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Research Article

Implementation of TAGE Method Using Seikkala Derivatives Applied to Two-Point Fuzzy Boundary Value Problems

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Iterative methods particularly the Two-Parameter Alternating Group Explicit (TAGE) methods are used to solve system of linear equations generated from the discretization of two-point fuzzy boundary value problems (FBVPs). The formulation and implementation of the TAGE method are also presented. Then numerical experiments are carried out onto two example problems to verify the effectiveness of the method. The results show that TAGE method is superior compared to GS method in the aspect of number of iterations, execution time, and Hausdorff distance.

1. Introduction

Fuzzy boundary value problems (FBVPs) and treating fuzzy differential equations were one of the major applications for fuzzy number arithmetic [1]. FBVPs can be approached by two types. For instance, the first approach addresses problems in which the boundary values are fuzzy where the solution is still in fuzzy function. Then the second approach is based on generating the fuzzy solution from the crisp solution [2]. To solve these problems, numerical methods obtain their approximate solution. Consequently, in this paper, let two-point linear FBVPs be defined in general form as follows:

$$\begin{gathered}\tilde{x}''(t)+p(t)\,\tilde{x}'(t)+q(t)\,\tilde{x}(t)=f(t),\quad t\in[a,b],\\ \tilde{x}(a)=\sigma,\\ \tilde{x}(b)=\omega,\end{gathered}\tag{1}$$

where $\tilde{x}(t)$ is a fuzzy function and $f(t)$, $p(t)$, and $q(t)$ are continuous functions on $[a,b]$, whereas, σ and ω are fuzzy numbers.

Based on the Seikkala derivative [3], (1) will be solved numerically by applying the second-order central finite difference scheme to discretize the two-point linear FBVPs into linear systems. Then the generated linear systems will be solved iteratively by using Two-Parameter Alternating Group Explicit (TAGE) method [4, 5]. By considering the Group Explicit (GE) method for the numerical solution of parabolic and elliptic problems, Evans [6, 7] discovered Alternating Group Explicit method. Later, Sukon and Evans [5] expanded this approach to initiate the TAGE method thus proving that this method is superior compared to AGE method. From previous studies, findings of the papers related to the TAGE iterative method and its variants [8–13] have shown that TAGE method has been widely used to solve the nonfuzzy problems. Due to the efficiency of the methods, this paper extends the application of TAGE iterative method in solving fuzzy problems. Since the fuzzy linear systems will be constructed, the iterative method becomes the natural option to get a fuzzy numerical solution of the problem.

The outline of the paper is organized as follows. Section 2 will discuss the finite difference method based on the second-order finite difference scheme in discretizing two-point FBVPs, while Section 3 presents the formulation and implementation of the TAGE methods in solving linear systems generated from the second-order finite difference scheme. Section 4 shows some numerical examples and conclusions are given in Section 5.

2. Finite Difference Approximation Equations

To be clear, let $\tilde{x}$ be a fuzzy subset of real numbers. It is characterized by the corresponding membership function evaluated at t, writing $\tilde{x}(t)$ as a number in $[0,1]$. α-cut of $\tilde{x}$, in which α is denoted as a crisp number, can be written as $\tilde{x}(\alpha)$ in $\{x \mid \tilde{x}(t) \geq \alpha\}$, for $0 < \alpha \leq 1$. The interval of the α-cut of fuzzy numbers will be written as $\tilde{x}(\alpha) = [\underline{x}(\alpha), \overline{x}(\alpha)]$, for all α, since they were always closed and bounded [14]. Suppose $(\underline{x}, \overline{x})$ is parametric form of fuzzy function x. For arbitrary positive integer n subdivide the interval $a \leq t \leq b$, whereas $t_i = a + ih$ $(i = 0, 1, 2, \ldots, n)$ for i and $h = (b-a)/n$.

