F_x$ are $(\tau_n, y'(\tau_n), y(\tau_n), x(\tau_n))$.

Expanding (37b) around $(t_n, y(t_n), x(t_n))$, if G_x^{-1} exists, gives the following:

$$\phi_n \simeq -G_x^{-1} G_y \epsilon_n; \tag{42}$$

substitute ϕ_n in (41):

$$\begin{aligned} &\epsilon_n \left(I + \frac{h}{\alpha_n} F_{y'}^{-1} \left(\beta_s F_y - \beta_s G_x^{-1} G_y F_x\right)\right) \simeq -\frac{h}{\alpha_n} \\ &\quad \cdot F_{y'}^{-1} \left(\left(T_4 + \beta_s sh\widetilde{\epsilon}_n\right) F_y + \left(T_5 + \beta_s sh\widetilde{\phi}_n\right) F_x + \overline{T}_1 F_{y'}\right), \\ &\epsilon_n \left(1 + O(h)\right) \simeq \gamma_3 = O\left(h^3\right), \end{aligned} \tag{43}$$

where

$$\begin{aligned} \gamma_3 &= -\frac{h}{\alpha_n} F_{y'}^{-1} \left(\left(T_4 + \beta_s sh\widetilde{\epsilon}_n\right) F_y + \left(T_5 + \beta_s sh\widetilde{\phi}_n\right) F_x + \overline{T}_1 F_{y'}\right), \\ \epsilon_n &= O\left(h^3\right); \end{aligned} \tag{44}$$

thus the method is of second order.

Modified Technique for One-Leg Twin Method. Here the arguments of F and G are different if the arguments of G are taken as $(\tau_n, y(\tau_n), x(\tau_n))$, and (37a) and (37b) become the following:

$$\begin{aligned} &F\left(\tau_n, \frac{1}{h}\rho y_n, \sigma y_n, \sigma x_n\right) = 0, \\ &G\left(\tau_n, \sigma y_n, \sigma x_n\right) = 0, \\ &G\left(\tau_n, \beta_s y_{n+s} - \beta_s \beta^* y_{n-1}, \beta_s x_{n+s} - \beta_s \beta^* x_{n-1}\right) = 0, \\ &0 = G\left(\tau_n, T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n + y(\tau_n), T_5 + \beta_s \phi_n + \beta_s sh\widetilde{\phi}_n + x(\tau_n)\right), \\ &0 \simeq G\left(\tau_n, y(\tau_n), x(\tau_n)\right) + \left(T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n\right) G_y + \left(T_5 + \beta_s \phi_n + \beta_s sh\widetilde{\phi}_n\right) G_x, \\ &\left(T_5 + \beta_s \phi_n + \beta_s sh\widetilde{\phi}_n\right) \simeq -\left(T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n\right) \cdot G_x^{-1} G_y. \end{aligned} \tag{45}$$

Substitute $(T_5 + \beta_s \phi_n + \beta_s sh\widetilde{\phi}_n)$ in (41):

$$\begin{aligned} 0 &\simeq \left(\overline{T}_1 + \frac{\alpha_n \epsilon_n}{h}\right) F_{y'} + \left(T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n\right) F_y - \left(T_4 + \beta_s \epsilon_n + \beta_s sh\widetilde{\epsilon}_n\right) G_x^{-1} G_y F_x, \\ \epsilon_n &\left(I + \frac{h}{\alpha_n} F_{y'}^{-1} \left(\beta_s F_y - \beta_s G_x^{-1} G_y F_x\right)\right) \simeq \gamma_4 = O\left(h^3\right), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \gamma_4 &= -\frac{h}{\alpha_n} F_{y'}^{-1} \left(\left(T_4 + \beta_s sh\widetilde{\epsilon}_n\right) F_y - \left(T_4 + \beta_s sh\widetilde{\epsilon}_n\right) G_x^{-1} G_y F_x + \overline{T}_1 F_{y'}\right), \\ \epsilon_n &\left(I + O(h)\right) \simeq \gamma_4 = O\left(h^3\right). \end{aligned} \tag{47}$$

