

Research Article

On the Convergence of a Nonlinear Boundary-Value Problem in a Perforated Domain

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We consider a family with respect to a small parameter of nonlinear boundary-value problems as well as the corresponding spectral problems in a domain perforated periodically along a part of the boundary. We prove the convergence of solution of the original problems to the solution of the respective homogenized problem in this domain.

1. Introduction

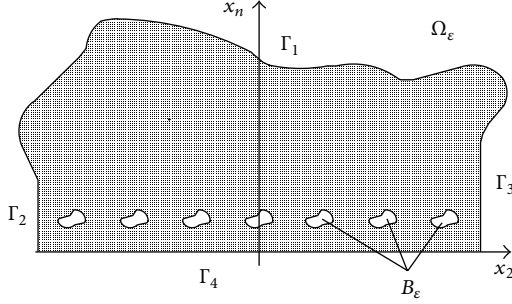
The paper is devoted to study of convergence of nonlinear boundary-value problems in a domain perforated along the boundary. There exist a lot of literatures, where boundary-value problems in perforated domains were studied. We refer to works [1–33]. In these papers and monographs the authors studied different kinds of perforation for linear as well as for nonlinear differential operators. Usually it is considered a family of problems depending on small parameter that characterizes the size of perforation. The main goal of the research is to find a homogenized (limit) model which is close to originally considered problems posed in the perforated domain. The general technique of homogenization method can be found in [23, 24, 27, 28].

The present paper will deal with convergence of boundary-value problems in perforated domains for nonlinear p -Laplace operator. Some problems for nonlinear operators were homogenized, for example, in [1, 20, 21, 30–33]. We consider a family of boundary-value problems in n -dimensional domain, $n > 2$, which is periodically perforated along the boundary by small sets. It is assumed that the diameter of each set and the distance between them have the same order. In our problem we suppose that the Dirichlet condition holds on the boundary of cavities, while the Neumann boundary condition is fulfilled on the boundary of the domain. We derive the limit (homogenized) problem for the original problems when the small parameter characterizing the size of perforation

tends to zero. Moreover, we establish the strong convergence in $W^{1,p}$ of the solutions for the considered problems to the corresponding solution of the limit problem. In addition we have obtained an estimate of the solution in a neighborhood of the eigenvalue of a corresponding spectral problem.

One of our goals is to prove the asymptotic behavior for the eigenvalue problem for p -Laplace operator in our perforated domain. Many authors considered spectral problems for p -Laplace operator; see, for example, [33–38]. These papers contain the results on qualitative properties of the p -Laplace spectral problems, convergence of eigenvalue problems, and some estimates for the difference between considered eigenvalues. The applications of our problem do not require the knowledge about the full spectrum of eigenvalues. Therefore we have proved the homogenization theorems only for the first eigenvalue of the spectral problem in perforated domain. More precisely, we have proved that the first eigenvalue of the spectral problems converges to the corresponding eigenvalue of the spectral limit problem. An analogous problem for linear elliptic operators for the two-dimensional domain was considered in [15] and for dimension three in [10].

The crucial point in our analysis is the validity of the Friedrichs inequality for functions in perforated domains. We prove this nontrivial result which is of an independent interest. Some papers devoted to this inequality in domains with microinhomogeneous structure are [8–10, 18, 19].

FIGURE 1: Structure of Ω_ε .

2. Preliminaries and the Main Results

Let $\Omega \subset \mathbb{R}^{n-1} \times \{x_n > 0\}$, $n > 2$, be a domain with boundary $\partial\Omega = \Gamma$. We assume that Γ is piece-wise smooth and consists of the parts Γ_i , $i = 1, 2, 3, 4$: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_4 = [-1/2; 1/2]^{n-1} \cap \{x_n = 0\}$, Γ_2, Γ_3 are orthogonal to $\{x_n = 0\}$ and belong to the planes $\{x_1 = -1/2\}$ and $\{x_1 = 1/2\}$ correspondingly, and $\Gamma \setminus \Gamma_4$ is a smooth surface. In the sequel $\varepsilon = 1/(2\mathcal{N} + 1)$ is a small parameter, $\mathcal{N} \in \mathbb{N}$, $\mathcal{N} \gg 1$.

Consider the set $B \subset \mathbb{R}^n$ belonging to the ball

$$K = \left\{ \xi : \sum_{i=1}^{n-1} \xi_i^2 + \left(\xi_n - \frac{1}{2} \right)^2 < a^2 \right\}, \quad 0 < a < \frac{1}{2} \quad (1)$$

and having a smooth boundary. If one multiplies each coordinate of B with parameter ε and does integer translations of this set along Γ_4 , we obtain the set denoted by B_ε . Let $\Gamma_\varepsilon = \partial B_\varepsilon$. Define the perforated domain Ω_ε as $\Omega \setminus \overline{B_\varepsilon}$. See the illustration for cut of Ω_ε on Figure 1.

Denote by $W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$ a set of functions from $W^{1,p}(\Omega_\varepsilon)$ with zero trace on Γ_ε . Analogously, by $W^{1,p}(\Omega, \Gamma_4)$ we define a set of functions from $W^{1,p}(\Omega)$ with zero trace on Γ_4 . We also consider the space $C_0^\infty(\overline{\Omega}, \Gamma_4)$, which is the set of functions from $C^\infty(\overline{\Omega})$, vanishing in the neighborhood of Γ_4 . Analogously, $C_0^\infty(\overline{\Omega}_\varepsilon, \Gamma_\varepsilon)$ denotes the set of functions from $C^\infty(\overline{\Omega}_\varepsilon)$, vanishing in a neighborhood of Γ_ε .

Remark 1. One can extend the functions $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$ into B_ε by zero. For the extended function we keep the same notation. It is true that u_ε belongs to $W^{1,p}(\Omega)$; see [24].

Definition 2. For $2 \leq p < n$ define the operator

$$\Delta u_p(x) \equiv \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(|\nabla u(x)|^{p-2} \frac{\partial u(x)}{\partial x_j} \right). \quad (2)$$

We consider the following spectral problem:

$$\begin{aligned} -\Delta_p u_\varepsilon(x) &= \lambda_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial_p \nu} &\equiv |\nabla u|^{p-2} (\nabla u, \nu) = 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (3)$$

where ν is the unit outward normal vector to the boundary of Ω .

Definition 3. One says that λ_ε is an eigenfunction to problem (3) if there exists $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon) \setminus \{0\}$, satisfying the integral identity

$$\begin{aligned} \int_{\Omega_\varepsilon} \sum_{j=1}^n |\nabla u_\varepsilon(x)|^{p-2} \frac{\partial u_\varepsilon(x)}{\partial x_j} \frac{\partial \varphi_\varepsilon(x)}{\partial x_j} dx \\ = \lambda_\varepsilon \int_{\Omega_\varepsilon} |u_\varepsilon|^{p-2} u_\varepsilon \varphi_\varepsilon dx \end{aligned} \quad (4)$$

for every $\varphi_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$. The couple $(u_\varepsilon, \lambda_\varepsilon)$ is called the solution to (3).

