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Research Article

Positive Solutions for Class of State Dependent Boundary Value Problems with Fractional Order Differential Operators

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We consider the following state dependent boundary-value problem $D^{\alpha}_{0+}y(t) - pD^{\beta}_{0+}g(t, y(\sigma(t))) + f(t, y(\tau(t))) = 0$, $0 < t < 1$; $y(0) = 0$, $\eta y(\sigma(1)) = y(1)$, where D^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha < 2$, $0 < \eta < 1$, $p \le 0$, $0 < \beta < 1$, $\beta + 1 - \alpha \ge 0$ the function g is defined as $g(t, u) : [0, 1] \times [0, \infty) \to [0, \infty)$, and $g(0, 0) = 0$ the function f is defined as $f(t, u) : [0, 1] \times [0, \infty) \to [0, \infty)\sigma(t), \tau(t)$ are continuous on t and $0 \le \sigma(t), \tau(t) \le t$. Using Banach contraction mapping principle and Leray-Schauder continuation principle, we obtain some sufficient conditions for the existence and uniqueness of the positive solutions for the above fractional order differential equations, which extend some references.

1. Introduction

Fractional order differential equations has useful applications in many fields, such as physics, mechanics, chemistry, engineering, biology, and so on. There has been a significant development in fractional differential equations (e.g., [1–9]). In the previous papers, some authors investigated fractional order partial differential equations [10–15]. For example, Wu [15] used the wavelet operational method for solving fractional partial differential equations numerically. Since it is one of the important fields to be concerned with the boundary value problems for fractional order differential equations, some authors considered the existence of positive solutions for fractional differential equations or systems with boundary value conditions [16–25] and the stability [26].

As early as 1994, Delbosco [27] investigated the nonlinear Dirichlet-type problem

$$x^{\alpha-1}\left(D^{\alpha}_{0+}y\right)(x) = f\left(y(x)\right), \quad 0 < x < 1,\ 1 < \alpha < 2, \\ y(0) = y'(1) = 0, \tag{1}$$

where D^{α} is α order Riemann-Liouville derivative. The author had proved that if f is a Lipschitz function, then the problem has at least one solution $y(x)$ in a certain subspace of $C[0, 1]$ in which the fractional derivative has a Hölder property.

Later, using some fixed point theorems, Bai and Lü [20] obtained the existence of positive solutions of the following equation with boundary value conditions

$$D^{\alpha}_{0+}y(t) + f\left(t, y(t)\right) = 0, \quad 0 < t < 1,\ 1 < \alpha < 2, \\ y(0) = y(1) = 0, \\ \text{or} \quad y(0) + y'(0) = y(1) + y'(1) = 0. \tag{2}$$

More recently, Bai [9] also considered the following boundary value problem

$$D^{\alpha}_{0+}y(t) + f\left(t, y(t)\right) = 0, \quad 0 < t < 1,\ 1 < \alpha \le 2, \\ y(0) = 0, \\ \beta y(\eta) = y(1). \tag{3}$$

By constructing a Green's function, and using contraction map principle, the author obtained some existence conditions of positive solutions for (3).

Motivated by the above references, we consider a state dependent boundary value problem with fractional order differential operators

$$\begin{gathered} D_{0+}^{\alpha} y(t) - p D_{0+}^{\beta} g(t, y(\sigma(t))) + f(t, y(\tau(t))) = 0, \\ 0 < t < 1, \\ y(0) = 0, \\ \eta y(\sigma(1)) = y(1), \end{gathered} \tag{4}$$

where D^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha < 2$, $0 < \beta < 1$, $0 < \eta < 1$, $p \le 0$, $\beta + 1 - \alpha \ge 0$, $1 - \eta\sigma^{\alpha-1}(1) > 0$; the function g is defined as $g(t, u) : [0, 1] \times [0, \infty) \to [0, \infty)$, and $g(0, 0) = 0$; the function f is defined as $f(t, u) : [0, 1] \times [0, \infty) \to [0, \infty)$ and $\sigma(t)$, $\tau(t)$ are continuous on t and $0 \le \sigma(t), \tau(t) \le t$.

By using Banach contraction mapping principle and Leray-Schauder continuation principle, we obtain some sufficient conditions for the existence and uniqueness of the positive solutions for boundary value problem (4). Furthermore, we give an example to illustrate our results.

2. Preliminary

In this section, we introduce some definitions and preliminary facts which are used in this paper.

Definition 1 (see [8, 16]). The fractional integral of order α with the lower limit t_0 for a function f is defined as

$$I^{\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > t_0,\ \alpha > 0, \tag{5}$$

provided the right-side is point-wise defined on $[t_0, \infty]$, where Γ is the Gamma function.

Definition 2 (see [8, 16]). Riemann-Liouville derivative of order α with the lower limit t_0 for a function $f : [0, \infty) \to \mathscr{R}$ can be written as

$$\begin{gathered} D^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \\ t > t_0,\ n-1 < \alpha < n. \end{gathered} \tag{6}$$

Definition 3 (see [8, 16]). Caputo's derivative of order α with the lower limit t_0 for a function $f : [0, \infty) \to \mathscr{R}$ can be written as

$$\begin{aligned} {}^{c}D^{\alpha}(f(t)) &= \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds \\ &= I^{n-\alpha} f^{(n)}(t), \quad t > t_0,\ n-1 < \alpha < n. \end{aligned} \tag{7}$$

It is well known that if $n - 1 \le \alpha \le n$, then $D^{\alpha} t^{\alpha-k} = 0$, $k = 1, 2, \ldots, n$. Furthermore, if $y(t) \in L^1[0, T]$ and $\alpha > 0$, then for $t \in [0, T]$, we have

$$D^{\alpha} I^{\alpha} y(t) = y(t), \tag{8}$$

which is with the semigroup property

$$I^{\delta} I^{\alpha} = I^{\delta+\alpha} = I^{\alpha} I^{\delta}, \tag{9}$$

for $\delta + \alpha > 0$ and $t \in [0, T]$.