Consequently, $\epsilon_n = O(h^3)$ and the global error $\overline{\epsilon}_n$ affecting y_n is $O(h^2)$. However, since

$$\begin{aligned} 0 &= G\left(\tau_n, \sigma y_n, \sigma x_n\right) \\ &= G\left(\tau_n, \sigma y(t_n) + \sigma\overline{\epsilon}_n, \sigma x(t_n) + \sigma\overline{\phi}_n\right) \\ &\simeq \left(\sigma y(t_n) + \sigma\overline{\epsilon}_n - y(\tau_n)\right) G_y + \left(\sigma x(t_n) + \sigma\overline{\phi}_n - x(\tau_n)\right) G_x \\ &\simeq \left(\sigma\overline{\epsilon}_n + T_4\right) G_y + \left(\sigma\overline{\phi}_n + T_5\right) G_x, \end{aligned} \tag{48}$$

the global error $\overline{\phi}_n$ is related to $\overline{\epsilon}_n$ by the difference

$$\sigma\overline{\phi}_n \simeq \left(\sigma\overline{\epsilon}_n + T_4\right) G_x^{-1} G_y - T_5 = O\left(h^2\right). \tag{49}$$

The solution $\overline{\phi}_n$ of this difference equation is also $O(h^2)$ since σ^{-1} is a bounded operator; thus the method is of second order. □

4. Numerical Tests

Here, some numerical results are presented to evaluate the performance of the proposed technique [11, 12].

Test 1. Consider the differential algebraic equations:

$$\begin{aligned} x'(t) &= 2(1-y)\sin(y) + \frac{x}{\sqrt{1-y}}, \\ 0 &= x^2 + (y-1)\cos^2(y), \end{aligned} \tag{50}$$

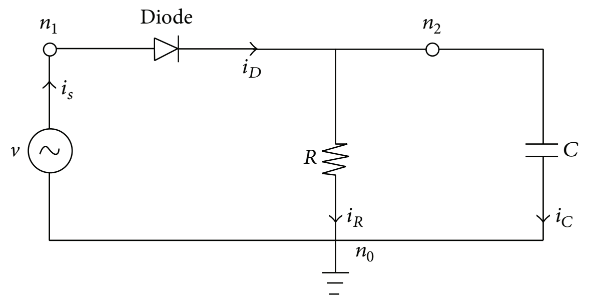

Figure 1: Rectifier circuit.

with the initial conditions $x(1) = 1$, $y(1) = 0$, and the exact solution is $x(t) = t\cos(1 - t^2)$, $y(t) = 1 - t^2$.

Test 2. Consider the nonlinear DAEs:

$$\begin{aligned} y_1' - y_2 - y_1 y_3 &= 0, \\ y_2' + \left(\frac{\pi}{3}\right)^2 y_1 - y_2 y_3 &= 0, \\ -\left(\frac{\pi}{3}\right)^2 y_1^2 - y_2^2 + 1 &= 0, \\ t &\in [0, 1], \end{aligned} \tag{51}$$

with the initial conditions $y_1(0) = 0$, $y_2(0) = 1$, and $y_3(0) = 0$.

The exact solutions are $y_1(t) = (3/\pi)\sin((\pi/3)t)$, $y_2(t) = \cos((\pi/3)t)$, and $y_3(t) = 0$.

Test 3. Consider the nonlinear DAEs:

$$\begin{aligned} x'(t) &= f(x, t) - B(x, t)\, y, \\ 0 &= g(x, y), \end{aligned} \tag{52}$$

with

$$\begin{aligned} f &= \begin{pmatrix} -x_1 + x_2 - \sin(t) - (1 + 2t) \\ 0 \end{pmatrix}, \\ B &= \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \\ x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\ g &= x_1^2 + x_1 (x_2 - \sin(t) - 1 + 2t), \end{aligned} \tag{53}$$

subject to the initial conditions $x_1(0) = 1$, $x_2(0) = 0$, and $y(t) = -1$.

The exact solution is $x_e = (1 - 2t, \sin(t))$, $y_e = -\cos(t)/(1 - 2t)$.

Test 4 (practical test). Consider rectifier diode circuit [13] in Figure 1 for transforming an AC voltage source into a DC voltage. It is designed in such a way that it damps the incoming sine-wave.

