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Research Article

A Note on Best Approximation in 0-Complete Partial Metric Spaces

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We study the existence and uniqueness of best proximity points in the setting of 0-complete partial metric spaces. We get our results by showing that the generalizations, which we have to consider, are obtained from the corresponding results in metric spaces. We introduce some new concepts and consider significant theorems to support this fact.

1. Introduction

Best approximation theory, in general, and best proximity point theory, in particular, have received a great attention in the last decades and have significant applications in convex optimization, differential inclusions, and optimal controls. We can say that these theories strongly relate nonlinear functional analysis with theory of functions and topologic studies.

Let (X, d) be a metric space and let A and B be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be cyclic if $T(A) \subset B$ and $T(B) \subset A$; see [1, 2]. Also, a point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$. In light of these concepts, Eldred and Veeramani [3] proved the following existence theorem (see preliminaries for basic notions in this theorem); see also [4–7].

Theorem 1 (see [3]). *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space and let T be a cyclic mapping on $A \cup B$. Suppose that there exists $r \in (0, 1)$ such that*

$$d(Ty, Tx) \leq r\ d(x, y) + (1 - r)\, d(A, B), \tag{1}$$

for all $x \in A$ and $y \in B$. Then T has a unique best proximity point z in A and $\{T^{2n}x\}$ converges to z for each $x \in A$.

On the other hand, the concept of metric space has been generalized in many directions. In this paper, we consider the concept of partial metric introduced by Matthews [8] as a part of the study of denotational semantics. The reader is referred to Bukatin et al. [9] for more details and motivation in introducing the new context, which is also leading to interesting research in foundations of topology; see, for instance, [10–18]. In particular, Romaguera [19] introduced the notions of 0-Cauchy sequence and 0-complete partial metric spaces and proved an interesting characterization of partial metric spaces in terms of 0-completeness. For other results on this specific topic, we refer to [19–21].

Also, we point out that some recent results [22–24] showed that a lot of fixed point theorems in partial metric spaces can be directly reduced to their known metric counterparts.

In this paper, we study the existence and uniqueness of best proximity points in the setting of 0-complete partial metric spaces. We get our results by showing that the generalizations, which we have to consider, are obtained from the corresponding results in metric spaces. We introduce some new concepts and consider significant theorems to support this fact. The overall motivation of this work is in underlining the strong relation between standard metric spaces and their generalizations to better target the research on this topic. Using the approach described in this paper, the reader will succeed in obtaining the extensions, to partial metric setting, of many recent results in the literature on best approximation theory.

2. Preliminaries

We collect some notions and notations needed in the sequel. Let $\mathbb{R}^+$ be the set of all nonnegative real numbers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{N}$ the set of all positive integers.

2.1. Best Approximation. We recall the notion of the property UC and some basic properties of this notion.

Let X be a Banach space. Then X is said to be uniformly convex if for every $0 < \epsilon \leq 2$ there exists some $\delta(\epsilon) > 0$ such that the conditions $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \epsilon$ imply that $\|(x + y)/2\| \leq 1 - \delta(\epsilon)$. On this basis, Suzuki et al. [25] introduced the notion of the property UC and extended Theorem 1 to metric spaces with the property UC.

Definition 2 (see [25]). Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC if the following holds.

If $\{x_n\}$ and $\{x_n'\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that

$$\lim_{n\to+\infty} d(x_n, y_n) = d(A, B),$$
$$\lim_{n\to+\infty} d(x_n', y_n) = d(A, B), \tag{2}$$

then

$$\lim_{n\to+\infty} d(x_n, x_n') = 0. \tag{3}$$

To clarify the value of the property UC, we give some examples.

Example 3 (see [26]). Let A and B be nonempty subsets of a uniformly convex Banach space X. Assume that A is convex. Then (A, B) has the property UC.

Example 4 (see [25]). Let A and B be nonempty subsets of a metric space (X, d) such that $d(A, B) = 0$. Then (A, B) satisfies the property UC.

Example 5 (see [25]). Let A, A', B, and B' be nonempty subsets of a metric space (X, d) such that $A \subset A'$, $B \subset B'$, and $d(A, B) = d(A', B')$. If (A', B') satisfies the property UC, then (A, B) satisfies the property UC.