We also need to introduce some Lemmas as follows, which will be used in the proof of our main theorems.

Lemma 4 (see [19, 20]). *Let $\alpha > 0$; then the fractional equation*

$$D^{\alpha}(h(t)) = 0, \tag{10}$$

has solutions

$$\begin{gathered} h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \\ c_i \in R, \quad i = 1, 2, \ldots, n, \quad n = [\alpha] + 1. \end{gathered} \tag{11}$$

Lemma 5 (see [19, 20]). *Let $\alpha > 0$; then*

$$I^{\alpha} D^{\alpha} h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{12}$$

for some $c_i \in \mathscr{R}$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 6 (see [28], the Banach contraction mapping theorem). *Let $T : M \to M$ be a contraction mapping of a complete metric space M. Then T has one and only one fixed point.*

Lemma 7 (see [28–30], the Leray-Schauder continuation principle). *Let X be a Banach space with $C \subset X$ being closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ is completely continuous. Then either*

(a) *T has a fixed point in $\overline{U}$, or*

(b) *there exists $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda T u$.*

Throughout this paper, we assume that $f, g \in C([0, 1] \times [0, \infty), [0, \infty,))$ and we satisfy the following

(H): (i) $f(t, u)$, $g(t, u)$ is Lebesgue measurable with respect to t on $[0, 1]$;

(ii) $f(t, u)$, $g(t, u)$ is continuous with respect to u on $[0, \infty)$.

3. Main Results

For convenience, we rewrite (4) as follows:

$$\begin{gathered} D_{0+}^{\alpha} y(t) = p D_{0+}^{\beta} g(t, y(\sigma(t))) - f(t, y(\tau(t))), \\ 0 < t < 1, \end{gathered} \tag{13}$$

$$\begin{gathered} y(0) = 0, \\ \eta y(\sigma(1)) = y(1). \end{gathered} \tag{14}$$

Integrating both sides of (13) of α order with respect to t, it follows that

$$\begin{aligned} y(t) &= p\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} g(s, y(\sigma(s)))\,ds \\ &\quad + c_0 t^{\alpha-1} + c_1 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t (t-s)^{\alpha-1} f(s, y(\tau(s)))\,ds, \quad 0 < t < 1. \end{aligned} \tag{15}$$

From (14) and (15), we have

$$c_1 = 0, \tag{16}$$

$$\begin{aligned} y(1) &= p\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1} g(s, y(\sigma(s)))\,ds \\ &\quad + c_0 - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, y(\tau(s)))\,ds, \\ &\quad 0 < t < 1, \\ y(\sigma(1)) &= p\frac{1}{\Gamma(\alpha-\beta)}\int_0^{\sigma(1)} [\sigma(1)-s]^{\alpha-\beta-1} g(s, y(\sigma(s)))\,ds \\ &\quad + c_0\sigma^{\alpha-1}(1) - \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^{\sigma(1)} [\sigma(1)-s]^{\alpha-1} f(s, y(\tau(s)))\,ds, \\ &\quad 0 < t < 1. \end{aligned} \tag{17}$$

Combining (14) with (17), we obtain

$$\begin{aligned} c_0 &= \frac{1}{\Gamma(\alpha)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-1} f(s, y(\tau(s)))\,ds \\ &\quad - \frac{\eta}{\Gamma(\alpha)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^{\sigma(1)} [\sigma(1)-s]^{\alpha-1} f(s, y(\tau(s)))\,ds \\ &\quad - \frac{p}{\Gamma(\alpha-\beta)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-\beta-1} g(s, y(\sigma(s)))\,ds \\ &\quad + \frac{p\eta}{\Gamma(\alpha-\beta)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^{\sigma(1)} [\sigma(1)-s]^{\alpha-\beta-1} g(s, y(\sigma(s)))\,ds, \\ &\quad 0 < t < 1. \end{aligned} \tag{18}$$