Table 1: The errors of solutions for the first test.

	t	h	$\mathrm{Er}(x(t))$	$\mathrm{Er}(y(t))$
HMS	2	0.01	$4.83216E-4$	$9.09651E-4$
	4	0.01	$1.19431E-2$	$4.42402E-3$
	2	0.001	$1.96883E-6$	$3.71621E-6$
	4	0.001	$5.36086E-4$	$1.98856E-4$
	2	0.0001	$1.68493E-7$	$3.18069E-7$
	4	0.0001	$3.80065E-7$	$1.40968E-7$
HOL	2	0.01	$2.69539E-4$	$5.05238E-4$
	4	0.01	$1.3771E-2$	$5.10544E-3$
	2	0.001	$4.43816E-7$	$8.37372E-7$
	4	0.001	$1.40624E-4$	$5.21576E-5$
	2	0.0001	$3.55151E-8$	$6.70428E-8$
	4	0.0001	$1.39345E-6$	$5.16839E-7$
HBDF	2	0.01	$1.99174E-2$	$3.53307E-2$
	4	0.01	$2.83319E-1$	$9.99465E-2$
	2	0.001	$6.02345E-5$	$1.13683E-4$
	4	0.001	$1.37187E-3$	$5.08695E-4$
	2	0.0001	$6.21028E-8$	$1.17233E-7$
	4	0.0001	$4.78321E-6$	$1.77411E-6$

In Figure 1, the circuit is a half wave rectifier circuit. The diode permits the flow of the current during the positive half cycle and stops the current flow during the negative half cycle. The capacitor is used to smooth the output voltage such that the output voltage at the load resistor R is close to a DC voltage:

$$v_{n_1}(t) = 10\sin(377t). \tag{54}$$

Kirchhoff's Current Law (KCL) at n_1

$$\begin{aligned} i_s(t) &= i_D(t), \\ i_D(t) &= I_s\left(\exp\left(38.685\left(v_{n_1}(t) - v_{n_2}(t)\right)\right) - 1\right). \end{aligned} \tag{55}$$

Assuming $I_s = 10^{-13} A$,

$$i_s(t) = I_s\left(\exp\left(38.685\left(v_{n_1}(t) - v_{n_2}(t)\right)\right) - 1\right). \tag{56}$$

KCL at n_2:

$$\begin{aligned} i_D(t) &= i_R(t) + i_c(t), \\ I_s\left(\exp\left(38.685\left(v_{n_1}(t) - v_{n_2}(t)\right)\right) - 1\right) &= \frac{v_{n_2}(t)}{R} + c\frac{d}{dt}v_{n_2}(t), \end{aligned} \tag{57}$$

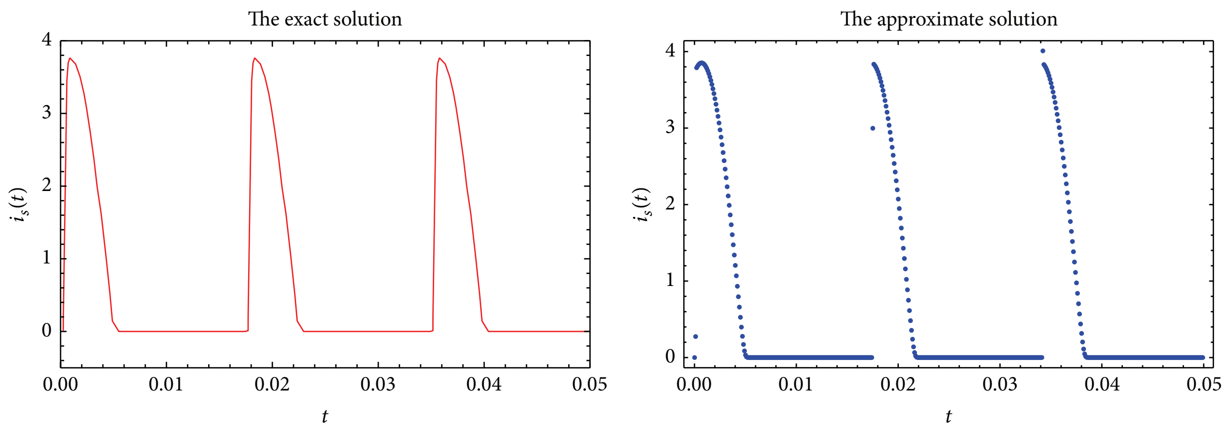

FIGURE 2: The exact and approximate solutions for the current of the diode circuit.