Theorem 6 (see [25]). *Let (X, d) be a metric space and let A and B be nonempty subsets of X such that (A, B) satisfies the property UC. Assume that A is complete and let T be a cyclic mapping on $A \cup B$. Assume that there exists $r \in [0, 1)$ such that*

$$d(Ty, Tx) \leq r \max\{d(x, y), d(x, Tx), d(Ty, y)\} + (1 - r)\, d(A, B), \tag{4}$$

for all $x \in A$ and $y \in B$. Then the following hold:

(i) *T has a unique best proximity point z in A;*

(ii) *z is a unique fixed point of T^2 in A;*

(iii) *$\{T^{2n}x\}$ converges to z for every $x \in A$;*

(iv) *T has at least one best proximity point in B;*

(v) *if (B, A) satisfies the property UC, then Tz is a unique best proximity point in B and $\{T^{2n}y\}$ converges to Tz for every $y \in B$.*

2.2. Partial Metric Spaces. We give the definitions and some characterizations of partial metric and partial metric space.

Definition 7 (see [8]). A partial metric on a nonempty set X is a mapping $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$, the following conditions are satisfied:

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(p2) $p(x, x) \leq p(x, y)$;

(p3) $p(x, y) = p(y, x)$;

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A nonempty set X equipped with a partial metric p is called partial metric space; we denote this space by (X, p).

Notice that if $p(x, y) = 0$, then (p1) and (p2) imply $x = y$, but the converse does not hold true in general.

Also, each partial metric p on X generates a T_0 topology γ_p on X which has as a base the family of the open balls (p-balls) $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\} \tag{5}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 8 (see [8, 27]). Let (X, p) be a partial metric space. Then a sequence $\{x_n\}$ is called

(i) convergent, with respect to γ_p, if there exists some x in X such that $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$;

(ii) Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to γ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

As shown in [19], one can introduce a weaker form of completeness in partial metric spaces. Precisely, we recall the following statements:

(i) a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$;

(ii) (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to the partial metric p, to a point $x \in X$ such that $p(x, x) = 0$;

(iii) if (X, p) is complete, then (X, p) is 0-complete, but the converse does not hold.

Example 9 (see [19]). The partial metric space $(\mathbb{Q} \cap \mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{Q} \cap \mathbb{R}^+$, is a 0-complete partial metric space which is not complete.

3. Fixed Point Results

Hitzler and Seda in [28] proved a useful tool: Proposition 10. Later on, Haghi et al. [22] used the same argument to show that many fixed point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces. Also, they considered some significant cases to support their work. Here we recall, without giving the proof, the key result of the above authors; the interested reader is referred to [22, 28] for more details.

Proposition 10 (see [22, 28]). *Let (X, p) be a partial metric space. Then the function $d : X \times X \to \mathbb{R}^+$ given by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ p(x, y), & \text{if } x \neq y, \end{cases} \quad (6)$$

is a metric on X such that (X, d) is complete if and only if (X, p) is 0-complete.

Inspired by this fact, we introduce two new notions in partial metric spaces and use these notions to extend the approach in [22] to best proximity point results.

Definition 11. Let (X, p) be a partial metric space and let A be a nonempty subset of X. Then, A is said to be a 0-closed subset of (X, p) if, for each sequence $\{x_n\}$ in A converging to a point $x \in X$ with $p(x, x) = 0$, we have $x \in A$.

Example 12. In the partial metric space $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$, any real line interval containing zero is 0-closed. Also the interval $[a, +\infty)$, for every $a > 0$, is 0-closed. We note that $[0, 1]$ is 0-closed but it is not closed.