Denote the value of x and $(\underline{x}, \overline{x})$ at the representative point t_i $(i = 0, 1, 2, \ldots, n)$ by x_i at $(\underline{x_i}, \overline{x_i})$. Thus, by using the second-order central finite difference scheme, problem (1) can be developed as

$$\underline{x_i''} \approx \frac{\underline{x_{i-1}} - 2\underline{x_i} + \underline{x_{i+1}}}{h^2}, \quad (2a)$$

$$\overline{x_i''} \approx \frac{\overline{x_{i-1}} - 2\overline{x_i} + \overline{x_{i+1}}}{h^2}, \quad (2b)$$

$$\underline{x_i'} \approx \frac{\underline{x_{i+1}} - \underline{x_{i-1}}}{2h}, \quad (3a)$$

$$\overline{x_i'} \approx \frac{\overline{x_{i+1}} - \overline{x_{i-1}}}{2h}, \quad (3b)$$

which give

$$\begin{aligned} x_i'' &= \left(\underline{x_i''}, \overline{x_i''}\right), \\ x_i' &= \left(\underline{x_i'}, \overline{x_i'}\right). \end{aligned} \quad (4)$$

By using parametric form of fuzzy function, (1) can be written as

$$\underline{x_i''} = \underline{f(t_i) - p(t_i)\, x_i' - q(t_i)\, x_i}, \quad (5a)$$

$$\overline{x_i''} = \overline{f(t_i) - p(t_i)\, x_i' - q(t_i)\, x_i}. \quad (5b)$$

Suppose that $p(t_i) > 0$ and $q(t_i) > 0$ for $i = 0, 1, 2, \ldots, n$. Then

$$\underline{x_i''} + p(t_i)\,\underline{x_i'} + q(t_i)\,\underline{x_i} = f(t_i), \quad (6a)$$

$$\overline{x_i''} + p(t_i)\,\overline{x_i'} + q(t_i)\,\overline{x_i} = f(t_i). \quad (6b)$$

By applying (2a) and (3a), (6a) will be reduced to

$$\begin{aligned} &\frac{\underline{x_{i-1}} - 2\underline{x_i} + \underline{x_{i+1}}}{h^2} + p(t_i)\frac{\underline{x_{i+1}} - \underline{x_{i-1}}}{2h} + q(t_i)\,\underline{x_i} \\ &\quad = f(t_i) \end{aligned} \quad (7a)$$

for $i = 1, 2, \ldots, n-1$. Meanwhile, by substituting (2b) and (3b) into (6b), we will have

$$\begin{aligned} &\frac{\overline{x_{i-1}} - 2\overline{x_i} + \overline{x_{i+1}}}{h^2} + p(t_i)\frac{\overline{x_{i+1}} - \overline{x_{i-1}}}{2h} + q(t_i)\,\overline{x_i} \\ &\quad = f(t_i). \end{aligned} \quad (7b)$$

Then, (7a) and (7b) can be rewritten as follows:

$$\begin{aligned} &(2 - hp(t_i))\,\underline{x_{i-1}} + \left(2h^2 q(t_i) - 4\right)\underline{x_i} \\ &\quad + (2 + hp(t_i))\,\underline{x_{i+1}} = 2h^2 f(t_i), \end{aligned} \quad (8a)$$

$$\begin{aligned} &(2 - hp(t_i))\,\overline{x_{i-1}} + \left(2h^2 q(t_i) - 4\right)\overline{x_i} \\ &\quad + (2 + hp(t_i))\,\overline{x_{i+1}} = 2h^2 f(t_i), \end{aligned} \quad (8b)$$

respectively, for $i = 1, 2, \ldots, n-1$. Since both of (8a) and (8b) have the same form in terms of the equation, except that, based on the interval of the α-cuts, the differences are identified only in the upper and lower bounds, it can be rewritten as

$$\rho_i x_{i-1} + \beta_i x_i + \varphi_i x_{i+1} = F_i \quad (9)$$

for $i = 1, 2, \ldots, n-1$, where

$$\begin{aligned} \rho_i &= 2 - hp(t_i), \\ \beta_i &= 2h^2 q(t_i) - 4, \\ \varphi_i &= 2 + hp(t_i), \\ F_i &= 2h^2 f(t_i). \end{aligned} \quad (10)$$

Now, we can express the second-order central finite difference approximation (9) in a matrix form as

$$Ax = b \quad (11)$$

with

$$\begin{aligned} A &= \begin{bmatrix} \beta_1 & \varphi_1 & & & & & \\ \rho_2 & \beta_2 & \varphi_2 & & & & \\ & \rho_3 & \beta_3 & \varphi_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \rho_{n-3} & \beta_{n-3} & \varphi_{n-3} & \\ & & & & \rho_{n-2} & \beta_{n-2} & \varphi_{n-2} \\ & & & & & \rho_{n-1} & \beta_{n-1} \end{bmatrix}_{(n-1)\times(n-1)}, \\ x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \end{bmatrix}^T, \\ b &= \begin{bmatrix} f_1 - \rho_1 x_0 & f_2 & \cdots & f_{n-2} & f_{n-1} - \varphi_{n-1} x_n \end{bmatrix}^T. \end{aligned} \quad (12)$$

Since this study will deal with an application of the method, the computational method of it will be diagonally dominant matrix and positive definite matrix [15].