According to (15) and (18), it follows that

$$\begin{aligned} y(t) &= \frac{p}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} g(s, y(\sigma(s)))\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-1} t^{\alpha-1} f(s, y(\tau(s)))\,ds \\ &\quad - \frac{\eta}{\Gamma(\alpha)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^{\sigma(1)} [\sigma(1)-s]^{\alpha-1} t^{\alpha-1} f(s, y(\tau(s)))\,ds \\ &\quad - \frac{p}{\Gamma(\alpha-\beta)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-\beta-1} t^{\alpha-1} g(s, y(\sigma(s)))\,ds \\ &\quad + \frac{p\eta}{\Gamma(\alpha-\beta)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \int_0^{\sigma(1)} [\sigma(1)-s]^{\alpha-\beta-1} t^{\alpha-1} g(s, y(\sigma(s)))\,ds \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(\tau(s)))\,ds \\ &= \frac{p}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} g(s, y(\sigma(s)))\,ds \\ &\quad - \frac{p}{\Gamma(\alpha-\beta)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \left(\int_0^t + \int_t^{\sigma(1)} + \int_{\sigma(1)}^1\right)(1-s)^{\alpha-\beta-1} t^{\alpha-1} g(s, y(\sigma(s)))\,ds \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(\tau(s)))\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)[1-\eta\sigma^{\alpha-1}(1)]} \\ &\quad \cdot \left(\int_0^t + \int_0^{\sigma(1)} + \int_{\sigma(1)}^1\right)(1-s)^{\alpha-1} t^{\alpha-1} f(s, y(\tau(s)))\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{p\eta}{\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]}\\
&\cdot\left(\int_0^t+\int_t^{\sigma(1)}\right)[\sigma(1)-s]^{\alpha-\beta-1}t^{\alpha-1}g(s,y(\sigma(s)))\,ds\\
&-\frac{\eta}{\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]}\\
&\cdot\left(\int_0^t+\int_t^{\sigma(1)}\right)[\sigma(1)-s]^{\alpha-1}t^{\alpha-1}f(s,y(\tau(s)))\,ds\\
&=p\int_0^t\frac{1}{\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]}\\
&\quad\cdot\Big\{\left[1-\eta\sigma^{\alpha-1}(1)\right](t-s)^{\alpha-\beta-1}\\
&\quad-(1-s)^{\alpha-\beta-1}t^{\alpha-1}\\
&\quad+\eta[\sigma(1)-s]^{\alpha-\beta-1}t^{\alpha-1}\\
&\quad\cdot g(s,y(\sigma(s)))\Big\}\,ds\\
&+p\int_t^{\sigma(1)}\Big(\big(-(1-s)^{\alpha-\beta-1}t^{\alpha-1}\\
&\qquad+\eta[\sigma(1)-s]^{\alpha-\beta-1}t^{\alpha-1}\big)\\
&\qquad\cdot\left(\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1}\Big)\\
&\qquad\cdot g(s,y(\sigma(s)))\,ds\\
&-p\int_{\sigma(1)}^1\frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]}\\
&\qquad\cdot g(s,y(\sigma(s)))\,ds\\
&-\int_0^t\Big(\Big(\left(1-\eta\sigma^{\alpha-1}(1)\right)(t-s)^{\alpha-1}\\
&\qquad-(1-s)^{\alpha-1}t^{\alpha-1}+\eta[\sigma(1)-s]^{\alpha-1}t^{\alpha-1}\Big)\\
&\qquad\cdot\left(\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1}\Big)\\
&\qquad\cdot f(s,y(\tau(s)))\,ds\\
&-\int_t^{\sigma(1)}\frac{-(1-s)^{\alpha-1}t^{\alpha-1}+\eta[\sigma(1)-s]^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]}\\
&\quad\cdot f(s,y(\tau(s)))\,ds\\
&+\int_{\sigma(1)}^1\frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]}f(s,y(\tau(s)))\,ds,\\
&\qquad\qquad 0<t<1.
\end{aligned}
\tag{19}
$$

Let

$$
G(t,s)=
\begin{cases}
\Big((1-s)^{\alpha-\beta-1}t^{\alpha-1}-\left(1-\eta\sigma^{\alpha-1}(1)\right)\\
\quad\cdot(t-s)^{\alpha-\beta-1}-\eta[\sigma(1)-s]^{\alpha-\beta-1}t^{\alpha-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0\le s\le t\le 1,\quad s\le\sigma(1);\\
\Big((1-s)^{\alpha-\beta-1}t^{\alpha-1}\\
\quad-\left(1-\eta\sigma^{\alpha-1}(1)\right)(t-s)^{\alpha-\beta-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0<\sigma(1)\le s\le t\le 1;\\
\Big((1-s)^{\alpha-\beta-1}t^{\alpha-1}-\eta[\sigma(1)-s]^{\alpha-\beta-1}t^{\alpha-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0<t\le s\le\sigma(1)\le 1;\\
\Big((1-s)^{\alpha-\beta-1}t^{\alpha-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha-\beta)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0\le t\le s\le 1,\quad \sigma(1)\le s,
\end{cases}
\tag{20}
$$

$$
\widetilde{G}(t,s)=
\begin{cases}
\Big((1-s)^{\alpha-1}t^{\alpha-1}-\left(1-\eta\sigma^{\alpha-1}(1)\right)(t-s)^{\alpha-1}\\
\quad-\eta[\sigma(1)-s]^{\alpha-1}t^{\alpha-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0\le s\le t\le 1,\quad s\le\sigma(1);\\
\Big((1-s)^{\alpha-1}t^{\alpha-1}\\
\quad-\left(1-\eta\sigma^{\alpha-1}(1)\right)(t-s)^{\alpha-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0<\sigma(1)\le s\le t\le 1;\\
\Big((1-s)^{\alpha-1}t^{\alpha-1}-\eta[\sigma(1)-s]^{\alpha-1}t^{\alpha-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0<t\le s\le\sigma(1)\le 1;\\
\Big((1-s)^{\alpha-1}t^{\alpha-1}\Big)\\
\quad\cdot\left(\Gamma(\alpha)\left[1-\eta\sigma^{\alpha-1}(1)\right]\right)^{-1},\\
\qquad 0\le t\le s\le 1,\quad \sigma(1)\le s.
\end{cases}
\tag{21}
$$

According to (19)–(21), it follows that

$$
\begin{aligned}
y(t)=&-p\int_0^1 G(t,s)\,g(s,y(\sigma(s)))\,ds\\
&+\int_0^1\widetilde{G}(t,s)\,f(s,y(\tau(s)))\,ds,
\end{aligned}
\tag{22}
$$

which means that if $y(t)$ satisfies (13)-(14), then it satisfies (22). It is easy to show that if $y(t)$ satisfies (22), then it also satisfies (13)-(14). Thus, the boundary value problem (13)-(14) is actually equivalent to integral equation (22). Therefore, we have the following.