Proposition 13. *Let (X, p) be a partial metric space and let A be a nonempty 0-closed subset of (X, p). Then A is a closed subset of the metric space (X, d), where d is the metric given in Proposition 10.*

Proof. Let A be any nonempty 0-closed subset of (X, p). Assume that a sequence $\{x_n\}$ in A converges, with respect to the metric d, to a point $x \in X$; that is $d(x_n, x) \to 0$ whenever $n \to +\infty$. Also, we assume that $x_n \neq x$ for all $n \in \mathbb{N}$. This implies that $p(x_n, x) = d(x_n, x)$ for all $n \in \mathbb{N}$. The triangular inequality gives us

$$\begin{aligned} p(x, x) &\leq p(x_n, x) + p(x_n, x) - p(x_n, x_n) \\ &\leq p(x_n, x) + p(x_n, x) \end{aligned} \quad (7)$$

and hence, for $n \to +\infty$, we deduce that $p(x, x) = 0$. It follows that the sequence $\{x_n\}$ converges to x in (X, p). Now, since A is 0-closed, we have $x \in A$ and so A is a closed subset of (X, d). □

In view of the results in [22], we recall some known results in partial metric spaces and relate these results to their metric counterparts. Obviously we start with the partial metric version of the Banach-Caccioppoli theorem, due to Matthews [8]. However, for convenience, we formulate this result in the setting of 0-complete partial metric spaces.

Theorem 14 (see [19]). *Let (X, p) be a 0-complete partial metric space and let $T : X \to X$ be a mapping. If there exists a real number $k \in [0, 1)$ such that*

$$p(Tx, Ty) \leq kp(x, y), \quad (8)$$

for all $x, y \in X$, then there exists a unique point $x \in X$ such that $x = Tx$. Moreover, $p(x, x) = 0$.

As known, the above statement affirms that the fixed point of any contraction (i.e., self-mapping satisfying condition (8)) has zero self-distance.

Then, we give the following short proof of the above theorem.

Proof. Let d be the metric defined in Proposition 10. Since T satisfies condition (8), then T satisfies the following condition:

$$d(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X. \quad (9)$$

Indeed, if $x, y \in X$ with $x \neq y$, then $d(x, y) = p(x, y)$; on the contrary, if $x = y$ the contraction condition is trivially satisfied. Thus, T is a contraction in the metric space (X, d). Now, the metric space (X, d) is complete in view of Proposition 10, since (X, p) is a 0-complete partial metric space. Consequently, the existence and uniqueness of a point $x \in X$ such that $x = Tx$ follows by application of the Banach-Caccioppoli fixed point theorem in metric space. Finally, from (8), by choosing $y = x$, we deduce that $p(x, x) = 0$. □

For our scope, we need to introduce a new notion of 0-continuity.

Definition 15. Let (X, p) be a partial metric space and let $T : X \to X$ be a mapping. We say that T is 0-continuous in $x \in X$ with $p(x, x) = 0$ if $p(Tx, Tx) = 0$ and for each sequence $\{x_n\}$ converging to x we have

$$\lim_{n \to +\infty} p(Tx_n, Tx) = p(Tx, Tx). \quad (10)$$

Also, T is 0-continuous mapping in X if T is 0-continuous in each $x \in X$ with $p(x, x) = 0$.

Remark 16. Any contraction in Banach-Caccioppoli sense, in a partial metric space, is a 0-continuous mapping.

Now, we give an example of a 0-continuous mapping that is not continuous.

Example 17. Let $(\mathbb{R}^+, p)$ be a partial metric space, where $p(x, y) = \max\{x, y\}$ for all $x \in \mathbb{R}^+$. Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be the mapping defined by

$$Tx = \begin{cases} \dfrac{x}{3}, & x \in \mathbb{R}^+ \cap \mathbb{Q}, \\ \dfrac{x}{2}, & \in \mathbb{R}^+ \setminus \mathbb{Q}. \end{cases} \quad (11)$$

We note that T is a contraction with $k = 1/2$ and hence T is a 0-continuous mapping. On the other hand, T is not continuous, for instance, in $x = 1$. In fact, for every $n \in \mathbb{N}$, let $x_n = 1 + (\sqrt{2}/n)$; then we have $x_n \to 1$ as $n \to +\infty$, but

$$\lim_{n\to+\infty} p(Tx_n, T1) = \lim_{n\to+\infty} \max\left\{\frac{1}{2}\left(1+\frac{\sqrt{2}}{n}\right), \frac{1}{3}\right\} = \frac{1}{2} \neq \frac{1}{3} = p(T1, T1). \tag{12}$$

Now, we recall that $T : (X, d) \to (X, d)$ is said to be a Banach operator if there exists a real number $k \in [0, 1)$ such that $d(Tx, T^2x) \le kd(x, Tx)$, for all $x \in X$ ([29], Definition 4). It is well known that every continuous Banach operator on a complete metric space (X, d) has a fixed point (see, for instance, [29], Corollary 2). Then, we give the following existence result in partial metric spaces.

