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*Research Article*

# A Boundary Value Problem for Bihypermonogenic Functions in Clifford Analysis

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This paper deals with a nonlinear boundary value problem for bihypermonogenic functions in Clifford analysis. The integrals of quasi-Cauchy's type and Plemelj formula for bihypermonogenic functions are firstly reviewed briefly. The nonlinear Riemmann boundary value problem for bihypermonogenic functions is discussed and the existence of solutions is obtained, which also indicates that the linear boundary value problem has a unique solution.

## 1. Introduction

Clifford algebra is an associative and noncommutative algebraic structure that was set up at the beginning of the twentieth century. Clifford analysis is an important branch of modern analysis, which studies the functions defined in $\mathbb{R}^{n+1}$ with the value in Clifford algebra space [1]. Clifford analysis possesses not only important theoretical value but also applicable value, which plays an important role in many fields, such as quantum mechanics, Maxwell equation, and Yang-Mills field. Since 1987, Xu [2, 3], Wen [4], Huang [5, 6], Qiao [7–9], and so forth have done a lot of work on boundary value problems for monogenic functions and biregular functions in Clifford analysis. Eriksson and Leutwiler [10–12] introduced hypermonogenic functions in Clifford analysis, studied some of its properties, and discussed the integral representation for hypermonogenic functions. Qiao [9] investigated the boundary value problems of hypermonogenic functions. In recent years, Zhang and Du [13, 14] discussed Riemann boundary value problems and singular integral equations in Clifford analysis. Bian et al. [15] obtained the integral formulas and Plemelj formula for bihypermonogenic functions. Yang et al. [16] studied a kind of boundary value problem for hypermonogenic function vector. Zhang and Gürlebeck [17] studied Riemann boundary value problems in Clifford analysis.

In this paper, based on the integral formulas and Plemelj formula for bihypermonogenic functions presented in [15], we study a nonlinear Riemann boundary value problem for bihypermongenic functions. We first review briefly the integrals of quasi-Cauchy's type and Plemelj formula for bihypermonogenic functions and then prove the existence of solutions of a nonlinear Riemann boundary value problem and derive the unique solution of the corresponding linear Riemann boundary value problem.

## 2. Preliminaries

Let $C\ell_{0,n}$ be a real Clifford algebra over an $n+1$-dimensional real vector space $R^{n+1}$ with orthogonal basis $e := \{e_0, e_1, \ldots, e_n\}$, satisfying the relation $e_i e_j + e_j e_i = -2\delta_{ij}$ $(i, j = 1, \ldots, n)$, where $\delta_{ij}$ is the usual Kronecker delta. Then $C\ell_{0,n}$ has its basis $e_0 = 1, e_1, \ldots, e_n; e_1e_2, \ldots, e_{n-1}e_n; \ldots; e_1 \cdots e_n$. Hence the real Clifford algebra is formed by the elements presented as $a = \sum_A x_A e_A$, $x_A \in \mathbb{R}$, where $A = \{i_1, i_2, \ldots, i_k \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$ or $A = \emptyset$ and $e_\emptyset = e_0$.

For $a \in Cl_{0,n}$, we give some calculations as follows:

$$a' = \sum_A x_A e'_A, \tag{1}$$

where $e'_A = (-1)^{|A|}e_A$ and $|A| = n_A$ is the cardinality of $A$; that is, when $A = \emptyset$, $|A| = 0$ and when $A = \{\alpha_1, \alpha_2, \dots, \alpha_h\} \neq \emptyset$, then $|A| = h$; $e'_0 = 1$, $e'_i = -e_i$, $i = 1, \dots, n$.

Recall that any element $x \in C\ell_{0,n}$ may be uniquely decomposed as $x = b + ce_n$, for $b, c \in C\ell_{0,n-1}$ (the Clifford algebra generated by $e_0, \dots, e_{n-1}$). Using this decomposition, we define the mappings $P : C\ell_{0,n} \to C\ell_{0,n-1}$ and $Q : C\ell_{0,n} \to C\ell_{0,n-1}$ by $Px = b$ and $Qx = c$. Note that if $x = \sum_A x_A e_A \in C\ell_{0,n}$, then

$$\begin{aligned} Px &= \sum_{n\notin A} x_A e_A, \qquad Qx = \sum_{n\in A} x_A e_{A\setminus\{n\}}, \\ P'x &= \sum_{n\notin A} x_A e'_A, \qquad Q'x = \sum_{n\in A} x_A e'_{A\setminus\{n\}}. \end{aligned} \tag{2}$$