We will show that the problem

$$\begin{aligned} -\Delta_p u_0(x) &= \lambda_0 |u_0|^{p-2} u_0 \quad \text{in } \Omega, \\ u_0 &= 0 \quad \text{on } \Gamma_4, \\ \frac{\partial u_0}{\partial_p \nu} &\equiv |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \overline{\Gamma_4} \end{aligned} \quad (5)$$

is homogenized (the limit one) for (3).

As usual, we understand the solution to this boundary-value problem in the weak sense, that is, iff $u_0 \in W^{1,p}(\Omega, \Gamma_4) \setminus \{0\}$ satisfies

$$\int_{\Omega} \sum_{j=1}^n |\nabla u_0(x)|^{p-2} \frac{\partial u_0(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_j} dx = \lambda_0 \int_{\Omega} |u_0|^{p-2} u_0 \varphi dx \quad (6)$$

for every $\varphi \in W^{1,p}(\Omega, \Gamma_4)$.

Moreover, we prove the following results.

Theorem 4. Assume that $F \in W^{-1,q}(\Omega)$, $1/p + 1/q = 1$, $2 \leq p < \infty$, $q > 1$, and K is an arbitrary compact set belonging to the complex plane \mathbb{C} ; K does not contain the eigenvalues of problem (5). Then the following statements hold:

- (1) There exists a number $\varepsilon_0 > 0$, such that the unique solution to the problem

$$\begin{aligned} -\Delta_p U_\varepsilon(x) &= \lambda |U_\varepsilon|^{p-2} U_\varepsilon + F \quad \text{in } \Omega_\varepsilon, \\ U_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon, \\ \frac{\partial U_\varepsilon}{\partial_p \nu} &\equiv |\nabla U_\varepsilon|^{p-2} (\nabla U_\varepsilon, \nu) = 0 \quad \text{on } \partial\Omega \end{aligned} \quad (7)$$

does exist for all $\varepsilon < \varepsilon_0$ and for all $\lambda \in K$. Moreover, the uniform (in ε and λ) estimate

$$\|U_\varepsilon\|_{W^{1,p}} \leq C \|F\|_{L_q} \quad (8)$$

is valid, where C does not depend on U_ε and F .

(2) It yields that

$$\|U_\varepsilon - U_0\|_{W^{1,p}} \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0, \quad (9)$$

where U_0 is the unique solution of the problem

$$\begin{aligned} -\Delta_p U_0(x) &= \lambda |U_0|^{p-2} U_0 + F \quad \text{in } \Omega, \\ U_0 &= 0 \quad \text{on } \Gamma_4, \\ \frac{\partial U_0}{\partial \nu} &\equiv |\nabla U_0|^{p-2} \frac{\partial U_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \bar{\Gamma}_4. \end{aligned} \quad (10)$$

Here the solutions to problems (7) and (10) are understood in the weak sense, that is, iff U_δ , $\delta = \{\varepsilon_i\}$ satisfies the integral identity

$$\begin{aligned} \int_\Omega \sum_{j=1}^n |\nabla U_\delta(x)|^{p-2} \frac{\partial U_\delta(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_j} dx \\ = \lambda \int_\Omega |U_\delta|^{p-2} U_\delta \varphi dx + \int_\Omega F \varphi dx \end{aligned} \quad (11)$$

for every $\varphi \in W^{1,p}(\Omega, \Gamma_4)$.

Theorem 5. The spectrum of problems (3) and (5) is nonempty closed set. Let $\lambda_\varepsilon^1, \lambda_0^1$ be the first eigenvalues of problems (3) and (5), respectively. Then

$$\lambda_\varepsilon^1 \rightarrow \lambda_0^1 \quad \text{as } \varepsilon \rightarrow 0. \quad (12)$$

Moreover, if u_ε, u_0 are corresponding eigenfunctions, normalized in L_p , then up to a subsequence,

$$\|u_\varepsilon - u_0\|_{W^{1,p}} \rightarrow 0. \quad (13)$$

The proofs of analogous theorems for linear boundary-value problems were given in [6, 12–15] for different types of singular perturbations. The following lemma, which is proved in Section 3.1, is necessary for our analysis.

Lemma 6. Let v_ε be a sequence of functions from $W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$ and assume that $v_\varepsilon \rightharpoonup v^*$ weakly in $W^{1,p}(\Omega)$ when $\varepsilon \rightarrow 0$. Then $v^* \in W^{1,p}(\Omega, \Gamma_4)$.

For the questions on existence of solutions to the discussed problems we will refer to the following general result (see [39]).

Theorem 7. Let V be reflexive separable Banach space. Assume that the operator $A : V \rightarrow V'$ has the following properties:

- (i) A is bounded and semicontinuous: $\|A\| < \infty$, $\forall u, v, w \in V$; the function $s \rightarrow (A(u+sv), w)$ is continuous as a function mapping \mathbb{R} into \mathbb{R} .
- (ii) A is monotone: $(A(u) - A(v), u - v) \geq 0 \quad \forall u, v \in V$.
- (iii) Consider

$$\frac{(A(u), u)}{\|u\|} \rightarrow +\infty \quad \text{as } \|u\| \rightarrow +\infty. \quad (14)$$

Then for any $f \in V'$ there exists $u \in V$ such that $A(u) = f$.

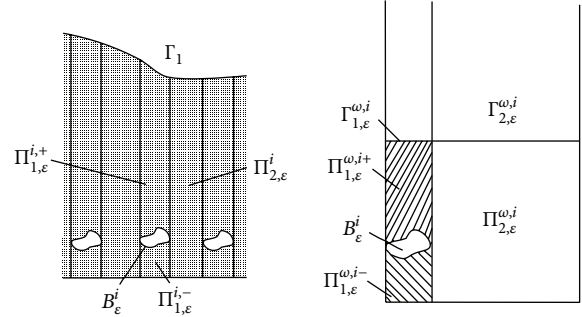


FIGURE 2: Parts of Ω_ε .

3. The Friedrichs Inequality

In our analysis we will need the Friedrichs inequality for functions $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$. We prove this result.

Theorem 8. The inequality

$$\int_{\Omega_\varepsilon} |v_\varepsilon|^p dx \leq K \int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^p dx \quad (15)$$

holds for any functions $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$, where the constant K does not depend on ε .

Proof. To demonstrate the technique of proving and avoiding the heavy n -dimensional notations we assume that $n = 2$. The case of arbitrary $n > 2$ can be done by repeating all lines of the present proof.