Theorem 18. *Let (X, p) be a 0-complete partial metric space and let $T : X \to X$ be a 0-continuous mapping such that*

$$p\left(Tx, T^2x\right) \le kp(x, Tx), \tag{13}$$

for all $x \in X$, where $k \in (0, 1)$. Then T has a fixed point in X.

Proof. Let d be the metric given in Proposition 10. Since T satisfies condition (13), then T is a Banach operator; that is

$$d\left(Tx, T^2x\right) \le kd(x, Tx), \quad \forall x \in X. \tag{14}$$

Indeed, if $x \neq Tx$, then $d(x, Tx) = p(x, Tx)$. Again, if $x = Tx$, the above contractive condition is trivially satisfied. This implies that, for each starting point $x_0 \in X$, the sequence $\{T^n x_0\}$ is Cauchy in (X, d) and hence 0-Cauchy in (X, p). Thus, there exists $x \in X$ such that

$$\lim_{n\to+\infty} p(x_n, x) = p(x, x) = 0. \tag{15}$$

Also, since T is 0-continuous, we deduce that

$$x = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} Tx_n = Tx. \tag{16}$$

It follows that x is a fixed point of T; that is $x = Tx$. □

Next, we show that this theorem can be viewed as a direct consequence of its metric counterpart. First, we need the following auxiliary result.

Proposition 19. *Let (X, p) be a partial metric space and let $T : X \to X$ be a 0-continuous mapping. Then T is continuous with respect to the metric d given in Proposition 10; that is $T : (X, d) \to (X, d)$ is continuous.*

Proof. We assume that T is 0-continuous mapping and show that T is continuous with respect to the metric d. Fix $x \in X$ and consider a sequence $\{x_n\}$ converging to x with respect to d. If $x_n = x_m$ for all $n \ge m$, then $x = x_m$ and so $d(Tx_n, Tx) = d(Tx, Tx) = 0$ for all $n \ge m$. On the contrary, assume that $x_n \neq x$ for all $n \in \mathbb{N}$ (the same holds if $x_n \neq x$ for infinite many values of n). Therefore, from $p(x_n, x) = d(x_n, x)$ we deduce that

$$\lim_{n\to+\infty} p(x_n, x) = p(x, x) = 0. \tag{17}$$

Since T is 0-continuous, then

$$\lim_{n\to+\infty} d(Tx_n, Tx) \le \lim_{n\to+\infty} p(Tx_n, Tx) = 0 \tag{18}$$

and so we conclude that $T : (X, d) \to (X, d)$ is continuous. □

We give the following short proof of Theorem 18.

Proof. Since $T : X \to X$ in Theorem 18 is 0-continuous, in virtue of Proposition 19, we have that T is continuous with respect to the metric d given in Proposition 10. Thus, we can apply Corollary 2 of [29] to conclude. □

Briefly, by using the same technique, we show that also the fixed point results for cyclic mappings in metric spaces can be translated in partial metric spaces. Firstly, we recall the following definition.

Definition 20. Let (X, p) be a partial metric space and let A and B be two nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is called cyclic mapping if $T(A) \subset B$ and $T(B) \subset A$. Also T is a cyclic contraction if there exists a real number $k \in [0, 1)$ such that

$$p(Tx, Ty) \le kp(x, y), \quad \forall x \in A, y \in B. \tag{19}$$

Now, we state and prove the following.

Theorem 21. *Let (X, p) be a 0-complete partial metric space and let A and B be two nonempty 0-closed subsets of X. If $T : A \cup B \to A \cup B$ is a cyclic contraction, then T has a unique fixed point in $A \cap B$.*

Proof. Let d be the metric given in Proposition 10. In virtue of Proposition 13, A and B are closed subsets of the metric space (X, d), since A and B are 0-closed sets. Obviously, since T satisfies condition (19), then T satisfies also the following condition:

$$d(Tx, Ty) \le kd(x, y), \quad \forall x \in A, y \in B. \tag{20}$$

Indeed, if $x, y \in X$ and $x \neq y$, then $d(x, y) = p(x, y)$. On the other hand, for $x = y$, the above condition is trivially true. We obtain that T is a cyclic contraction with respect to the metric d. Therefore, from Theorem 1.1 of [2] (in view of the comments in the introduction section of [2]) readily follows the existence and uniqueness of a point $x \in A \cap B$ such that $x = Tx$. Also, from (19) for $y = Tx$, we get $p(x, x) = 0$. □

4. Best Proximity Point Results

In the light of extending the technique in the previous section to obtain best proximity point results in partial metric space, we recall some notions and notations.