We also introduce the Dirac operator

$$D_l f = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}, \qquad D_r f = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} e_i, \tag{3}$$

and the modified Dirac operator

$$\begin{aligned} M^l f(x) &= D_l f(x) + (n-1)\frac{Q'f}{x_n}, \\ M^r f(x) &= D_r f(x) + (n-1)\frac{Qf}{x_n}. \end{aligned} \tag{4}$$

Denote by $\Omega = \Omega_1 \times \Omega_2$ an open connected set in the Euclidean space $\mathbb{R}^{m+1} \times \mathbb{R}^{k+1}$, $1 \le m \le n$, $1 \le k \le n$. Define a set $\mathcal{F}_\Omega^{(r)}$ to consist of all functions

$$f(x,y) = \sum_{A\subset\{1,\dots,m\}} \sum_{B\subset\{m+1,\dots,m+k\}} f_{A,B}(x,y)\, e_A e_B, \tag{5}$$

with values in $C\ell_{0,k+m}$ for which $f_{A,B}(x,y) \in \mathscr{C}^r(\Omega)$.

*Definition 1.* Let $f \in \mathcal{F}_\Omega^{(1)}$ and $x \in \mathbb{R}^{m+1} \setminus \{x_m = 0\}$, $y \in \mathbb{R}^{k+1} \setminus \{y_k = 0\}$. A function $f(x,y)$ is called *bihypermonogenic* on $\Omega$, if

$$M_x^l f(x,y) = 0, \qquad M_y^r f(x,y) = 0, \tag{6}$$

for any $(x,y) \in \Omega$, where

$$\begin{aligned} M_x^l f(x,y) = {} & \frac{\partial f}{\partial x_0}(x,y) + \sum_{i=1}^{m} e_i \frac{\partial f}{\partial x_i}(x,y) \\ & + (m-1)\frac{Q'_x(f(x,y))}{x_m} \end{aligned} \tag{7}$$

is the left modified Dirac operator in $C\ell_{0,m}$ calculated with respect to $x \in \mathbb{R}^{m+1} \setminus \{x_m = 0\}$ and

$$\begin{aligned} M_y^r f(x,y) = {} & \frac{\partial f}{\partial y_0}(x,y) + \sum_{i=1}^{k} \frac{\partial f}{\partial y_i}(x,y)\, e_{i+m} \\ & + (k-1)\frac{Q_y(f(x,y))}{y_k} \end{aligned} \tag{8}$$

is the right modified Dirac operator in the Clifford algebra generated by $e_0, e_{m+1}, \dots, e_{m+k}$ calculated with respect to $y \in \mathbb{R}^{k+1} \setminus \{y_k = 0\}$, where

$$\begin{aligned} Q'_x f(x,y) &= \sum_{m\in A} \sum_{B} f_{A,B}(x,y)\, e'_{A\setminus\{m\}} e_B \\ &= \sum_{m\in A} \sum_{B} (-1)^{|A\setminus\{m\}|} f_{A,B}(x,y)\, e_{A\setminus\{m\}} e_B, \\ Q_y f(x,y) &= \sum_{A} \sum_{m+k\in B} f_{A,B}(x,y)\, e_A e_{B\setminus\{m+k\}}. \end{aligned} \tag{9}$$

## 3. The Cauchy Integral Formula and Plemelj Formula

In this section, we give some simple review on the Cauchy integral formula and Plemelj formula for bihypermonogenic functions obtained by us and presented in [15]. We first give some notations which will be used in the following analysis.

A function $f(x,y) : \partial\Omega_1 \times \partial\Omega_2 \to Cl_{0,n}$ is said to be Hölder continuous on $\partial\Omega_1 \times \partial\Omega_2$, if $f(x,y)$ satisfies

$$\begin{gathered} |f(x_1,y_1) - f(x_2,y_2)| \le M_1 |(x_1,y_1) - (x_2,y_2)|^\beta, \\ (x_1,y_1), (x_2,y_2) \in \partial\Omega_1 \times \partial\Omega_2, \quad (0 < \beta < 1). \end{gathered} \tag{10}$$