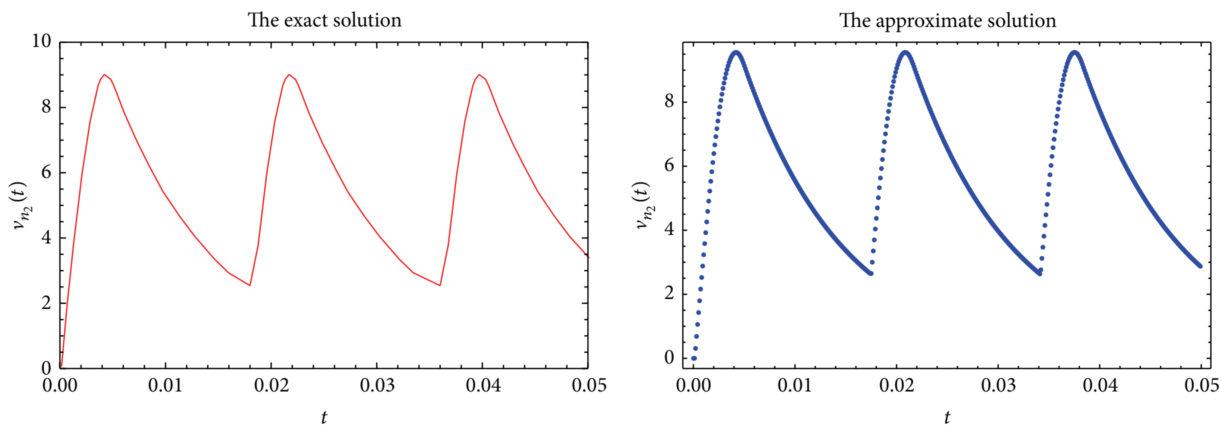

FIGURE 3: The exact and numerical solutions for the voltage ($v_{n_2}(t)$) of the diode circuit.

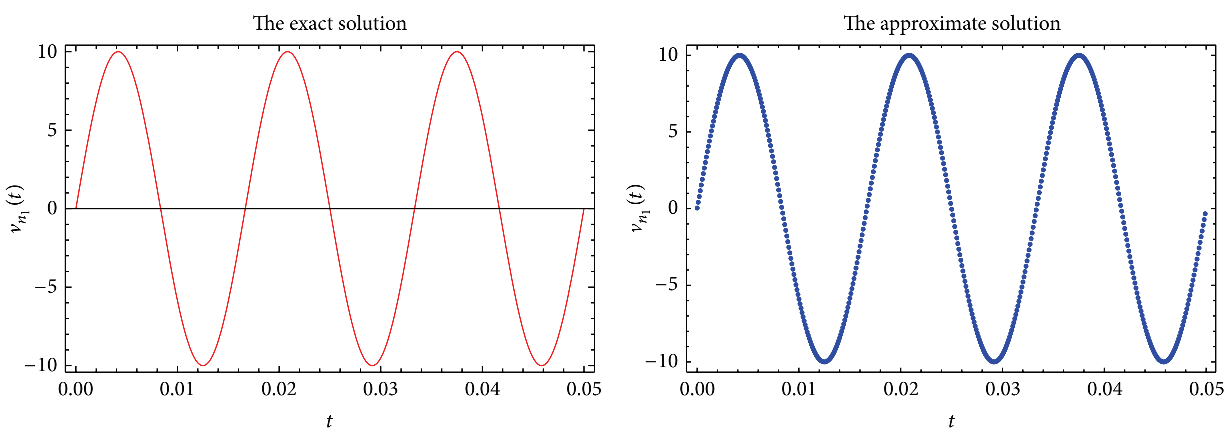

FIGURE 4: The exact and approximate solutions of the voltage ($v_{n_1}(t)$) of the diode circuit.

where i_s is the supply current, i_D is the diode current, i_R is the current in the resistor R, i_C is the current in the capacitor C, I_s is the reverse current of the diode, n_1 and n_2 are node 1 and node 2, n_0 is the reference node, and v is the supply voltage. We solve this circuit with hybrid formula (6a) and (6b) method and draw the exact and the numerical solutions in Figures 2–4.