Definition 22. Let (X, p) be a partial metric space and let A and B be two nonempty subsets of X. For all $x \in X$, put

$$p(x, A) := \inf\{p(x, a) : a \in A\}, \quad p(A, B) := \inf\{p(a, b) : a \in A, b \in B\}. \tag{21}$$

If d is the metric related to the partial metric p in Proposition 10, then $d(x, A) = p(x, A)$ if $x \notin A$ and $d(A, B) = p(A, B)$ if $A \cap B = \emptyset$.

Then we adapt the property UC in partial metric spaces.

Definition 23. Let A and B be nonempty subsets of a partial metric space (X, p). Then (A, B) is said to satisfy the property UC_p if the following holds.

If $\{x_n\}$ and $\{x'_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that

$$\lim_{n\to+\infty} p(x_n, y_n) = p(A, B), \quad \lim_{n\to+\infty} p(x'_n, y_n) = p(A, B), \tag{22}$$

then

$$\lim_{n\to+\infty} p(x_n, x'_n) = 0. \tag{23}$$

Following the approach in [22], we present an auxiliary lemma; see Lemma 2.2 of [22].

Lemma 24. *Let (X, p) be a partial metric space and let A and B be two nonempty subsets of X. Assume that $T : A\cup B \to A\cup B$ is a cyclic mapping and d is the metric given in Proposition 10. Then, for all $x \in A$ and $y \in B$ with $x \neq y$, we have $M_d(x, y) = M_p(x, y)$, where*

$$M_d(x, y) := \max\{d(x, y), d(x, Tx), d(y, Ty)\}, \quad M_p(x, y) := \max\{p(x, y), p(x, Tx), p(y, Ty)\}. \tag{24}$$

Proof. Let $x \in A$ and $y \in B$; if $x \neq y$, $x \neq Tx$ and $y \neq Ty$, then

$$\begin{aligned} M_d(x, y) &= \max\{d(x, y), d(x, Tx), d(y, Ty)\} \\ &= \max\{p(x, y), p(x, Tx), p(y, Ty)\} \\ &= M_p(x, y). \end{aligned} \tag{25}$$

Also, if $x \neq y$, $x \neq Tx$ and $y = Ty$, in view of the fact that $p(y, Ty) \leq p(x, y)$, we deduce

$$\begin{aligned} M_d(x, y) &= \max\{d(x, y), d(x, Tx)\} \\ &= \max\{p(x, y), p(x, Tx)\} \\ &= \max\{p(x, y), p(x, Tx), p(y, Ty)\} \\ &= M_p(x, y). \end{aligned} \tag{26}$$

Analogous reasoning shows that $M_d(x, y) = M_p(x, y)$ in the case $x \neq y$, $x = Tx$ and $y \neq Ty$, and in the case $x \neq y$, $x = Tx$ and $y = Ty$. This concludes the proof. However, notice that for all $x, y \in X$ we have

$$d(x, y) \leq p(x, y), d(x, Tx) \leq p(x, Tx), \quad d(y, Ty) \leq p(y, Ty). \tag{27}$$

This implies that

$$M_d(x, y) \leq M_p(x, y) \tag{28}$$

for all $x, y \in X$. □

A direct consequence of Lemma 24 is the following.

If A and B are nonempty and disjoint subsets of a partial metric space (X, p), then

$$\begin{aligned} &rM_d(x, y) + (1 - r) d(A, B) \\ &\quad = rM_p(x, y) + (1 - r) p(A, B), \end{aligned} \tag{29}$$

for all $x \in A$, $y \in B$, where $r \in [0, 1)$.

We are ready to state and prove the following best approximation result in partial metric spaces.