3. Two-Parameter Alternating Group Explicit Iterative Method

Based on previous study conducted by Evans, clearly we can see that they have discussed theoretically how to compute the value of parameter r given by Mohanty et al. [9–13]. In this paper, the optimum value of parameters r_1 and r_2 will be calculated by implementing several numerical experiments, so those optimum values will be found if the number of iterations is smaller.

Family of AGE can be considered efficient to two-step method to solve linear system. None of the researchers had been trying to apply this method in solving fuzzy problem generated from discretization of fuzzy partial difference equation. This paper will discuss the application of this iterative method which will solve the fuzzy linear system given by (1). Consider a class of methods mentioned in [4, 5] which is based on the splitting of the matrix A into the sum of its constituent symmetric and positive definite matrices, as follows:

$$A = G_1 + G_2, \tag{13}$$

where

$$G_1 = \left[\begin{array}{cc|cc|c|cc} g_1 & \varphi_1 & & & & & \\ \rho_2 & g_2 & & & & & \\ \hline & & g_3 & \varphi_3 & & & \\ & & \rho_4 & g_4 & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & g_{n-2} & \varphi_{n-2} \\ & & & & & \rho_{n-1} & g_{n-1} \end{array}\right],$$

$$G_2 = \left[\begin{array}{c|cc|c|cc|c} g_1 & & & & & & \\ \hline & g_2 & \varphi_2 & & & & \\ & \rho_3 & g_3 & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & g_{n-3} & \varphi_{n-3} & \\ & & & & \rho_{n-2} & g_{n-2} & \\ \hline & & & & & & g_{n-1} \end{array}\right] \tag{14}$$

if n is odd. Similarly, we define the following matrices:

$$G_1 = \left[\begin{array}{cc|c|cc|c} g_1 & \varphi_1 & & & & \\ \rho_2 & g_2 & & & & \\ \hline & & \ddots & & & \\ \hline & & & g_{n-3} & \varphi_{n-3} & \\ & & & \rho_{n-2} & g_{n-2} & \\ \hline & & & & & g_{n-1} \end{array}\right],$$

$$G_2 = \left[\begin{array}{c|cc|c|cc} g_1 & & & & & \\ \hline & g_2 & \varphi_2 & & & \\ & \rho_3 & g_3 & & & \\ \hline & & & \ddots & & \\ \hline & & & & g_{n-2} & \varphi_{n-2} \\ & & & & \rho_{n-1} & g_{n-1} \end{array}\right] \tag{15}$$

if n is even, with $g_i = \beta_i/2$ $(i = 1, 2, \ldots, n-1)$. In this paper, we only consider that case n is even.

Then (11) becomes

$$(G_1 + G_2)\, x = b. \tag{16}$$

Thus, the explicit form of TAGE method can be written as

$$\begin{aligned} x^{(k+1/2)} &= (G_1 + r_1 I)^{-1} \left[b - (G_2 - r_1 I)\, x^{(k)} \right], \\ x^{(k+1)} &= (G_2 + r_2 I)^{-1} \left[b - (G_1 - r_2 I)\, x^{(k+1/2)} \right], \end{aligned} \tag{17}$$

where $r_1, r_2 > 0$ are the acceleration parameters, and a pair of $(G_1 + r_1 I)$ and $(G_2 + r_2 I)$ are invertible. From (17), therefore, the implementation of TAGE method is presented in Algorithm 1.

Algorithm 1 (TAGE method).

(i) Initialize $\widetilde{U}^{(0)} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.

(ii) For $i = 1p, 2p, \ldots, n - p$, initialize parameters ρ_i, β_i, φ_i, f_i, r_1, r_2, G_1, and G_2.

(iii) *First Sweep*. For $i = 1p, 2p, \ldots, n - p$, compute

$$x^{(k+1/2)} = (G_1 + r_1 I)^{-1} \left[b - (G_2 - r_1 I)\, x^{(k)} \right]. \tag{18}$$

(iv) *Second Sweep*. For $i = 1p, 2p, \ldots, n - p$, compute

$$x^{(k+1)} = (G_2 + r_2 I)^{-1} \left[b - (G_1 - r_2 I)\, x^{(k+1/2)} \right]. \tag{19}$$

(v) *Convergence Test*. If the convergence criterion, that is, $\|\widetilde{U}^{(k+1)} - \widetilde{U}^{k}\|_\infty \leq \varepsilon$, is satisfied, go to Step (vi). Otherwise go back to Step (ii).