Lemma 8. *Problem (13)-(14) is equivalent to (22).*

Lemma 9. *For any $(t,s)\in(0,1)\times(0,1)$, $G(t,s),\widetilde{G}(t,s)$ are continuous and $G(t,s)>0$, $\widetilde{G}(t,s)>0$.*

Proof. It is obvious that $G(t,s), \widetilde{G}(t,s)$ are continuous on $[0,1]\times[0,1]$. We first prove that $G(t,s)>0$ on $[0,1]\times[0,1]$. Let

$$\begin{aligned} &g_1(t,s) \\ &\quad = (1-s)^{\alpha-\beta-1}t^{\alpha-1} - \left[1-\eta\sigma^{\alpha-1}(1)\right](t-s)^{\alpha-\beta-1} \\ &\qquad - \eta[\sigma(1)-s]^{\alpha-\beta-1}t^{\alpha-1}, \\ &\qquad\qquad 0\le s\le t\le 1, \quad s\le\sigma(1); \\ &g_2(t,s) \\ &\quad = (1-s)^{\alpha-\beta-1}t^{\alpha-1} - \left[1-\eta\sigma^{\alpha-1}(1)\right](t-s)^{\alpha-\beta-1}, \\ &\qquad\qquad 0<\sigma(1)\le s\le t\le 1; \\ &g_3(t,s) \\ &\quad = (1-s)^{\alpha-\beta-1}t^{\alpha-1} - \eta[\sigma(1)-s]^{\alpha-\beta-1}t^{\alpha-1}, \\ &\qquad\qquad 0<t\le s\le\sigma(1)\le 1; \\ &g_4(t,s) = (1-s)^{\alpha-\beta-1}t^{\alpha-1}, \quad 0\le t\le s\le 1,\ \sigma(1)\le s. \end{aligned} \tag{23}$$

We first show that $g_1(t,s)>0$, $0\le s\le t\le 1$, $s\le\sigma(1)$. We rewrite $g_1(t,s)$ as follows:

$$\begin{aligned} g_1(t,s) &= t^{\alpha-1}\Bigg((1-s)^{\alpha-\beta-1} - \left[1-\eta\sigma^{\alpha-1}(1)\right] \\ &\quad \cdot \left(1-\frac{s}{t}\right)^{\alpha-\beta-1} t^{-\beta} - \eta[\sigma(1)-s]^{\alpha-\beta-1}\Bigg). \end{aligned} \tag{24}$$

Let

$$\begin{aligned} h_1(t,s) &= (1-s)^{\alpha-\beta-1} \\ &\quad - \left[1-\eta\sigma^{\alpha-1}(1)\right]\left(1-\frac{s}{t}\right)^{\alpha-\beta-1} t^{-\beta} \\ &\quad - \eta\left[\sigma(1)-s\right]^{\alpha-\beta-1}. \end{aligned} \tag{25}$$

Since $0<\eta\sigma^{\alpha-1}(1)<1$, then

$$h_1(s,s) = (1-s)^{\alpha-\beta-1} - \eta\left[\sigma(1)-s\right]^{\alpha-\beta-1} > 0. \tag{26}$$

Differentiating both sides of (25) with respect to t, it follows that

$$\begin{aligned} &\frac{\partial h_1(t,s)}{\partial t} \\ &\quad = \left[1-\eta\sigma^{\alpha-1}(1)\right]\left(1-\frac{s}{t}\right)^{\alpha-\beta-2} t^{-\beta-2} \\ &\qquad \cdot \left[(\beta-\alpha+1)s + \left(1-\frac{s}{t}\right)t\beta\right] \\ &\quad = \left[1-\eta\sigma^{\alpha-1}(1)\right]\left(1-\frac{s}{t}\right)^{\alpha-\beta-2} t^{-\beta-2}\left(s-\alpha s+t\beta\right) \ge 0, \\ &\qquad\qquad 0<s<t\le 1; \end{aligned} \tag{27}$$

which means that $h_1(t,s)$ is nondecreasing with respect to t on $[s,1]$. Thus, for any $t\in[s,1]$, $h_1(t,s)\ge h_1(s,s)>0$, therefore, $g_1(t,s)=t^{\alpha-1}h_1(t,s)>0$.

Using the similar method, we can prove that $g_2(t,s)>0$, and it is obvious that $g_3(t,s)>0$, $g_4(t,s)>0$. Hence, combining (20) and (23), we obtain that $G(t,s)>0$.

Now, we prove that $\widetilde{G}(t,s)>0$. Denote

$$\begin{aligned} &\widetilde{g}_1(t,s) \\ &\quad = (1-s)^{\alpha-1}t^{\alpha-1} - \left[1-\eta\sigma^{\alpha-1}(1)\right](t-s)^{\alpha-1} \\ &\qquad - \eta[\sigma(1)-s]^{\alpha-1}t^{\alpha-1}, \\ &\qquad\qquad 0\le s\le t\le 1, \quad s\le\sigma(1); \\ &\widetilde{g}_2(t,s) \\ &\quad = (1-s)^{\alpha-1}t^{\alpha-1} - \left[1-\eta\sigma^{\alpha-1}(1)\right](t-s)^{\alpha-1}, \\ &\qquad\qquad 0<\sigma(1)\le s\le t\le 1; \\ &\widetilde{g}_3(t,s) \\ &\quad = (1-s)^{\alpha-1}t^{\alpha-1} - \eta[\sigma(1)-s]^{\alpha-1}t^{\alpha-1}, \\ &\qquad\qquad 0<t\le s\le\sigma(1)\le 1; \\ &\widetilde{g}_4(t,s) = (1-s)^{\alpha-1}t^{\alpha-1}, \quad 0\le t\le s\le 1,\ \sigma(1)\le s. \end{aligned} \tag{28}$$

Let

$$\begin{aligned} \widetilde{h}_1(t,s) &= (1-s)^{\alpha-1} - \left[1-\eta\sigma^{\alpha-1}(1)\right]\left(1-\frac{s}{t}\right)^{\alpha-1} \\ &\quad - \eta\left[\sigma(1)-s\right]^{\alpha-1}. \end{aligned} \tag{29}$$