The above tests are solved by the hybrid multistep method in (6a) and (6b) and its hybrid twin one-leg method (7) (with $k = 2$, $\beta^* = -0.4$, and $s = -0.4$) and compared with HBDF

Table 2: The errors of solutions for the second test.

	t	h	$\mathrm{Er}(y_1(t))$	$\mathrm{Er}(y_2(t))$	$\mathrm{Er}(y_3(t))$
HMS	0.5	0.01	$4.44120E-6$	$2.67822E-6$	$5.5219E-8$
	1	0.01	$5.17837E-6$	$9.38061E-6$	$5.5219E-8$
	0.5	0.001	$4.51402E-8$	$2.72911E-8$	$5.54467E-11$
	1	0.001	$5.21721E-8$	$9.46285E-8$	$5.52825E-11$
	0.5	0.0001	$4.52156E-10$	$2.73373E-10$	$2.00697E-12$
	1	0.0001	$5.22150E-10$	$9.47077E-10$	$1.29584E-12$
HOL	0.5	0.01	$1.10985E-5$	$6.7033E-6$	$2.27002E-7$
	1	0.01	$1.29380E-5$	$2.34557E-5$	$2.27002E-7$
	0.5	0.001	$1.1285E-7$	$6.82285E-8$	$2.27173E-10$
	1	0.001	$1.3043E-7$	$2.36571E-7$	$2.27181E-10$
	0.5	0.0001	$1.13038E-9$	$6.8343E-10$	$2.18404E-12$
	1	0.0001	$1.30538E-9$	$2.36771E-9$	$1.15477E-12$
HBDF	0.5	0.01	$2.89796E-5$	$1.75273E-5$	$9.67763E-7$
	1	0.01	$3.40875E-5$	$6.18349E-5$	$9.67763E-7$
	0.5	0.001	$2.99655E-7$	$1.81172E-7$	$9.67982E-10$
	1	0.001	$3.46637E-7$	$6.28731E-7$	$9.67637E-10$
	0.5	0.0001	$3.00632E-9$	$1.81762E-9$	$8.01653E-13$
	1	0.0001	$3.47202E-9$	$6.29754E-9$	$5.0231E-13$

Table 3: The errors of solutions for the third test.

	t	h	$\mathrm{Er}(x_1(t))$	$\mathrm{Er}(x_2(t))$	$\mathrm{Er}(y(t))$
HMS	1	0.01	$1.0116E-5$	$1.011656E-5$	$5.88302E-5$
	3	0.01	$1.58952E-6$	$1.58952E-6$	$2.0603E-6$
	1	0.001	$1.03830E-7$	$1.03831E-7$	$6.02688E-7$
	3	0.001	$1.61765E-8$	$1.61766E-8$	$2.07402E-8$
	1	0.0001	$1.04072E-9$	$1.04072E-9$	$6.0429E-9$
	3	0.0001	$1.64812E-10$	$1.64812E-10$	$2.12782E-10$
HOL	1	0.01	$4.47718E-6$	$4.47718E-6$	$1.0808E-4$
	3	0.01	$4.05837E-6$	$4.05837E-6$	$5.02559E-6$
	1	0.001	$5.35857E-8$	$5.35858E-8$	$8.50777E-8$
	3	0.001	$3.97665E-8$	$3.97665E-8$	$5.01833E-8$
	1	0.0001	$9.21768E-10$	$9.21768E-10$	$9.21768E-10$
	3	0.0001	$3.94626E-10$	$3.94624E-10$	$5.00513E-10$
HBDF	1	0.01	$5.77901E-7$	$5.77901E-7$	$676543E-5$
	3	0.01	$1.07426E-6$	$1.07427E-6$	$1.2481E-5$
	1	0.001	$2.5353E-8$	$2.5353E-8$	$7.15841E-7$
	3	0.001	$1.1663E-8$	$1.1663E-8$	$1.24319E-7$
	1	0.0001	$2.71828E-10$	$2.71828E-10$	$7.19898E-9$
	3	0.0001	$1.11241E-10$	$1.11242E-10$	$1.23669E-9$

method [3] (with $k = 2$ and $s = 0.3$) at different values of t. In solving the tests, Newton-Raphson method is used and the initial guesses are obtained by an interpolating polynomial. The errors of solutions of tests 1, 2, and 3 are tabulated in Tables 1, 2, and 3.

5. Conclusion

This paper focuses on the implementation of hybrid multistep classes and their twin one-leg classes on implicit mixed differential algebraic equations. The orders of convergence for these classes are discussed. Numerical tests are introduced, which show that the introduced methods give better results than HBDF.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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