Let the length of projection of B on axis ξ_1 equal b . Denote

$$B_\varepsilon^i = \{x \in \Omega : \varepsilon^{-1}(x_1 - i, x_2) \in B\}, \quad i \in \mathbb{Z}. \quad (16)$$

Represent the domain Ω_ε as follows:

$$\Omega_\varepsilon = \bigcup_{i=-\mathcal{N}}^{\mathcal{N}} (\Pi_{1,\varepsilon}^{i+} \cup \Pi_{1,\varepsilon}^{i-}) \cup \bigcup_{i=-\mathcal{N}-1}^{\mathcal{N}} \Pi_{2,\varepsilon}^i \cup R_\varepsilon, \quad (17)$$

where $\Pi_{1,\varepsilon}^{i+}$ is the domain (a part of the vertical strip) of the length $b\varepsilon$, bounded from below by the upper part of the boundary B_ε^i and bounded from above by a part of the boundary Γ_1 ; the domain $\Pi_{1,\varepsilon}^{i-}$ is the strip of the length $b\varepsilon$, bounded from above by the lower part of B_ε^i and bounded from below by the line $x_2 = 0$; $\Pi_{2,\varepsilon}^i$, $i = -\mathcal{N}, \dots, \mathcal{N} - 1$, is the strip of the length $(1 - b)\varepsilon$, bounded from below by the line $x_2 = 0$ and bounded from above by a part of Γ_1 and having the common vertical bounds with $\Pi_{1,\varepsilon}^{i+}, \Pi_{1,\varepsilon}^{i+1+}$ (see Figure 2); the domains $\Pi_{2,\varepsilon}^{-\mathcal{N}}$ and $\Pi_{2,\varepsilon}^{\mathcal{N}}$ are vertical strips bounded from below by $x_2 = 0$ and bounded from above by a part of Γ_1 and having the common vertical bounds with $\Pi_{1,\varepsilon}^{-\mathcal{N}-1}$ and $\Pi_{1,\varepsilon}^{\mathcal{N}+1}$, correspondingly. The sum of widths for $\Pi_{2,\varepsilon}^{-\mathcal{N}-1}$ and $\Pi_{2,\varepsilon}^{\mathcal{N}}$ is $(1 - b)\varepsilon$. Finally, R_ε is the remaining part of Ω_ε having the bound belonging to Γ_1 .

Let $\omega > \varepsilon$. Define by $\Pi_{1,\varepsilon}^{\omega,i+}$ a domain $\Pi_{1,\varepsilon}^{i+} \cap \{x_2 < \omega\}$ (see Figure 2). Moreover, let $\Gamma_{1,\varepsilon}^{\omega,i}$ be the segment of the line $x_n = \omega$,

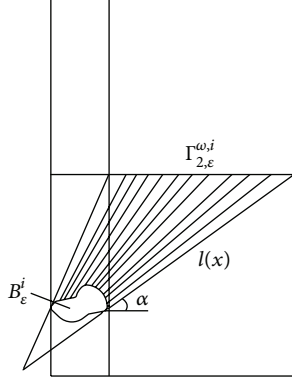


FIGURE 3: Tangential lines.

which is upper bound for $\Pi_{1,\epsilon}^{\omega,i+}$. Analogously we define the sets $\Gamma_{2,\epsilon}^{\omega,i}$ and $\Pi_{2,\epsilon}^{\omega,i}$. Without loss of generality we can assume that v_ϵ is real-valued function. Denote by $C_0^1(\overline{\Omega}_\epsilon, \Gamma_\epsilon)$ functions from $C^1(\overline{\Omega}_\epsilon)$, vanishing in the neighborhood of Γ_ϵ . Assume first that $v_\epsilon \in C_0^1(\overline{\Omega}_\epsilon, \Gamma_\epsilon)$. Let $(x_1, \omega) \in \Pi_{1,\epsilon}^{i+}$, and point (x_1, \tilde{x}_2) belongs to the bottom of $\Pi_{1,\epsilon}^{i+}$. Since $v_\epsilon(x_1, \tilde{x}_2) = 0$, then the Newton-Leibnitz formula gives

$$v_\epsilon(x_1, \omega) = \int_{\tilde{x}_2}^{\omega} \frac{\partial v_\epsilon}{\partial x_2} dx_2. \quad (18)$$

Taking the power p on both sides of that equality and using the Hölder inequality, we have

$$\begin{aligned} |v_\epsilon(x_1, \omega)|^p &= \left| \int_{\tilde{x}_2}^{\omega} \frac{\partial v_\epsilon}{\partial x_2} dx_2 \right|^p \leq \omega^{p/q} \int_{\tilde{x}_2}^{\omega} \left| \frac{\partial v_\epsilon}{\partial x_2} \right|^p dx_2 \\ &\leq \omega^{p/q} \int_{\tilde{x}_2}^{\omega} |\nabla v_\epsilon|^p dx_2. \end{aligned} \quad (19)$$

Integrate both sides of that inequality over $\Gamma_{1,\epsilon}^{\omega,i}$ with respect to x_1 and assume that the function v_ϵ is extended by zero into $B_\epsilon^i \cup \Pi_{1,\epsilon}^{i-}$. One gets

$$\int_{\Gamma_{1,\epsilon}^{\omega,i}} |v_\epsilon(x_1, \omega)|^p dx_1 \leq \omega^{p/q} \int_{\Pi_{1,\epsilon}^{\omega,i+}} |\nabla v_\epsilon|^p dx. \quad (20)$$

Consider now the rectangle $\Pi_{2,\epsilon}^i$, $i = -\mathcal{N}, \dots, \mathcal{N}$, which touches B_ϵ^i from the right. Draw the tangential lines to B_ϵ^i from both ends of segment $\Gamma_{2,\epsilon}^{\omega,i}$ (see Figure 3).

It is easy to see that the angle α between the tangential lines and Ox_1 belongs to $(\alpha_0, \pi/2)$, where $\alpha_0 > 0$ does not depend on ϵ and ω . We remember that our assumption was $\omega > \epsilon$. This follows from the fact that the diameter of the set and the distance between them are of the same order. Connect all points of $\Gamma_{2,\epsilon}^{\omega,i}$ with boundary of B_ϵ^i such that the intersection of these lines coincides with the intersection point of tangents. Thus, we have a beam of lines with directors $l(x)$. The angle between each line and Ox_1 belongs to $(\alpha_0, \pi/2)$.