(vi) Display approximate solutions.

4. Numerical Experiments

Two examples of FBVPs are considered to verify the effectiveness of GS, AGE, and TAGE methods. For comparison purposes, three parameters were observed that are number of iterations, execution time (in seconds), and Hausdorff distance (as mentioned in Definition 2). Based on these two problems, numerical results for GS, AGE, and TAGE methods have been recorded in Tables 1 to 5.

Definition 2 (see [16]). Given two minimum bounding rectangles P and Q, a lower bound of the Hausdorff distance from the elements confined by P to the elements confined by Q is defined as

$$\begin{aligned} &\mathrm{HausDistLB}\,(P, Q) \\ &\quad = \mathrm{Max}\left\{ \mathrm{MinDist}\left(f_\alpha, Q\right) : f_\alpha \in \mathrm{Faces\ Of}\ (P) \right\}. \end{aligned} \tag{20}$$

Problem 1. Consider

$$x''(t) = \widetilde{k}\,(-6t), \quad t \in (0, 1), \tag{21}$$

where $\widetilde{k}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$ with the boundary conditions $\widetilde{x}(0) = 0$ and $\widetilde{x}(1) = 1$. The exact solutions for

$$\underline{x''}\,(t; \alpha) = \underline{k}\,(\alpha)\,(-6t), \tag{22a}$$

Table 1: Comparison of three parameters between GS, AGE, and TAGE methods at $\alpha = 0.00$.

		Methods	n 512	1024	2048	4096	8192
Problem 1	Number of iterations	GS	681711	2431928	8548735	29480437	99066551
		AGE	96747	354438	1279808	4549671	15883620
		TAGE	77377	279463	876061	2879619	10383345
	Execution time	GS	48.94	211.19	989.91	5719.20	32465.10
		AGE	8.00	39.00	202.00	1310.00	8125.00
		TAGE	7.00	31.00	141.00	822.00	5342.00
	Hausdorff distance	GS	$2.6560e-06$	$1.0624e-05$	$4.2497e-05$	$1.6999e-04$	$6.7995e-04$
		AGE	$3.2355e-07$	$1.3084e-06$	$5.2674e-06$	$2.1148e-05$	$8.4787e-05$
		TAGE	$2.4955e-07$	$9.8491e-07$	$3.0435e-06$	$1.0607e-05$	$4.5689e-05$
Problem 2	Number of iterations	GS	475487	1692329	5930853	20369573	68062962
		AGE	67638	247434	891667	3161503	10997813
		TAGE	53492	187245	671456	2122064	7505046
	Execution time	GS	35.27	155.77	764.09	4457.31	26063.40
		AGE	6.00	27.00	141.00	912.00	5676.00
		TAGE	5.00	20.00	107.00	608.00	3887.00
	Hausdorff distance	GS	$2.4952e-06$	$7.7115e-06$	$3.0279e-05$	$1.2097e-04$	$4.8386e-04$
		AGE	$8.3545e-07$	$1.0823e-06$	$3.7861e-06$	$1.5058e-05$	$6.0327e-05$
		TAGE	$7.7615e-07$	$8.1054e-07$	$2.6440e-06$	$1.0853e-05$	$3.2986e-05$

Table 2: Comparison of three parameters between GS, AGE, and TAGE methods at $\alpha = 0.25$.