Since $0<\eta\sigma^{\alpha-1}(1)<1$, then

$$\begin{aligned} \widetilde{h}_1(1,s) &\ge (1-s)^{\alpha-1} - \left[1-\eta\sigma^{\alpha-1}(1)\right](1-s)^{\alpha-1} \\ &\quad - \eta\left[\sigma(1)-s\right]^{\alpha-1} \\ &= \eta\sigma^{\alpha-1}(1)(1-s)^{\alpha-1} - \eta\left[\sigma(1)-s\right]^{\alpha-1} \\ &= \eta\left[\sigma(1)-s\sigma(1)\right]^{\alpha-1} - \eta\left[\sigma(1)-s\right]^{\alpha-1} \\ &\ge \eta\left[\sigma(1)-s\right]^{\alpha-1} - \eta\left[\sigma(1)-s\right]^{\alpha-1} = 0. \end{aligned} \tag{30}$$

Differentiating both sides of (29) with respect to t, it follows that

$$\begin{aligned} \frac{\partial\widetilde{h}_1(t,s)}{\partial t} &= -(\alpha-1)\left[1-\eta\sigma^{\alpha-1}(1)\right]\left(1-\frac{s}{t}\right)^{\alpha-2} st^{-2} \\ &\le 0, \quad 0<s<t\le 1; \end{aligned} \tag{31}$$

which means that $\widetilde{h}_1(t,s)$ is nonincreasing with respect to t on $[s,1]$. Thus, for any $t\in[s,1]$, $\widetilde{h}_1(t,s)\ge h_1(1,s)>0$, therefore, $\widetilde{g}_1(t,s)=t^{\alpha-1}\widetilde{h}_1(t,s)>0$.

Using the similar method, we can prove that $\widetilde{g}_2(t,s)>0$, and the case that $\widetilde{g}_3(t,s)>0$, $\widetilde{g}_4(t,s)>0$ is obvious. Combining (21) and (28) and using the above argument, we obtain that $\widetilde{G}(t,s)>0$. The proof is complete. □

Lemma 10. *For any $(t,s) \in (0,1)\times(0,1)$, $G(t,s), \widetilde{G}(t,s)$ are nondecreasing functions with respect to $t \in (0,1)$; that is, for any $t \in (0,1)$, $G(t,s) > G(s,s), \widetilde{G}(t,s) > \widetilde{G}(s,s)$.*

Proof. According to the proof of Lemma 9, we notice that

$$\begin{gathered} \frac{\partial g_1(t,s)}{\partial t} = (\alpha-1)t^{\alpha-2}h_1(t,s) + t^{\alpha-1}\frac{\partial h_1(t,s)}{\partial t} > 0, \\ 0 < s < t \leq 1, \\ \frac{\partial g_2(t,s)}{\partial t} > 0, \\ \frac{\partial g_3(t,s)}{\partial t} > 0, \\ \frac{\partial g_4(t,s)}{\partial t} > 0. \end{gathered} \tag{32}$$

At the same time, for $s < t < 1$, we have

$$\begin{aligned} &\frac{\partial \widetilde{g}_1(t,s)}{\partial t} \\ &\quad = (\alpha-1)(1-s)^{\alpha-1}t^{\alpha-2} - \left[1-\eta\sigma^{\alpha-1}(1)\right] \\ &\qquad \cdot (t-s)^{\alpha-2} - \eta[\sigma(1)-s]^{\alpha-1}t^{\alpha-2} \\ &\quad \geq (\alpha-1)t^{\alpha-2}\left[(1-s)^{\alpha-1} - \eta[\sigma(1)-s]^{\alpha-1}\right] \geq 0, \\ &\frac{\partial \widetilde{g}_2(t,s)}{\partial t} > 0, \\ &\frac{\partial \widetilde{g}_3(t,s)}{\partial t} > 0, \\ &\frac{\partial \widetilde{g}_4(t,s)}{\partial t} > 0. \end{aligned} \tag{33}$$

The proof is complete. □

Now, we present our main results.

Theorem 11. *Assume that (H) holds. Suppose that there are two functions $\lambda(t), \mu(t) \in C([0,1],[0,\infty))$ such that*

$$\begin{gathered} \left|f(t,u) - f(t,v)\right| \leq \lambda(t)|u-v|, \\ \text{for } t \in [0,1],\ u,v \in [0,\infty), \\ \left|g(t,u) - g(t,v)\right| \leq \mu(t)|u-v|, \\ \text{for } t \in [0,1],\ u,v \in [0,\infty). \end{gathered} \tag{34}$$

If

$$-p\int_0^1 G(1,s)\mu(s)\,ds + \int_0^1 \widetilde{G}(1,s)\lambda(s) < 1, \tag{35}$$

then the problem (13)-(14) has a unique positive solution.