Let $(x_1, \omega) \in \Gamma_{2,\epsilon}^{\omega,i}$, and $(\tilde{x}_1, \tilde{x}_2) \in \partial B_\epsilon^i \cap l(x_1, \omega)$. Since $v_\epsilon(\tilde{x}_1, \tilde{x}_2) = 0$, then

$$v_\epsilon(x_1, \omega) = \int_{(\tilde{x}_1, \tilde{x}_2)}^{(x_1, \omega)} \frac{\partial v_\epsilon}{\partial l} dl. \quad (21)$$

Analogously to (19), taking the power p on both sides of that formula and using the Hölder inequality, one obtains

$$\begin{aligned} |v_\epsilon(x_1, \omega)|^p &\leq \left| \frac{\omega}{\sin \alpha_0} \right|^{p/q} \int_{(\tilde{x}_1, \tilde{x}_2)}^{(x_1, \omega)} \left| \frac{\partial v_\epsilon}{\partial l} \right|^p dl \\ &\leq \left| \frac{\omega}{\sin \alpha_0} \right|^{p/q} \int_{(\tilde{x}_1, \tilde{x}_2)}^{(x_1, \omega)} |\nabla v_\epsilon|^p dl. \end{aligned} \quad (22)$$

Integrating both sides of that inequality over $\Gamma_{2,\epsilon}^{\omega,i}$ with respect to x_1 and replacing the right-hand side by the greater integral, we get

$$\begin{aligned} \int_{\Gamma_{2,\epsilon}^{\omega,i}} |v_\epsilon(x_1, \omega)|^p dx_1 &\leq \left| \frac{\omega}{\sin \alpha_0} \right|^{p/q} \int_{\Pi_{2,\epsilon}^{\omega,i} \cup \Pi_{1,\epsilon}^{\omega,i+} \cup \Pi_{1,\epsilon}^{\omega,i-}} |\nabla v_\epsilon|^p dx, \\ i &= -\mathcal{N}, \dots, \mathcal{N}. \end{aligned} \quad (23)$$

It remains to estimate the integral over $\Gamma_{2,\epsilon}^{\omega, -\mathcal{N}-1}$. We use the same technique as for $\Gamma_{2,\epsilon}^{\omega,i}$, $i = -\mathcal{N}, \dots, \mathcal{N}$. In this case one needs to consider the domain $B_\epsilon^{-\mathcal{N}}$ which is bordered with the strip $\Pi_{2,\epsilon}^{-\mathcal{N}-1}$ from the right. Analogously, we get

$$\begin{aligned} \int_{\Gamma_{2,\epsilon}^{\omega, -\mathcal{N}-1}} |v_\epsilon(x_1, \omega)|^p dx_1 &\leq C \omega^{p/q} \int_{\Pi_{2,\epsilon}^{\omega, -\mathcal{N}-1} \cup \Pi_{1,\epsilon}^{\omega, -\mathcal{N}-1+} \cup \Pi_{1,\epsilon}^{\omega, -\mathcal{N}-1-}} |\nabla v_\epsilon|^p dx. \end{aligned} \quad (24)$$

Let $\Gamma_4^\omega = \overline{\Omega} \cap \{x_2 = \omega\}$ be the segment connecting the points $(-1/2, \omega)$ and $(1/2, \omega)$; it means that $\Gamma_4^\omega = \bigcup_i \Gamma_{1,\epsilon}^{\omega,i} \cup \Gamma_{2,\epsilon}^{\omega,i}$. Summing up inequalities (20), (23), and (24), one gets

$$\int_{\Gamma_4^\omega} |v_\epsilon(x)|^p dx_1 \leq C_1 \omega^{p/q} \int_{\Omega_\epsilon} |\nabla v_\epsilon|^p dx. \quad (25)$$

Finally, integrating (25) with respect to ω , we obtain that

$$\int_{\Omega_\epsilon} |v_\epsilon(x)|^p dx_1 dx_2 \leq K \int_{\Omega_\epsilon} |\nabla v_\epsilon|^p dx. \quad (26)$$

Approximating the functions from $W^{1,p}(\Omega_\epsilon, \Gamma_\epsilon)$ by smooth functions, we conclude that inequality (26) is valid for $v_\epsilon \in W^{1,p}(\Omega_\epsilon, \Gamma_\epsilon)$. The proof is complete. \square

Remark 9. (1) Extending functions from $W^{1,p}(\Omega_\epsilon, \Gamma_\epsilon)$ by zero into $\bigcup_i B_\epsilon^i$ we obtain the Friedrichs inequality in Ω :

$$\|v_\epsilon\|_{L_p(\Omega)}^p \leq K \|\nabla v_\epsilon\|_{L_p(\Omega)}^p \quad \text{for } v_\epsilon \in W^{1,p}(\Omega_\epsilon, \Gamma_\epsilon). \quad (27)$$

(2) The validity of Friedrichs inequality means that one can introduce in $W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$ the norm $\|v\|_{W^{1,p}} \equiv \|\nabla v\|_{L_p}$ which is equivalent to

$$\|v\|_{W^{1,p}} \equiv \|v\|_{L_p} + \|\nabla v\|_{L_p}. \quad (28)$$

3.1. Proof of Lemma 6

Proof. First we point out that with the same method of proof inequality (25) is valid also for the case of an arbitrary $n > 2$, where

$$\Gamma_4^\omega = \overline{\Omega} \cap \{x_n = \omega\}. \quad (29)$$

Approximating the functions from $W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$ by smooth functions, we conclude that inequality (25) is valid for $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon)$. Using (25) and keeping in mind the uniform boundedness of the sequence v_ε , we obtain that

$$\int_{\Gamma_4^\omega} |v_\varepsilon(x_1, \dots, x_{n-1}, \omega)|^p dx \leq \omega^{p/q} C. \quad (30)$$

Now, we can pass to the limit in (30) when $\varepsilon \rightarrow 0$ and find that

$$\int_{\Gamma_4^\omega} |v^*|^p dx \leq \omega^{p/q} C_3. \quad (31)$$

Due to the fact that ω is an arbitrary small positive number and $v^* \in W^{1,p}(\Omega)$, it follows from (31) that $v^* = 0$ on Γ_4 . The proof is complete. \square

4. Proof of Theorem 4

For the proof we need the following lemma.

Lemma 10. *Let Q be an arbitrary compact set in the complex plane and $\lambda \in Q$. Suppose that the estimate*

$$\|U_\varepsilon\|_p \leq C \|F\|_q \quad (32)$$

holds uniformly in ε and λ for any solution U_ε of the boundary-value problem (7), which is normalized in $L_p(\Omega)$. Then estimate (32) holds also for any solution of problem (7) when $\lambda \in Q$.