Denote by $H(\partial\Omega_1 \times \partial\Omega_2, \beta)$ the set of all Hölder continuous functions on $\partial\Omega_1 \times \partial\Omega_2$ with the index $\beta$. For any $f \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$, define the norm in $H(\partial\Omega_1 \times \partial\Omega_2, \beta)$ as $\|f\|_\beta = C(f, \partial\Omega_1 \times \partial\Omega_2) + H(f, \partial\Omega_1 \times \partial\Omega_2, \beta)$, where

$$\begin{gathered} C(f, \partial\Omega_1 \times \partial\Omega_2) = \sup_{\substack{(u_i,v_i)\in\partial\Omega_1\times\partial\Omega_2 \\ (u_1,v_1)\neq(u_2,v_2)}} \frac{|f(u_1,v_1) - f(u_2,v_2)|}{|(u_1,v_1) - (u_2,v_2)|^\beta}, \\ H(f, \partial\Omega_1 \times \partial\Omega_2, \beta) = \max_{(u,v)\in\partial\Omega_1\times\partial\Omega_2} |f(u,v)|, \\ f \in H(\partial\Omega_1 \times \partial\Omega_2, \beta). \end{gathered} \tag{11}$$

Furthermore, for any $f, g \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$, we have

$$\|fg\|_\beta \le J_0 \|f\|_\beta \|g\|_\beta. \tag{12}$$

**Theorem 2** (see [15]). *Let $\Omega'$ and $\Omega''$ be open subsets of $\mathbb{R}_+^{m+1}$ and $\mathbb{R}_+^{k+1}$, respectively. Suppose that $\Omega_1$ and $\Omega_2$ satisfy $\overline{\Omega_1} \subset \Omega'$ and $\overline{\Omega_2} \subset \Omega''$, respectively. The boundaries $\partial\Omega_1$, $\partial\Omega_2$ of $\Omega_1$, $\Omega_2$ are differentiable, oriented, compact Liapunov surfaces. If*

*$\varphi(x,y)$ is a bihypermonogenic function in $\Omega'\times\Omega''$, $x\in\Omega_1$, $y\in\Omega_2$, then*

$$\begin{aligned}\varphi(x,y) &= \lambda\int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,x)\,d\sigma_m(u)\,\varphi(u,v)\,d\boldsymbol{\sigma}_k(v)\,\mathbf{E}_k(v,y)\\ &- \lambda\int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,x)\,d\sigma_m(u)\,\widetilde{\varphi(u,v)}\widetilde{d\boldsymbol{\sigma}_k(v)}\mathbf{F}_k(v,y)\\ &- \lambda\int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,x)\,\widehat{d\sigma_m(u)}\widehat{\varphi(u,v)}d\boldsymbol{\sigma}_k(v)\,\mathbf{E}_k(v,y)\\ &+ \lambda\int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,x)\,\widehat{d\sigma_m(u)}\widetilde{\widehat{\varphi(u,v)}}\widetilde{d\boldsymbol{\sigma}_k(v)}\mathbf{F}_k(v,y),\end{aligned} \tag{13}$$

*where*

$$\begin{aligned} d\sigma_m &= dx_1\wedge dx_2\wedge\cdots dx_m\\ &\quad+\sum_{i=1}^{m}(-1)^i e_i\,dx_0\wedge\cdots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\cdots dx_m,\\ d\boldsymbol{\sigma}_k &= dy_1\wedge dy_2\wedge\cdots dy_k\\ &\quad+\sum_{i=1}^{k}(-1)^i e_{i+m}\,dy_0\wedge dy_1\wedge\cdots\wedge dy_{i-1}\wedge dy_{i+1}\\ &\quad\wedge\cdots dy_k,\\ v &= v_0+v_1e_{m+1}+\cdots+v_ke_{m+k},\\ y &= y_0+y_1e_{m+1}+\cdots+y_ke_{m+k},\\ \lambda &= \frac{2^{m-1}x_m^{m-1}2^{k-1}y_k^{k-1}}{\omega_{m+1}\omega_{k+1}},\\ E_l(u,x) &= \frac{(u-x)^{-1}}{|u-x|^{l-1}|u-\widehat{x}|^{l-1}},\quad u,x\in\mathbb{R}^{m+1},\\ F_l(u,x) &= \frac{(\widehat{u}-x)^{-1}}{|u-x|^{l-1}|u-\widehat{x}|^{l-1}},\quad u,x\in\mathbb{R}^{m+1},\\ \mathbf{E}_l(v,y) &= \frac{(v-y)^{-1}}{|v-y|^{l-1}|v-\widetilde{y}|^{l-1}},\\ \mathbf{F}_l(v,y) &= \frac{(\widetilde{v}-y)^{-1}}{|v-y|^{l-1}|v-\widetilde{y}|^{l-1}},\end{aligned} \tag{14}$$

*and the involutions ^ and ~ are defined by*

$$\begin{aligned}&\widehat{e_i}=e_i,\quad i\in\{0,1,\ldots,m+k\}\setminus\{m\},\\ &\widehat{e_m}=-e_m,\quad \widehat{ab}=\widehat{a}\widehat{b},\\ &\widetilde{e_i}=e_i,\quad i\in\{0,1,\ldots,m+k-1\},\\ &\widetilde{e_{m+k}}=-e_{m+k},\quad \widetilde{ab}=\widetilde{a}\widetilde{b}.\end{aligned} \tag{15}$$