Proof. Set

$$\Omega = \left\{y(t) \in C[0,1] \mid y(t) \geq 0, \text{ for } t \in [0,1]\right\}, \tag{36}$$

with the maximum norm

$$\|y\| = \max_{0\leq t\leq 1}\left|y(t)\right|. \tag{37}$$

It is easy to show that Ω is a complete metric space. We denote a operator T as follows:

$$\begin{aligned} Ty(t) = &-p\int_0^1 G(t,s)\,g(s,y(\sigma(s)))\,ds \\ &+ \int_0^1 \widetilde{G}(t,s)\,f(s,y(\tau(s)))\,ds. \end{aligned} \tag{38}$$

From Lemma 9 and the conditions $p \leq 0$, $f > 0$, $g > 0$, it follows that T maps Ω into itself and we only need to prove the contraction. In fact, according to (H) and (34), for any $u, v \in \Omega$, we have

$$\begin{aligned} &\|Tu - Tv\| \\ &\quad = \max_{0\leq t\leq 1}\bigg|-p\int_0^1 G(t,s)\,g(s,u(\sigma(s)))\,ds \\ &\qquad + p\int_0^1 G(t,s)\,g(s,v(\sigma(s)))\,ds \\ &\qquad + \int_0^1 \widetilde{G}(t,s)\,f(s,u(\tau(s)))\,ds \\ &\qquad - \int_0^1 \widetilde{G}(t,s)\,f(s,v(\tau(s)))\,ds\bigg| \\ &\quad \leq -p\int_0^1 G(t,s)\mu(s)\left|u(\sigma(s)) - v(\sigma(s))\right|ds \\ &\qquad + \int_0^1 \widetilde{G}(t,s)\lambda(s)\left|u(\tau(s)) - v(\tau(s))\right|ds \\ &\quad \leq \left[-p\int_0^1 G(t,s)\mu(s)\,ds + \int_0^1 \widetilde{G}(t,s)\lambda(s)\right]\|u-v\| \\ &\quad \leq \left[-p\int_0^1 G(1,s)\mu(s)\,ds + \int_0^1 \widetilde{G}(1,s)\lambda(s)\right]\|u-v\| \\ &\quad < \|u-v\|, \end{aligned} \tag{39}$$

which means that

$$|Tu - Tv| < \|u-v\|. \tag{40}$$

By the Banach contraction mapping principle (Lemma 6), we obtain that T has a unique fixed point $\widetilde{y}(t)$ which is a positive solution of (13)-(14). The proof is complete. □

Remark 12. If $p = 0$, $f(t, y(\tau(t))) = f(y(t))$, then problem (13)-(14) is problem (3).

Theorem 13. *Assume that (H) holds. Suppose that there exists four nonnegative real-valued functions $m, n, l, q \in L^1[0,1]$ such that*

$$f(t,u) \le n(t) + m(t)u, \quad \text{for almost every } t \in [0,1] \text{ and all } u \in [0,\infty), \tag{41}$$

$$g(t,u) \le l(t) + q(t)u, \quad \text{for almost every } t \in [0,1] \text{ and all } u \in [0,\infty). \tag{42}$$

If

$$\int_0^1 \left[\widetilde{G}(1,s)m(s) - pG(1,s)q(s)\right] ds < 1, \tag{43}$$

then the problem (13)-(14) has at least one positive solution.

Proof. We also consider the operator T defined in (38). We divide the proof into four steps.

Step 1. $T : \Omega \to \Omega$ is continuous.

Let $y_n(t)$ be a sequence in Ω such that $y_n(t) \to y(t)$ as $n \to \infty$. Noticing that $f(t,y), g(t,y)$ are continuous with respect to y, then for each $t \in [0,1]$, we have

$$\begin{aligned} &\lim_{n\to\infty} f(t, y_n(\tau(t))) = f(t, y(\tau(t))), \\ &\lim_{n\to\infty} (t, y_n(\sigma(t))) = g(t, y(\sigma(t))); \end{aligned} \tag{44}$$

thus,

$$\begin{aligned} &\lim_{n\to\infty} \sup_{t\in[0,1]} \left|f(t, y_n(\tau(t))) - f(t, y(\tau(t)))\right| = 0, \\ &\lim_{n\to\infty} \sup_{t\in[0,1]} \left|g(t, y_n(\sigma(t))) - g(t, y(\sigma(t)))\right| = 0, \end{aligned} \tag{45}$$

which implies that

$$\begin{aligned} &\left|Ty_n(t) - Ty(t)\right| \\ &= \left| -p\int_0^1 G(t,s)\, g(s, y_n(\sigma(s)))\, ds \right. \\ &\quad + p\int_0^1 G(t,s)\, g(s, y(\sigma(s)))\, ds \\ &\quad + \int_0^1 \widetilde{G}(t,s)\, f(s, y_n(\sigma(s)))\, ds \\ &\quad \left. - \int_0^1 \widetilde{G}(t,s)\, f(s, y(\sigma(s)))\, ds \right| \\ &\le -p\int_0^1 G(t,s)\left|g(s, y_n(\sigma(s))) - g(s, y(\sigma(s)))\right| ds \\ &\quad + \int_0^1 \widetilde{G}(t,s)\left|f(s, y_n(\sigma(s))) - f(s, y(\sigma(s)))\right| ds \\ &\le -p\int_0^1 G(1,s)\, ds \sup_{t\in[0,1]} \left|g(s, y_n(\sigma(s))) - g(s, y(\sigma(s)))\right| \\ &\quad + \int_0^1 \widetilde{G}(1,s)\, ds \sup_{t\in[0,1]} \left|f(s, y_n(\sigma(s))) - f(s, y(\sigma(s)))\right| \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{46}$$

Step 2. T maps bounded sets into bounded sets in Ω.

Indeed, it suffices to show that for any $\gamma > 0$, there exists a positive constant $\eta^* > 0$ such that for $y \in B_\gamma = \{y \in \Omega : \|y\| \le \gamma\}$, we have $\|Ty\| \le \eta^*$. From (38)–(43), we have

$$\begin{aligned} &\|Ty\| \\ &= \max_{0\le t\le 1} \left| -p\int_0^1 G(t,s)\, g(s, y(\sigma(s)))\, ds \right. \\ &\quad \left. + \int_0^1 \widetilde{G}(t,s)\, f(s, y(\tau(s)))\, ds \right| \\ &\le -p\int_0^1 G(t,s)\left[l(s) + q(s)\, y(\sigma(s))\right] ds \\ &\quad + \int_0^1 \widetilde{G}(t,s)\left[n(s) + m(s)\, y(\tau(s))\right] ds \\ &\le \int_0^1 \left[-pG(1,s)\, l(s) + \widetilde{G}(1,s)\, n(s)\right] ds \\ &\quad + \int_0^1 \left[-pG(1,s)\, q(s) + \widetilde{G}(1,s)\, m(s)\right] ds\, \|y\| \\ &\le \int_0^1 \left[-pG(1,s)\, l(s) + \widetilde{G}(1,s)\, n(s)\right] ds + \gamma := \eta^*, \end{aligned} \tag{47}$$

which means

$$\|Ty\| \le \eta^*. \tag{48}$$

Step 3. T maps bounded sets into equicontinuous sets in Ω.