Proof. Let us remember first that we denote by $\|\cdot\|_p$ the norm in L_p space. If $\|U_\varepsilon\|_p \neq 1$, then by setting $V_\varepsilon = U_\varepsilon / \|U_\varepsilon\|_p$, we obtain that

$$\|V_\varepsilon\|_p = 1 \quad (33)$$

and the function V_ε satisfies the identity

$$\begin{aligned} & \int_{\Omega} \sum_{j=1}^n |\nabla V_\varepsilon(x)|^{p-2} \frac{\partial V_\varepsilon(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_j} dx \\ &= \lambda \int_{\Omega} |V_\varepsilon|^{p-2} V_\varepsilon \varphi dx + \int_{\Omega} F_\varepsilon \varphi dx, \end{aligned} \quad (34)$$

where

$$F_\varepsilon = \frac{F}{\|U_\varepsilon\|_p^{p-1}}. \quad (35)$$

Hence, due to the assumptions we see that the estimate

$$\|V_\varepsilon\|_p \leq C \|F_\varepsilon\|_q \quad (36)$$

holds for V_ε . Multiplying the last inequality by $\|U_\varepsilon\|_p$ and using (33) and (35), we obtain the estimate (32) (probably with a different constant) for any U_ε . The proof is complete. \square

4.1. Proof of Part I

Step 1. The existence of the solution to problem (7) can be proved with help of Theorem 7. Indeed, we take $V = W^{1,p}(\Omega, \Gamma_\varepsilon)$. Introduce the operator $A(u) = -\sum_i (\partial/\partial x_i)(|\partial u/\partial x_i|^{p-2} (\partial u/\partial x_i))$. For $u, v \in W^{1,p}(\Omega, \Gamma_\varepsilon)$ we define

$$\begin{aligned} (A(u), v) &= - \int_{\Omega} \sum_i \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) v dx \\ &= \int_{\Omega} \sum_i \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx. \end{aligned} \quad (37)$$

Moreover, for $u \in W^{1,p}(\Omega, \Gamma_\varepsilon)$ it holds that $\lambda |u|^{p-1} u \in L_q(\Omega, \Gamma_\varepsilon)$, and for $F \in L_q(\Omega)$ we can introduce the functional on $W^{1,p}(\Omega, \Gamma_\varepsilon)$:

$$(\lambda |u|^{p-1} u + F, v) = \int_{\Omega} \lambda |u|^{p-1} uv dx + \int_{\Omega} Fv dx. \quad (38)$$

It is clear that U_ε is the solution to (7) iff $(A(U_\varepsilon), v) = (\lambda |U_\varepsilon|^{p-1} U_\varepsilon + F, v)$ for any $v \in W^{1,p}(\Omega, \Gamma_\varepsilon)$. Let us verify the properties of operator A . We take $u \in W^{1,p}$ with $\|u\|_{W^{1,p}} = 1$. Then

$$\|A(u)\|_{W^{-1,q}} = (A(u), u) = \|\nabla u\|_{L_p}^p \leq C \|u\|_{W^{1,p}}^p \leq C. \quad (39)$$

One can check the semicontinuity and monotonicity of operator A either directly or by using the following general result (see [39]).

Proposition 11. *If convex functional $v \rightarrow J(v)$ is differentiable in Gato sense, that is, there exists such continuous linear mapping $v \rightarrow J'(u) \cdot v$ of space V onto \mathbb{R} such that*

$$\lim_{s \rightarrow 0} \frac{1}{s} (J(u + sv) - J(u)) = J'(u) \cdot v, \quad (40)$$

then the mapping $u \rightarrow J'(u)$ is monotone and semicontinuous.

Let us define the functional

$$J(v) = \frac{1}{p} \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial v}{\partial x_i} \right|^p dx \quad (41)$$

on the space $V = W^{1,p}(\Omega, \Gamma_\varepsilon)$. It is easy to check that J is convex and differentiable in Gato sense and $J'(u) = A(u)$. Hence, A is semicontinuous and monotone operator. Finally, taking into account Remark 9 we calculate that

$$\frac{(A(u), u)}{\|u\|} = \frac{\|\nabla u\|_{L_p}^p}{\|u\|_{W^{1,p}}^{p-1}} = C \|u\|_{W^{1,p}}^{p-1} \longrightarrow +\infty \quad (42)$$

as $\|u\|_{W^{1,p}} \longrightarrow +\infty$.

Therefore by Theorem 7 for any $F \in L_q$ there exists a function U_ε satisfying $A(U_\varepsilon) = \lambda|U_\varepsilon|^{p-2} + F$.

Step 2. Let us derive now estimate (8). Substitute φ with U_ε in (11). Then it follows that

$$\begin{aligned} \int_\Omega |\nabla U_\varepsilon(x)|^p dx &= \lambda \int_\Omega |U_\varepsilon|^p dx + \int_\Omega F U_\varepsilon dx, \\ \|U_\varepsilon\|_{W^{1,p}}^p &= (\lambda + 1) \int_\Omega |U_\varepsilon|^p dx + \int_\Omega F U_\varepsilon dx \\ &\leq |\lambda + 1| \|U_\varepsilon\|_{L_p}^p + \|F\|_{L_q} \|U_\varepsilon\|_{L_p} \\ &\leq C \|U_\varepsilon\|_{W^{1,p}} \left(\|U_\varepsilon\|_{L_p}^{p-1} + \|F\|_{L_p} \right). \end{aligned} \quad (43)$$

Hence,

$$\|U_\varepsilon\|_{W^{1,p}}^{p-1} \leq C \left(\|U_\varepsilon\|_{L_p}^{p-1} + \|F\|_{L_q} \right). \quad (44)$$

Assume now that estimate (8) does not hold, that is, that there exists a sequence $\{\varepsilon_k\}$ $\varepsilon_k \rightarrow 0$ when $k \rightarrow \infty$, $F_k \in L_q(\Omega)$ and λ_k such that the inequality

$$\|U_{\varepsilon_k}\|_{W^{1,p}} > k \|F_k\|_q \quad (45)$$

holds for the solutions to problem (7), where

$$\begin{aligned} \varepsilon &= \varepsilon_k, \\ \lambda &= \lambda_k, \\ F &= F_k. \end{aligned} \quad (46)$$

Due to Lemma 10 we may assume without loss of generality that the sequence $\{U_\varepsilon\}$ is normalized in $L_p(\Omega)$, that is, that

$$\|U_{\varepsilon_k}\|_p = 1. \quad (47)$$

Then, by using (44) and (45), we obtain that

$$\begin{aligned} \|U_{\varepsilon_k}\|_{W^{1,p}} &\leq C, \\ \|F_k\|_q &< \frac{C}{k}, \end{aligned} \quad (48)$$

when $k \rightarrow \infty$.