		Methods	n 512	1024	2048	4096	8192
Problem 1	Number of iterations	GS	682475	2434982	8560953	29529307	99262033
		AGE	96840	354815	1281323	4555751	15908020
		TAGE	77449	279746	876948	2882382	10399116
	Execution time	GS	49.07	211.36	991.23	5874.81	32551.12
		AGE	9.00	39.00	202.00	1301.00	8164.00
		TAGE	7.00	31.00	141.00	827.00	5402.00
	Hausdorff distance	GS	$2.6560e-06$	$1.0624e-05$	$4.2497e-05$	$1.6999e-04$	$6.7995e-04$
		AGE	$3.2355e-07$	$1.3083e-06$	$5.2675e-06$	$2.1148e-05$	$8.4786e-05$
		TAGE	$2.4955e-07$	$9.8490e-07$	$3.0417e-06$	$1.0452e-05$	$4.5526e-05$
Problem 2	Number of iterations	GS	476030	1694502	5939547	20404350	68202066
		AGE	67704	247701	892745	3165828	11015151
		TAGE	53543	187435	672208	2124610	7514448
	Execution time	GS	35.26	155.79	756.06	4465.35	25999.98
		AGE	6.00	27.00	142.00	903.00	5652.00
		TAGE	5.00	21.00	106.00	605.00	3893.00
	Hausdorff distance	GS	$2.4650e-06$	$7.7039e-06$	$3.0277e-05$	$1.2097e-04$	$4.8386e-04$
		AGE	$8.0517e-07$	$1.0748e-06$	$3.7841e-06$	$1.5058e-05$	$6.0327e-05$
		TAGE	$7.4595e-07$	$8.0323e-07$	$2.6424e-06$	$1.0749e-05$	$3.2626e-05$

$$\overline{x''}(t;\alpha) = \overline{k}(\alpha)(-6t) \tag{22b}$$

are

$$\underline{x}(t;\alpha) = \underline{k}(\alpha)\left[-t^3 + 2t\right], \tag{23a}$$

$$\overline{x}(t;\alpha) = \overline{k}(\alpha)\left[-t^3 + 2t\right], \tag{23b}$$

respectively.

Problem 2 (see [17]). Consider

$$x''(t) - 4x(t) = \tilde{k}(4\cosh(1)), \quad t \in (0,1), \tag{24}$$

where $\tilde{k}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$ with the boundary conditions $\tilde{x}(0) = 0$ and $\tilde{x}(1) = 0$. The exact solutions for

$$\underline{x''}(t;\alpha) - 4\underline{x}(t;\alpha) = \underline{k}(\alpha)(4\cosh(1)), \tag{25a}$$

Table 3: Comparison of three parameters between GS, AGE, and TAGE methods at $\alpha = 0.50$.

		Methods	n 512	1024	2048	4096	8192
Problem 1	Number of iterations	GS	683007	2437112	8569470	29563373	99398298
		AGE	96905	355076	1282378	4559989	15925021
		TAGE	77499	279944	877567	2884304	10408307
	Execution time	GS	49.25	210.43	988.93	5784.36	32665.34
		AGE	9.00	39.00	203.00	1311.00	8152.00
		TAGE	6.00	31.00	141.00	837.00	5397.00
	Hausdorff distance	GS	$2.6560e-06$	$1.0624e-05$	$4.2497e-05$	$1.6999e-04$	$6.7995e-04$
		AGE	$3.2355e-07$	$1.3084e-06$	$5.2675e-06$	$2.1148e-05$	$8.4785e-05$
		TAGE	$2.4955e-07$	$9.8489e-07$	$3.0398e-06$	$1.0293e-05$	$4.5445e-05$
Problem 2	Number of iterations	GS	476410	1696018	5945607	20428592	68299033
		AGE	67751	247888	893496	3168843	11027246
		TAGE	53578	187569	672733	2126364	7520993
	Execution time	GS	35.40	155.80	757.38	4585.51	26078.03
		AGE	6.00	27.00	141.00	912.00	5696.00
		TAGE	5.00	21.00	107.00	620.00	3900.00
	Hausdorff distance	GS	$2.4346e-06$	$7.6963e-06$	$3.0275e-05$	$1.2097e-04$	$4.8386e-04$
		AGE	$7.7486e-07$	$1.0672e-06$	$3.7823e-06$	$1.5057e-05$	$6.0327e-05$
		TAGE	$7.1585e-07$	$7.9587e-07$	$2.6408e-06$	$1.0642e-05$	$3.2256e-05$

Table 4: Comparison of three parameters between GS, AGE, and TAGE methods at $\alpha = 0.75$.