For any $t_1, t_2 \in [0,1], t_1 < t_2$, and for each $y(t) \in B_\gamma$, we have

$$\begin{aligned} &\left|Ty(t_1) - Ty(t_2)\right| \\ &= \left| -p\int_0^1 \left[G(t_1,s) - G(t_2,s)\right] g(s, y(\sigma(s)))\, ds \right. \\ &\quad \left. + \int_0^1 \left[\widetilde{G}(t_1,s) - \widetilde{G}(t_2,s)\right] f(s, y(\tau(s)))\, ds \right| \end{aligned}$$

$$\begin{aligned}&\leq -p\int_0^1 \left|G(t_1,s)-G(t_2,s)\right|\left[l(s)+q(s)\,y(\sigma(s))\right]ds\\&\quad+\int_0^1\left|\widetilde{G}(t_1,s)-\widetilde{G}(t_2,s)\right|\left[n(t)+m(t)\,y(\tau(s))\right]ds\\&\leq -p\int_0^1\left|G(t_1,s)-G(t_2,s)\right|\left[l(s)+q(s)\,\gamma\right]ds\\&\quad+\int_0^1\left|\widetilde{G}(t_1,s)-\widetilde{G}(t_2,s)\right|\left[n(t)+m(t)\,\gamma\right]ds.\end{aligned}\tag{49}$$

Because $G(t,s), \widetilde{G}(t,s)$ are continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$, which means that for any $\varepsilon>0$, there exists $\delta>0$, when $|t_2-t_1|<\delta$, $s\in[0,1]$,

$$\begin{aligned}\left|G(t_1,s)-G(t_2,s)\right|&<\frac{\varepsilon}{2(-p)\int_0^1\left[l(s)+q(s)\,\gamma\right]ds},\\\left|\widetilde{G}(t_1,s)-\widetilde{G}(t_2,s)\right|&<\frac{\varepsilon}{2\int_0^1\left[n(s)+m(s)\,\gamma\right]ds}.\end{aligned}\tag{50}$$

Thus

$$\left|Ty(t_1)-Ty(t_2)\right|<\varepsilon,\tag{51}$$

which means that $\{Ty : y\in B_\gamma\}$ is equicontinuous.

Step 4. A priori bounds.

Let

$$\begin{gathered}U=\{y\in\Omega:\|y\|<r\},\\\text{where } r=\frac{\int_0^1\left[\widetilde{G}(1,s)\,n(s)-pG(1,s)\,l(s)\right]ds}{1-\int_0^1\left[\widetilde{G}(1,s)\,m(s)-pG(1,s)\,q(s)\right]ds}.\end{gathered}\tag{52}$$

Assume that there exists $y\in U$ and $0<\lambda<1$ such that $y=\lambda Ty$ and we claim that $\|y\|\neq r$. In fact,

$$\begin{aligned}y(t)&=\lambda(Ty)(t)\\&=\lambda\left[-p\int_0^1 G(t,s)\,g(s,y(\sigma(s)))\,ds\right.\\&\quad\left.+\int_0^1\widetilde{G}(t,s)\,f(s,y(\tau(s)))\,ds\right],\end{aligned}\tag{53}$$

which implies that

$$\begin{aligned}&|y(t)|\\&=\lambda\left|-p\int_0^1 G(t,s)\,g(s,y(\sigma(s)))\,ds\right.\\&\qquad\left.+\int_0^1\widetilde{G}(t,s)\,f(s,y(\tau(s)))\,ds\right|\\&<-p\int_0^1 G(t,s)\left[l(s)+q(s)\,y(\sigma(s))\right]ds\\&\quad+\int_0^1\widetilde{G}(t,s)\left[n(s)+m(s)\,y(\tau(s))\right]ds\\&\leq\int_0^1\left[-pG(1,s)\,l(s)+\widetilde{G}(1,s)\,n(s)\right]ds\\&\quad+\int_0^1\left[-pG(1,s)\,q(s)+\widetilde{G}(1,s)\,m(s)\right]ds\,\|y\|\\&\leq\int_0^1\left[-pG(1,s)\,l(s)+\widetilde{G}(1,s)\,n(s)\right]ds\\&\quad+r\int_0^1\left[-pG(1,s)\,q(s)+\widetilde{G}(1,s)\,m(s)\right]ds=r,\end{aligned}\tag{54}$$

which means that

$$\|y\|<r.\tag{55}$$

That is, there is no $y\in\partial U$, such that $y=\lambda Ty$ for $0<\lambda<1$.