*Definition 3* (see [15]). The integral

$$\begin{aligned}\phi(t_1,t_2) &= \lambda\int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\,d\sigma_m(u)\,\varphi(u,v)\,d\boldsymbol{\sigma}_k(v)\,\mathbf{E}_k(v,t_2)\\ &- \lambda\int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\,d\sigma_m(u)\,\widetilde{\varphi(u,v)}\widetilde{\mathbf{d}\boldsymbol{\sigma}_k(v)}\mathbf{F}_k(v,t_2)\\ &- \lambda\int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\,\widehat{d\sigma_m(u)}\widehat{\varphi(u,v)}d\boldsymbol{\sigma}_k(v)\,\mathbf{E}_k(v,t_2)\\ &+ \lambda\int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\,\widehat{d\sigma_m(u)}\widetilde{\widehat{\varphi(u,v)}}\widetilde{\mathbf{d}\boldsymbol{\sigma}_k(v)}\mathbf{F}_k(v,t_2)\end{aligned} \tag{16}$$

is called a singular integral on $\partial\Omega_1\times\partial\Omega_2$, where $\lambda$, $E_m(u,t_1)$, $\mathbf{E}_k(v,t_2)$, $F_m(u,t_1)$, and $\mathbf{F}_k(v,t_1)$ are given in Theorem 2.

*Definition 4* (see [15]). Let $\delta>0$ be a constant and $\lambda_\delta = B_1(t_1,\delta)\times B_2(t_2,\delta)$, where $B_i(t_i,\delta)$ $(i=1,2)$ are balls with the center at $t_i$ and the radius $\delta>0$. Denote

$$\begin{aligned}\phi_\delta(t_1,t_2) &= \lambda\int_{\partial\Omega_1\times\partial\Omega_2\setminus\lambda_\delta} E_m(u,t_1)\,d\sigma_m(u)\,\varphi(u,v)\,d\boldsymbol{\sigma}_k(v)\,\mathbf{E}_k(v,t_2)\\ &- \lambda\int_{\partial\Omega_1\times\partial\Omega_2\setminus\lambda_\delta} E_m(u,t_1)\,d\sigma_m(u)\,\widetilde{\varphi(u,v)}\widetilde{\mathbf{d}\boldsymbol{\sigma}_k(v)}\mathbf{F}_k(v,t_2)\\ &- \lambda\int_{\partial\Omega_1\times\partial\Omega_2\setminus\lambda_\delta} F_m(u,t_1)\,\widehat{d\sigma_m(\mu)}\widehat{\varphi(u,v)}d\boldsymbol{\sigma}_k(v)\,\mathbf{E}_k(v,t_2)\\ &+ \lambda\int_{\partial\Omega_1\times\partial\Omega_2\setminus\lambda_\delta} F_m(u,t_1)\,\widehat{d\sigma_m(u)}\widetilde{\widehat{\varphi(u,v)}}\widetilde{\mathbf{d}\boldsymbol{\sigma}_k(v)}\mathbf{F}_k(v,t_2).\end{aligned} \tag{17}$$

If $\lim_{\delta\to 0}\phi_\delta(t_1,t_2)=I$, then $I$ is called the Cauchy principal value of a singular integral, denoted by $I=\phi(t_1,t_2)$.

**Lemma 5** (see [11]). *Let $\Omega$ be an open subset of $\mathbb{R}_+^{n+1}=\{x=(x_0,x_1,\ldots,x_n)\mid x_n>0\}$ and let $K$ be an $n+1$-chain satisfying $\overline{K}\subset\Omega$; then*

$$\begin{aligned}&\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\left(\int_{\partial K}E_n(x,y)\,d\sigma(x)-\int_{\partial K}F_n(x,y)\,\widehat{d\sigma(x)}\right)\\ &\quad=\begin{cases}1, & y\in K,\\ 0, & y\in\mathbb{R}_+^{n+1}-\overline{K}.\end{cases}\end{aligned} \tag{18}$$

**Lemma 6** (see [11]). *Let $\Omega, K$ and $\partial K$ be as in Lemma 5 and $y\in\partial K$; then*

$$\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\left(\int_{\partial K}E_n(x,y)\,d\sigma(x)-\int_{\partial K}F_n(x,y)\,\widehat{d\sigma(x)}\right)=\frac{1}{2}. \tag{19}$$