Hence, there exists a subsequence $\{k'\}$ of indexes $\{k\}$ and $U^* \in W^{1,p}(\Omega)$, λ^* such that

$$\begin{aligned} \lambda_{k'} &\longrightarrow \lambda^* \in K, \\ U_{\varepsilon_{k'}} &\rightharpoonup U^* \text{ in } W^{1,p}(\Omega) \text{ when } k' \longrightarrow +\infty. \end{aligned} \quad (49)$$

By (47) and (49) we have that

$$U^* \neq 0. \quad (50)$$

Moreover, since the operator A is bounded, it follows that $\|A(U_{\varepsilon_{k'}})\|_{W^{-1,q}} \leq C$. Therefore we conclude the existence of $\chi \in W^{-1,q}$ such that $A(U_{\varepsilon_{k'}}) \rightharpoonup \chi$ weakly in $W^{-1,q}$. Suppose that v is an arbitrary fixed function from $C_0^\infty(\overline{\Omega}, \Gamma_\varepsilon)$. Then $v \in W^{1,p}(\Omega, \Gamma_\varepsilon)$ for all small ε . Substitute in the integral identity (11) v as a test function; $U = U_{\varepsilon_{k'}}$, $\lambda = \lambda_{k'}$, and $F = F_{k'}$, when $\varepsilon = \varepsilon_{k'}$. We have that

$$\int_\Omega \sum_{j=1}^n |\nabla U_{\varepsilon_{k'}}(x)|^{p-2} \frac{\partial U_{\varepsilon_{k'}}(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j} dx \quad (51)$$

$$= \lambda_{k'} \int_\Omega |U_{\varepsilon_{k'}}|^{p-2} U_{\varepsilon_{k'}} v dx + \int_\Omega F_{k'} v dx.$$

Now we want to pass to the limit as $\varepsilon_{k'} \rightarrow 0$. Due to the definition of $A(\cdot)$, we see that

$$(\chi, v) = \lim_{\varepsilon_{k'} \rightarrow 0} \left(\lambda_{k'} |U_{\varepsilon_{k'}}|^{p-2} U_{\varepsilon_{k'}} + F_{k'}, v \right). \quad (52)$$

From the other hand,

$$\begin{aligned} (A(U_{\varepsilon_{k'}}), U_{\varepsilon_{k'}}) &\longrightarrow (\chi, U^*), \\ (|U_{\varepsilon_{k'}}|^{p-2} U_{\varepsilon_{k'}}, U_{\varepsilon_{k'}}) &\longrightarrow (|U^*|^{p-2} U^*, U^*). \end{aligned} \quad (53)$$

Let us show now that $\chi = A(U^*)$. The monotonicity of operator A implies

$$\begin{aligned} (A(U_{\varepsilon_{k'}}) - A(U), U_{\varepsilon_{k'}} - U) &\geq 0 \\ \forall U &\in W^{1,p}(\Omega, \Gamma_4). \end{aligned} \quad (54)$$

Passing to the limit, it yields that

$$(\chi - A(U), U^* - U) \geq 0 \quad \forall U \in W^{1,p}(\Omega). \quad (55)$$

If we take $U = U^* - sW$, $s > 0$, $W \in W^{1,p}(\Omega, \Gamma_4)$, we get

$$\begin{aligned} s(\chi - A(U^* - sW), W) &\geq 0 \\ \implies (\chi - A(U^* - sW), W) &\geq 0. \end{aligned} \quad (56)$$

Let $s \rightarrow 0$. Then

$$(\chi - A(U^* - sW), W) \geq 0 \quad \forall W \in W^{1,p}(\Omega, \Gamma_4). \quad (57)$$

This exactly means that $\chi = A(U^*)$. Thus, passing to the limit in the integral identity (51) as $\varepsilon_{k'} \rightarrow 0$, using (48), (49), and Lemma 6, we obtain that

$$\begin{aligned} \int_\Omega |\nabla U^*(x)|^{p-2} \nabla U^*(x) \nabla v(x) dx \\ = \lambda^* \int_\Omega |U^*|^{p-2} U^* v dx \quad \text{for } v \in C^\infty(\Omega, \Gamma_4). \end{aligned} \quad (58)$$

From the density of the embedding $C^\infty(\overline{\Omega}, \Gamma_\varepsilon)$ into $W^{1,p}(\Omega, \Gamma_4)$ we can conclude that this inequality holds also for $v \in W^{1,p}(\Omega, \Gamma_4)$. It follows from Lemma 6 that $U^* \in W^{1,p}(\Omega, \Gamma_4)$. Due to the fact that $U^* \neq 0$ and v is an arbitrary function from $W^{1,p}(\Omega, \Gamma_4)$, it follows that $\lambda^* \in K$ is the eigenvalue of the limit problem (5). But we assumed that K did not contain the eigenvalues of the limit problem (5). This contradiction proves estimate (8).

Step 3. Let us prove now the uniqueness of the solution of (7). Let U_ε^1 and U_ε^2 be two different solutions. Then

$$\begin{aligned} U_\varepsilon^1 - U_\varepsilon^2 &= 0 \quad \text{on } \Gamma_\varepsilon, \\ |\nabla U_\varepsilon^1|^{p-2} \frac{\partial U_\varepsilon^1}{\partial \nu} - |\nabla U_\varepsilon^2|^{p-2} \frac{\partial U_\varepsilon^2}{\partial \nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (59)$$

This together with equation

$$\begin{aligned} (A(U_\varepsilon^1) - A(U_\varepsilon^2), U_\varepsilon^1 - U_\varepsilon^2) &= 0 \\ &= \sum_i \int_\Omega \left(\left| \frac{\partial U_\varepsilon^1}{\partial x_i} \right|^{p-2} \frac{\partial U_\varepsilon^1}{\partial x_i} - \left| \frac{\partial U_\varepsilon^2}{\partial x_i} \right|^{p-2} \frac{\partial U_\varepsilon^2}{\partial x_i} \right) \\ &\quad \cdot \left(\frac{\partial U_\varepsilon^1}{\partial x_i} - \frac{\partial U_\varepsilon^2}{\partial x_i} \right) dx \end{aligned} \quad (60)$$

implies that

$$\frac{\partial U_\varepsilon^1}{\partial x_i} = \frac{\partial U_\varepsilon^2}{\partial x_i} \implies U_\varepsilon^1 = U_\varepsilon^2 \quad (61)$$

due to the boundary conditions. The uniqueness of the solution to (10) can be proved identically.

4.2. Proof of Part 2. Let $\lambda \in K$ be an arbitrary fixed number and assume that the sequence $\{\varepsilon_k\} \rightarrow 0$ when $k \rightarrow +\infty$. Thinking analogously as in Part 1 of the proof, we have existence of a subsequence $\{k'\}$ and a function U^* such that $U_{\varepsilon_{k'}} \rightharpoonup U^*$ in $W^{1,p}(\Omega, \Gamma_4)$ (and strongly in $L_p(\Omega)$) when $k' \rightarrow +\infty$. In addition, $A(U_{\varepsilon_{k'}}) \rightharpoonup \chi$ weakly in $W^{-1,q}$. It can be shown exactly in the same way that $\chi = A(U^*)$.