From Step 1 to Step 3, it follows that T is completely continuous. Along with Step 4 and Lemma 7, it follows that T has at least a fixed point in U. The proof is complete. □

4. Examples

Example 1. Consider

$$\begin{gathered}D^{3/2}x(t)+\frac{1}{2}D^{3/4}\left(\frac{t^2x(t/2)}{10\left[1+x(t/2)\right]}\right)\\=\frac{e^{-t}x(t/2)}{(9+e^t)\left[1+x(t/2)\right]},\quad t\in[0,1],\\x(0)=0,\\x(1)=x\left(\frac{1}{2}\right),\end{gathered}\tag{56}$$

where

$$\begin{aligned} n &= 2, \\ p &= -\frac{1}{2}, \\ \alpha &= \frac{3}{2}, \\ \beta &= \frac{3}{4}, \\ \eta &= 1, \\ \tau(t) = \sigma(t) &= \frac{t}{2}, \\ g &= \frac{t^2 x(t/2)}{10[1 + x(t/2)]}, \\ f &= \frac{e^{-t} x(t/2)}{(9 + e^t)[1 + x(t/2)]}. \end{aligned} \tag{57}$$

It is easy to show that

$$\begin{aligned} \lambda(t) &= \frac{e^{-t}}{9 + e^t}, \\ \mu(t) &= \frac{t^2}{10}. \end{aligned} \tag{58}$$

Let

$$h(x) = \frac{x}{1 + x}, \quad x > 0; \tag{59}$$

then

$$\begin{aligned} |h(x) - h(y)| &= \left| \frac{x}{1 + x} - \frac{y}{1 + y} \right| = \frac{|x - y|}{(1 + x)(1 + y)} \\ &< |x - y|, \quad \text{for } x, y > 0; \end{aligned} \tag{60}$$

thus,

$$\begin{aligned} |f(t, u) - f(t, v)| &\le \lambda(t)|u - v|, \\ |g(t, u) - g(t, v)| &\le \mu(t)|u - v|, \end{aligned} \tag{61}$$

which satisfies (28), (29). Because

$$\begin{aligned} G(1,s) &= \begin{cases} \Big((1-s)^{-1/4} - \big(1 - 1/\sqrt{2}\big) \\ \quad \cdot (1-s)^{-1/4} - (1/2 - s)^{-1/4}\Big) \\ \quad \cdot \Big(\Gamma(3/4)\big(1 - 1/\sqrt{2}\big)\Big)^{-1}, \\ \qquad 0 \le s \le \dfrac{1}{2}; \\ \Big((1-s)^{-1/4} - \big(1 - 1/\sqrt{2}\big)(1-s)^{-1/4}\Big) \\ \quad \cdot \Big(\Gamma(3/4)\big(1 - 1/\sqrt{2}\big)\Big)^{-1}, \\ \qquad \dfrac{1}{2} \le s \le 1, \end{cases} \\ \widetilde{G}(1,s) &= \begin{cases} \Big((1-s)^{1/2} - \big(1 - 1/\sqrt{2}\big) \\ \quad \cdot (1-s)^{1/2} - (1/2 - s)^{1/2}\Big) \\ \quad \cdot \Big(\Gamma(3/2)\big(1 - 1/\sqrt{2}\big)\Big)^{-1}, \\ \qquad 0 \le s \le \dfrac{1}{2}; \\ \Big((1-s)^{1/2} - \big(1 - 1/\sqrt{2}\big)(1-s)^{1/2}\Big) \\ \quad \cdot \Big(\Gamma(3/2)\big(1 - 1/\sqrt{2}\big)\Big)^{-1}, \\ \qquad \dfrac{1}{2} \le s \le 1, \end{cases} \end{aligned} \tag{62}$$

thus

$$\begin{aligned} &- p \int_0^1 G(1,s)\mu(s)\,ds + \int_0^1 \widetilde{G}(1,s)\lambda(s) \\ &= \frac{1}{2} \int_0^{1/2} \frac{1/\sqrt{2}(1-s)^{-1/4} - (1/2 - s)^{-1/4}}{\Gamma(3/4)\big(1 - 1/\sqrt{2}\big)} \frac{s^2}{10} ds \\ &\quad + \frac{1}{2} \int_{1/2}^1 \frac{1/\sqrt{2}(1-s)^{-1/4}}{\Gamma(3/4)\big(1 - 1/\sqrt{2}\big)} \frac{s^2}{10} ds \\ &\quad + \int_0^{1/2} \frac{1/\sqrt{2}(1-s)^{1/2} - (1/2 - s)^{1/2}}{\Gamma(3/2)\big(1 - 1/\sqrt{2}\big)} \frac{e^{-s}}{9 + e^s} ds \\ &\quad + \int_{1/2}^1 \frac{1/\sqrt{2}(1-s)^{1/2}}{\Gamma(3/2)\big(1 - 1/\sqrt{2}\big)} \frac{e^{-s}}{9 + e^s} ds \\ &< \frac{1}{2} \int_0^{1/2} \frac{(1-s)^{-1/4}}{\Gamma(3/4)\big(\sqrt{2} - 1\big)} \frac{s^2}{10} ds \\ &\quad + \frac{1}{2} \int_{1/2}^1 \frac{(1-s)^{-1/4}}{\Gamma(3/4)\big(\sqrt{2} - 1\big)} \frac{s^2}{10} ds \\ &\quad + \int_0^{1/2} \frac{(1-s)^{1/2}}{\Gamma(3/2)\big(\sqrt{2} - 1\big)} \frac{e^{-s}}{9 + e^s} ds \\ &\quad + \int_{1/2}^1 \frac{(1-s)^{1/2}}{\Gamma(3/2)\big(\sqrt{2} - 1\big)} \frac{e^{-s}}{9 + e^s} ds \end{aligned}$$

$$< \frac{1}{20\Gamma(3/4)\left(\sqrt{2}-1\right)} + \frac{1}{9\Gamma(3/2)\left(\sqrt{2}-1\right)} \approx 0.4014 < 1; \quad (63)$$

which satisfies Theorem 11. Thus (56) has a unique positive solution on $[0, 1]$.

Conflict of Interests

The authors declare that they have no competing interests.

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