		Methods	n 512	1024	2048	4096	8192
Problem 1	Number of iterations	GS	683321	2438369	8574499	29583490	99478766
		AGE	96944	355232	1283001	4562489	15935054
		TAGE	77528	280061	877932	2885438	10414635
	Execution time	GS	49.22	210.33	1026.58	5771.53	32617.94
		AGE	8.00	39.00	203.00	1298.00	8186.00
		TAGE	7.00	31.00	141.00	835.00	5395.00
	Hausdorff distance	GS	$2.6560e-06$	$1.0624e-05$	$4.2497e-05$	$1.6999e-04$	$6.7995e-04$
		AGE	$3.2355e-07$	$1.3083e-06$	$5.2675e-06$	$2.1148e-05$	$8.4786e-05$
		TAGE	$2.4958e-07$	$9.8485e-07$	$3.0382e-06$	$1.0131e-05$	$4.5365e-05$
Problem 2	Number of iterations	GS	476633	1696912	5949186	20442908	68356295
		AGE	67778	247998	893940	3170624	11034378
		TAGE	53599	187647	673042	2127413	7524856
	Execution time	GS	35.42	155.72	757.27	4364.75	26127.43
		AGE	6.00	27.00	141.00	914.00	5706.00
		TAGE	4.00	20.00	107.00	612.00	3937.00
	Hausdorff distance	GS	$2.4044e-06$	$7.6888e-06$	$3.0273e-05$	$1.2097e-04$	$4.8386e-04$
		AGE	$7.4463e-07$	$1.0596e-06$	$3.7803e-06$	$1.5057e-05$	$6.0327e-05$
		TAGE	$6.8564e-07$	$7.8857e-07$	$2.6393e-06$	$1.0531e-05$	$3.1874e-05$

$$\overline{x''}(t;\alpha) - 4\overline{x}(t;\alpha) = \overline{k}(\alpha)(4\cosh(1)) \tag{25b}$$

are

$$\underline{x}(t;\alpha) = \underline{k}(\alpha)[\cosh(2t-1) - \cosh(1)], \tag{26a}$$

$$\overline{x}(t;\alpha) = \overline{k}(\alpha)[\cosh(2t-1) - \cosh(1)], \tag{26b}$$

respectively.

5. Conclusions

In this paper, TAGE method was used to solve linear systems which arise from the discretization of two-point FBVPs using the second-order central finite difference scheme. The results show that TAGE method is more superior in terms of the number of iterations, execution time, and Hausdorff distance compared to the AGE and GS methods. Since TAGE is well suited for parallel computation, it can be considered as a main

Table 5: Comparison of three parameters between GS, AGE, and TAGE methods at $\alpha = 1.00$.

		Methods	n 512	1024	2048	4096	8192
Problem 1	Number of iterations	GS	683426	2438784	8576162	29590144	99505380
		AGE	96956	355282	1283208	4563320	15938400
		TAGE	77538	280098	878054	2885812	10416768
	Execution time	GS	49.45	210.66	809.53	5758.67	32519.13
		AGE	9.00	39.00	202.00	1313.00	8221.00
		TAGE	7.00	31.00	141.00	817.00	5383.00
	Hausdorff distance	GS	$2.6559e-06$	$1.0624e-05$	$4.2497e-05$	$1.6999e-04$	$6.7995e-04$
		AGE	$3.2354e-07$	$1.3084e-06$	$5.2674e-06$	$2.1148e-05$	$8.4783e-05$
		TAGE	$2.4955e-07$	$9.8492e-07$	$3.0366e-06$	$9.9651e-06$	$4.5321e-05$
Problem 2	Number of iterations	GS	476706	1697208	5950370	20447642	68375230
		AGE	67786	248034	894086	3171216	11036748
		TAGE	53606	187674	673146	2127768	7526132
	Execution time	GS	35.43	155.72	755.20	4615.31	25815.45
		AGE	6.00	27.00	141.00	915.00	5662.00
		TAGE	4.00	21.00	107.00	613.00	3941.00
	Hausdorff distance	GS	$2.3742e-06$	$7.6812e-06$	$3.0271e-05$	$1.2097e-04$	$4.8386e-04$
		AGE	$7.1441e-07$	$1.0521e-06$	$3.7785e-06$	$1.5056e-05$	$6.0326e-05$
		TAGE	$6.5552e-07$	$7.8117e-07$	$2.6376e-06$	$1.0417e-05$	$3.1481e-05$

advantage because this method has groups of independent task which can be implemented simultaneously. It is hoped that the capability of the proposed method will be helpful for the further investigation in solving any multidimensional fuzzy partial differential equations [18]. Basically the results of this paper can be classified as one of full-sweep iteration. Apart from the concept of the full-sweep iteration, further investigation of half-sweep [19–24] and quarter-sweep [25–27] iterations can also be considered in order to speed up the convergence rate of the standard proposed iterative methods. Other than that, further study will be extended to solve nonlinear problem by combining Newton-Raphson method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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