Theorem 25. *Let (X, p) be a partial metric space and let A and B be nonempty and disjoint subsets of X such that (A, B) satisfies the property UC_p. Assume that A is 0-complete and T is a cyclic mapping on $A \cup B$. Also suppose that there exists a real number $r \in [0, 1)$ such that*

$$p(Ty, Tx) \leq rM_p(x, y) + (1 - r) p(A, B) \tag{30}$$

for all $x \in A$ and $y \in B$. Then the following hold:

(i) *T has a unique best proximity point z in A;*

(ii) *z is a unique fixed point of T^2 in A;*

(iii) *$\{T^{2n}x\}$ converges to z for every $x \in A$;*

(iv) *T has at least one best proximity point in B;*

(v) *if (B, A) satisfies the property UC_p, then Tz is a unique best proximity point in B and $\{T^{2n}y\}$ converges to Tz for every $y \in B$.*

Proof. Let d be the metric given in Proposition 10. Since (A, B) has the property UC_p in (X, p) and $A \cap B = \emptyset$, then (A, B) has the property UC in (X, d). Also, since A is 0-complete, then A is complete in (X, d). It follows that all the hypotheses of Theorem 6 are satisfied with respect to the metric d and consequently also the assertions (i)–(v) in Theorem 25 hold true. □

Recently, Caballero et al. [30] proved the following theorem for noncyclic mappings.

Theorem 26 (see [30]). *Let (X, d) be a complete metric space and let A and B be nonempty closed subsets of X such that $A_0 := \{x \in A : d(x, y) = d(A, B)$ for some $y \in B\} \neq \emptyset$ and*

$$\begin{matrix} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{matrix} \Longrightarrow d(x_1, x_2) = d(y_1, y_2), \tag{31}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0 := \{y \in B : d(x, y) = d(A, B)$ for some $x \in A\}$. Let $T : A \to B$ be a mapping such that $T(A_0) \subseteq B_0$ and

$$d(Tx, Ty) \leq \beta(d(x, y)) d(x, y), \tag{32}$$

for all $x, y \in A$ *and some function* $\beta : \mathbb{R}^+ \to [0,1)$ *satisfying the condition:* $\beta(t_n) \to 1$ *implies* $t_n \to 0$. *Then* T *has a unique best proximity point* $z \in A$.

From the above theorem we are ready to derive the partial metric counterpart.

Theorem 27. *Let* (X, p) *be a* 0*-complete partial metric space and let* A *and* B *be nonempty and disjoint* 0*-closed subsets of* X *such that* $A_0 := \{x \in A : p(x,y) = p(A,B)$ *for some* $y \in B\} \neq \emptyset$ *and*

$$\begin{aligned} p(x_1, y_1) &= p(A,B) \\ p(x_2, y_2) &= p(A,B) \end{aligned} \Longrightarrow p(x_1, x_2) = p(y_1, y_2), \quad (33)$$

where $x_1, x_2 \in A_0$ *and* $y_1, y_2 \in B_0 := \{y \in B : p(x,y) = p(A,B)$ *for some* $x \in A\}$. *Let* $T : A \to B$ *be a mapping such that* $T(A_0) \subseteq B_0$ *and*

$$p(Tx, Ty) \leq \beta(p(x,y))\, p(x,y), \quad (34)$$

for all $x, y \in A$ *and some function* $\beta : \mathbb{R}^+ \to [0,1)$ *satisfying the condition:* $\beta(t_n) \to 1$ *implies* $t_n \to 0$. *Then* T *has a unique best proximity point* $z \in A$.

Proof. Let d be the metric given in Proposition 10 and hence, since (X, p) is 0-complete, (X, d) is complete. In virtue of Proposition 13, A and B are closed subsets of the metric space (X, d), since A and B are 0-closed sets. Also, since $A \cap B = \emptyset$, then (31) and (32) hold true. It follows that all the hypotheses of Theorem 26 are satisfied with respect to the metric d and consequently also Theorem 27 holds true. □

Very recently, Gabeleh and Shahzad [31] proved the following theorem for a pair of mappings.