Now by using Lemma 6 we obtain that the limit function $U^* \in W^{1,p}(\Omega, \Gamma_4)$. Passing to the limit in the integral identity (51) by means of the same reasoning as in the proof of Part 1 of the theorem, we get that

$$\begin{aligned} \int_\Omega |\nabla U^*(x)|^{p-2} \nabla U^*(x) \nabla v(x) dx \\ = \lambda \int |U^*|^{p-2} U^* v dx + \int_\Omega F v dx, \end{aligned} \quad (62)$$

which coincides with the integral identity of problem (10). Since the solution to problem (10) is unique we conclude that $U^* = U_0$. In addition, from (49) we find that

$$\begin{aligned} U_{\varepsilon_k} &\longrightarrow U_0 \text{ strongly in } L_p(\Omega) \\ &\text{and weakly in } W^{1,p}(\Omega), \\ A(U_{\varepsilon_k}) &\rightharpoonup A(U_0) \text{ in } W^{-1,q} \text{ when } \varepsilon \longrightarrow 0. \end{aligned} \quad (63)$$

From the integral identities we have that

$$\begin{aligned} \int_\Omega \left(|\nabla U_{\varepsilon_k}|^{p-2} \nabla U_{\varepsilon_k} - |\nabla U_0|^{p-2} \nabla U_0 \right) \\ \cdot \nabla (U_{\varepsilon_k} - U_0) dx \\ = \lambda \int_\Omega |U_{\varepsilon_k}|^{p-2} U_{\varepsilon_k} U_0 dx \\ - \int_\Omega |\nabla U_0|^{p-2} \nabla U_0 (\nabla U_{\varepsilon_k} - \nabla U_0) dx \\ + \int_\Omega F (U_{\varepsilon_k} - U_0) dx. \end{aligned} \quad (64)$$

Due to convergences (63) the right-hand side tends to zero in the limit; therefore

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_\Omega \left(|\nabla U_{\varepsilon_k}|^{p-2} \nabla U_{\varepsilon_k} - |\nabla U_0|^{p-2} \nabla U_0 \right) \\ \cdot \nabla (U_{\varepsilon_k} - U_0) dx = 0. \end{aligned} \quad (65)$$

By using the inequality

$$2^{1-p} |a - b|^p \leq (|a|^{p-2} a - |b|^{p-2} b) \cdot (a - b), \quad p \geq 2, \quad (66)$$

with $a = \nabla U_{\varepsilon_k}$ and $b = \nabla U_0$ we can show the strong convergence of gradients:

$$\lim_{k \rightarrow \infty} \int_\Omega |\nabla U_{\varepsilon_k} - \nabla U_0|^p dx = 0. \quad (67)$$

Thus, we have proved that, up to a subsequence,

$$\|U_\varepsilon - U_0\|_{W^{1,p}} \longrightarrow 0. \quad (68)$$

The proof is complete.

5. Proof of Theorem 5

Define by Σ the spectrum of problem

$$\begin{aligned} -\Delta_p u(x) &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \gamma, \text{ where } \gamma \subset \partial\Omega, \\ \frac{\partial u}{\partial_p \nu} &\equiv |\nabla u|^{p-2} (\nabla u, \nu) = 0 \quad \text{on } \partial\Omega \setminus \bar{\gamma}. \end{aligned} \quad (69)$$

That is, $\Sigma = \{\lambda \in \mathbb{R} : \text{the problem (69) has the nontrivial weak solution}\}$.

Theorem 12. *The spectrum Σ is nonempty closed set, $\Sigma \subset (0, \infty)$.*

Proof. To show that the spectrum is nonempty, one needs to prove the existence of weak solution to problem (69). We can make use of Theorem 7. We take $V = W^{1,p}(\Omega, \gamma)$ and

$A(u) = -\sum_i (\partial/\partial x_i)(|\partial u/\partial x_i|^{p-2}(\partial u/\partial x_i))$. For $u, v \in W^{1,p}(\Omega, \gamma)$ we define

$$\begin{aligned} (A(u), v) &= - \int_{\Omega} \sum_i \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) v dx \\ &= \int_{\Omega} \sum_i \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx. \end{aligned} \quad (70)$$

One needs to verify the properties of operator A . First, the boundedness is fulfilled since

$$(A(u), u) \leq C \|u\|_{W^{1,p}}^p. \quad (71)$$

To check the semicontinuity, monotonicity of the operator, and the property $(A(u), u)/\|u\| \rightarrow +\infty$ as $\|u\|_{W^{1,p}} \rightarrow +\infty$ one can in the same way as in the proof of Theorem 4. Let us omit the details.

As soon as we conclude that the conditions of Theorem 7 are fulfilled, we conclude the existence of the solution. Thus, the spectrum is nonempty. Check now the positiveness of Σ . Let $\lambda \in \Sigma$ and the corresponding eigenfunction $u \in W^{1,p}(\Omega, \gamma)$. It yields that

$$\begin{aligned} \int_{\Omega} |\nabla u|^p dx &= \lambda \int_{\Omega} |u|^p dx, \\ \text{therefore } \lambda &= \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} > 0. \end{aligned} \quad (72)$$

Let $\lambda_k \in \Sigma$ correspond to eigenfunctions u_k and $\lambda_k \rightarrow \lambda$. Without loss of generality one may assume that $\|u_k\|_{L_p} = 1$. Since $\{\lambda_k\}$ is bounded, it follows from

$$\int_{\Omega} |\nabla u_k|^p dx = \lambda_k \int_{\Omega} |u_k|^p dx \quad (73)$$

that $\|u_k\|_{W^{1,p}(\Omega, \gamma)} \leq C$ which implies the existence of $u \in W^{1,p}(\Omega, \gamma)$ such that up to a subsequence

$$\begin{aligned} u_k &\rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \gamma), \\ u_k &\rightarrow u \text{ strongly in } L_p(\Omega). \end{aligned} \quad (74)$$

Moreover,

$$\|A(u_k)\| = (A(u_k), u_k) \leq C \|u_k\|_{L_p}^{p-1}, \quad (75)$$

and therefore there exists χ such that

$$A(u_k) \rightharpoonup \chi \text{ weakly in } W^{-1,q}(\Omega, \gamma). \quad (76)$$

These convergences imply that $\|u\|_{L_p} = 1$ and

$$\int_{\Omega} \chi \nabla v dx = \lambda \int_{\Omega} |u|^{p-2} u v dx \quad \text{for } v \in W^{1,p}(\Omega, \gamma). \quad (77)$$

Thus it remains to show that

$$\int_{\Omega} \chi \nabla v dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx. \quad (78)$$

For every $v \in W^{1,p}(\Omega, \gamma)$ it holds that

$$\begin{aligned} 0 &\leq \int_{\Omega} (|\nabla u_k|^{p-2} \nabla u_k - |\nabla v|^{p-2} \nabla v, \nabla u_k - \nabla v) dx \\ &= \int_{\Omega} \lambda_k |u_k|^{p-2} u_k (u_k - v) dx \\ &\quad - \int_{\Omega} |\nabla v|^{p-2} \nabla v (\nabla u_k - \nabla v) dx. \end{aligned} \quad (79)$$