Theorem 28 (see [31]). *Let* (X, d) *be a complete metric space, let* A *and* B *be nonempty closed subsets of* X *with* A_0, B_0 *as in Theorem 26 and let* $\eta : [0,1) \to (1/2, 1]$ *be defined by* $\eta(r) = (1+r)^{-1}$. *Assume that* $S : A \to A$ *and* $T : A \to B$ *are two mappings satisfying the following conditions:*

(i) *there exists* $r \in [0,1)$ *such that* $\eta(r)[d(Sx, Tx) - d(A,B)] \leq d(x,y)$ *implies* $d(Tx, Ty) \leq r d(x,y)$ *for all* $x, y \in A$;

(ii) $T(A_0) \subseteq B_0$ *and* $A_0 \subseteq S(A_0)$;

(iii) S *is an isometry; that is* $d(u,v) = d(Su, Sv)$ *for all* $u, v \in A$.

If the pair (A, B) *satisfies the condition*

$$\begin{aligned} d(x_1, y_1) &= d(A,B) \\ d(x_2, y_2) &= d(A,B) \end{aligned} \Longrightarrow d(x_1, x_2) \leq d(y_1, y_2), \quad (35)$$

where $x_1, x_2 \in A_0$ *and* $y_1, y_2 \in B_0$, *then there exists a unique* $z \in A$ *such that* $d(Sz, Tz) = d(A,B)$. *Moreover, for any* $x_0 \in A_0$, *the iterative sequence* $\{x_n\}$, *defined by the algorithm* $d(Sx_{n+1}, Tx_n) = d(A,B)$, *converges to* z.

Also from this theorem we are ready to derive the partial metric counterpart.

Theorem 29. *Let* (X, p) *be a* 0*-complete partial metric space, let* A *and* B *be nonempty and disjoint* 0*-closed subsets of* X *with* A_0, B_0 *as in Theorem 27, and let* $\eta : [0,1) \to (1/2, 1]$ *be defined by* $\eta(r) = (1+r)^{-1}$. *Assume that* $S : A \to A$ *and* $T : A \to B$ *are two mappings satisfying the following conditions:*

(i) *there exists* $r \in [0,1)$ *such that* $\eta(r)[p(Sx, Tx) - p(A,B)] \leq p(x,y)$ *implies* $p(Tx, Ty) \leq r p(x,y)$ *for all* $x, y \in A$;

(ii) $T(A_0) \subseteq B_0$ *and* $A_0 \subseteq S(A_0)$;

(iii) S *is an injective function such that* $p(u,v) = p(Su, Sv)$ *for all* $u, v \in A$.

If the pair (A, B) *satisfies the condition*

$$\begin{aligned} p(x_1, y_1) &= p(A,B) \\ p(x_2, y_2) &= p(A,B) \end{aligned} \Longrightarrow p(x_1, x_2) \leq p(y_1, y_2), \quad (36)$$

where $x_1, x_2 \in A_0$ *and* $y_1, y_2 \in B_0$, *then there exists a unique* $z \in A$ *such that* $p(Sz, Tz) = p(A,B)$. *Moreover, for any* $x_0 \in A_0$, *the iterative sequence* $\{x_n\}$, *defined by the algorithm* $p(Sx_{n+1}, Tx_n) = p(A,B)$, *converges to* z.

Proof. Let d be the metric given in Proposition 10; since (X, p) is 0-complete, then (X, d) is complete. In virtue of Proposition 13, A and B are closed subsets of the metric space (X, d), since A and B are 0-closed sets. Also, since $A \cap B = \emptyset$, then conditions (i) and (iii) of Theorem 28 and (35) hold true. It follows that all the hypotheses of Theorem 28 are satisfied with respect to the metric d and consequently also Theorem 29 holds true. □

5. Conclusion

The development of best approximation theory is an actual and relevant topic in solving various minimization problems. In particular, the constructive proofs of the existence and uniqueness of best proximity points consent to obtain algorithms for approaching these problems. On the other hand, over the years, an interesting matter was to introduce generalized notions of distance to suit better specific problems (i.e., here we consider partial metrics used in Computer Science). Then, the approach described in this paper suggests how to obtain theoretical results in generalized metric spaces from the corresponding results in standard metric spaces. As shown above, the approach has several advantages, such as the ability to define and use new notions for simplifying proofs and calculations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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