Passing to the limit as $k \rightarrow \infty$, one obtains that

$$\begin{aligned} 0 &\leq \int_{\Omega} \lambda |u|^{p-2} u (u - v) dx \\ &\quad - \int_{\Omega} |\nabla v|^{p-2} \nabla v (\nabla u - \nabla v) dx \\ &= \int_{\Omega} \chi (\nabla u - \nabla v) dx - \int_{\Omega} |\nabla v|^{p-2} \nabla v (\nabla u - \nabla v) dx. \end{aligned} \quad (80)$$

Take now $v = u - tw$, $w \in W^{1,p}(\Omega, \gamma)$, $t \in \mathbb{R}_+$. Then

$$0 \leq \int_{\Omega} (\chi - |\nabla(u - tw)|^{p-2} \nabla(u - tw)) \nabla w dx. \quad (81)$$

Passing to the limit as $t \rightarrow +0$, we get

$$0 \leq \int_{\Omega} (\chi - |\nabla u|^{p-2} \nabla u) \nabla w dx. \quad (82)$$

From this inequality we deduce the validity of (78). \square

Let us mention the properties of the first eigenvalue to spectral problem (69). The following theorem directly follows from the results in [34].

Theorem 13. *The smallest eigenvalue*

$$\lambda^1 := \inf_{u \in W^{1,p}(\Omega, \gamma) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx} \quad (83)$$

is simple and isolated. Moreover, if u_1, u_2 are two eigenfunctions corresponding to λ^1 , then they are proportional: $u_1 = ku_2$.

Now we come back to our spectral problems (3) and (5). The existence of normalized solution and properties of spectrum follows by Theorems 12 and 13. The uniqueness of the normalized solution is a direct consequence of Theorem 13. Concerning the regularity, the known fact (see [40]) says that if eigenfunctions $u \in W_0^{1,p}$, then $u \in L^\infty$. The statements of Theorem 5 can be proved similarly to the proof of statement (2) of Theorem 4.

Indeed, since λ_ε^1 is isolated, that is, $\exists \delta > 0 : |\lambda_\varepsilon^1 - \lambda_\varepsilon^2| > \delta$ (λ_ε^2 is the second eigenvalue to (3)), there exists a compact containing λ_ε^1 . Hence, there exists a subsequence $\{\varepsilon_k\}_{k=1}^\infty \rightarrow 0$ such that $\lambda_{\varepsilon_k}^1 \rightarrow \lambda^*$. The boundedness of u_{ε_k} in L_p implies the existence of limit u^* (strong in L_p and weak in $W^{1,p}$).

The boundedness of operator $A(u_{\varepsilon_k})$ in the dual space $W^{-1,q}$ gives the existence of weak limit

$$\begin{aligned}\chi &= \lim_{k \rightarrow \infty} A(u_{\varepsilon_k}) \\ &= \lim_{k \rightarrow \infty} \frac{\partial}{\partial x_j} \left(\sum_{j=1}^n |\nabla u_{\varepsilon_k}(x)|^{p-2} \frac{\partial u_{\varepsilon_k}(x)}{\partial x_j} \right).\end{aligned}\quad (84)$$

Thus, passing to the limit in the identity

$$\begin{aligned}&\int_{\Omega} \sum_{j=1}^n |\nabla u_{\varepsilon_k}(x)|^{p-2} \frac{\partial u_{\varepsilon_k}(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j} dx \\ &= \lambda_{\varepsilon_k}^1 \int_{\Omega} |u_{\varepsilon_k}|^{p-2} u_{\varepsilon_k} v dx,\end{aligned}\quad (85)$$

we obtain

$$\begin{aligned}&\int_{\Omega} \sum_{j=1}^n |\nabla u^*(x)|^{p-2} \frac{\partial u^*(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j} dx \\ &= \lambda^* \int_{\Omega} |u^*|^{p-2} u^* v dx.\end{aligned}\quad (86)$$

Having in mind the same arguments as in the proof of Theorem 4 one can show that $\lambda^* = \lambda_0^1$, $u^* = u_0$, and $\chi = A(u_0) = -\sum_i (\partial/\partial x_i)(|\partial u_0/\partial x_i|^{p-2}(\partial u_0/\partial x_i))$ which means that problem (5) is the limit one for (3). Analogously as in Theorem 4 one can derive that

$$\|u_{\varepsilon_k} - u_0\|_{W^{1,p}} \rightarrow 0. \quad (87)$$

Thus, Theorem 5 is completely proved.

The question on convergence of the full spectrum for similar p -Laplace boundary-value problem is studied, for example, in [33, 35]. We do not cover this studying for our problem since in the applications of the results of the present paper we will use only the fact about converging of the first eigenvalues.

6. The Estimate of the Solution in a Neighborhood of the Eigenvalue

Let us derive an estimate for the solution to problem (7) in a neighborhood of λ_0^1 .

Lemma 14. *Let λ be close to λ_0^1 and λ_{ε}^1 converges to λ_0^1 . Then the following estimate holds:*

$$\|U_{\varepsilon}\|_{W^{1,p}}^p \leq C \frac{\|F\|_{L_q}^2}{|\lambda_{\varepsilon}^1 - \lambda|}. \quad (88)$$

Proof. Since the function $U_{\varepsilon} \in W^{1,p}(\Omega_{\varepsilon}, \Gamma_{\varepsilon})$, it satisfies Friedrichs inequality $\|U_{\varepsilon}\|_{L_p}^p \leq K \|\nabla U_{\varepsilon}\|_{L_p}^p$. Moreover, it is clear that the best constant in the inequality is $1/\lambda_{\varepsilon}^1$, where λ_{ε}^1 is the first eigenvalue to (3). Thus,

$$\|U_{\varepsilon}\|_{L_p}^p \leq \frac{1}{\lambda_{\varepsilon}^1} \|\nabla U_{\varepsilon}\|_{L_p}^p. \quad (89)$$

By using estimates (43), (8), (89), integral identity with $v = U_{\varepsilon}$, and equivalent norm in $W^{1,p}$, it yields that

$$\begin{aligned}\|\nabla U_{\varepsilon}\|_{L_p}^p &\leq \lambda \|U_{\varepsilon}\|_{L_p}^p + \|F\|_{L_p} \|U_{\varepsilon}\|_{L_p} \\ &\leq \frac{\lambda}{\lambda_{\varepsilon}^1} \|\nabla U_{\varepsilon}\|_{L_p}^p + \|F\|_{L_q}^2 \\ &\Rightarrow |\lambda_{\varepsilon}^1 - \lambda| \|\nabla U_{\varepsilon}\|_{L_p}^p \leq C \|F\|_{L_q}^2,\end{aligned}\quad (90)$$

where $|\lambda_{\varepsilon}^1| < C = \text{const.}$

Replacing $\|\nabla U_{\varepsilon}\|_{L_p}^p$ with the equivalent norm $\|U_{\varepsilon}\|_{W^{1,p}}^p$, we obtain (88). \square

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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