**Theorem 7** (see [15]). *If $\varphi(u,v) \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$, then there exists the Cauchy principal value of singular integrals and*

$$\begin{aligned}\phi(t_1,t_2) &= -\frac{1}{4}\varphi(t_1,t_2) + X_1(t_1,t_2) + X_2(t_1,t_2) + X_3(t_1,t_2) \\ &\quad + X_4(t_1,t_2) + \frac{1}{4}(P_1\varphi + P_2\varphi) + \frac{1}{4}(Q_1\varphi + Q_2\varphi),\end{aligned} \tag{20}$$

*where*

$$\begin{aligned}
X_1(t_1,t_2) &= \lambda \int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\, d\sigma_m(u)\, \psi_1(u,v)\, d\boldsymbol{\sigma}_k(v) \times \mathbf{E}_k(v,t_2),\\
X_2(t_1,t_2) &= -\lambda \int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\, d\sigma_m(u)\, \psi_2(u,v)\, \widetilde{d\boldsymbol{\sigma}_k(v)} \times \mathbf{F}_k(v,t_2),\\
X_3(t_1,t_2) &= -\lambda \int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\psi_3(u,v)\, d\boldsymbol{\sigma}_k(v) \times \mathbf{E}_k(v,t_2),\\
X_4(t_1,t_2) &= \lambda \int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\psi_4(u,v)\, \widetilde{d\boldsymbol{\sigma}_k(v)} \times \mathbf{F}_k(v,t_2),
\end{aligned} \tag{21}$$

$$\psi_1(u,v) = \varphi(u,v) - \varphi(u,t_2) - \varphi(t_1,v) + \varphi(t_1,t_2),$$

$$\psi_2(u,v) = \widetilde{\varphi(u,v)} - \varphi(u,t_2) - \widetilde{\varphi(t_1,v)} + \varphi(t_1,t_2),$$

$$\psi_3(u,v) = \widehat{\varphi(u,v)} - \widehat{\varphi(u,t_2)} - \varphi(t_1,v) + \varphi(t_1,t_2),$$

$$\psi_4(u,v) = \widetilde{\widehat{\varphi(u,v)}} - \widehat{\varphi(u,t_2)} - \widetilde{\varphi(t_1,v)} + \varphi(t_1,t_2),$$

$$P_1\varphi = 2\lambda_1 \int_{\partial\Omega_1} E_m(u,t_1)\, d\sigma_m(u)\, \varphi(u,t_2),$$

$$P_2\varphi = 2\lambda_2 \int_{\partial\Omega_2} \varphi(t_1,v)\, d\boldsymbol{\sigma}_k(v)\, \mathbf{E}_k(v,t_2),$$

$$Q_1\varphi = -2\lambda_1 \int_{\partial\Omega_1} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\widehat{\varphi(u,t_2)},$$

$$Q_2\varphi = -2\lambda_2 \int_{\partial\Omega_2} \widetilde{\varphi(t_1,v)}\, \widetilde{d\boldsymbol{\sigma}_k(v)}\mathbf{F}_k(v,t_2),$$

$$\lambda = \lambda_1\lambda_2, \qquad \lambda_1 = \frac{2^{m-1}x_m^{m-1}}{\omega_{m+1}}, \qquad \lambda_2 = \frac{2^{k-1}y_k^{k-1}}{\omega_{k+1}}.$$

Set $\Omega_i^+ = \Omega_i$, $(i = 1,2)$, $\Omega_1^- = \mathbb{R}_+^{m+1} \setminus \overline{\Omega_1}$, $\Omega_2^- = \mathbb{R}_+^{k+1} \setminus \overline{\Omega_2}$, and denote $x(\in \Omega_1^\pm) \to t_1 \in \partial\Omega_1$ by $x \to t_1^\pm$. Moreover denote $y(\in \Omega_2^\pm) \to t_2 \in \partial\Omega_2$ by $y \to t_2^\pm$ and denote by $\phi^{\pm\pm}(t_1,t_2)$ the limits of $\phi(x,y)$ when $(x,y) \to (t_1^\pm, t_2^\pm)$. Then we have the following important theorem.

**Theorem 8** (see [15]). *If $\varphi(u,v) \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$, then*

$$\begin{aligned}
\phi^{++}(t_1,t_2) &= \frac{1}{4}[\varphi(t_1,t_2) + P_1(\varphi) + P_2(\varphi) + Q_1(\varphi) + Q_2(\varphi) + P_3(\varphi)],\\
\phi^{+-}(t_1,t_2) &= \frac{1}{4}[-\varphi(t_1,t_2) - P_1(\varphi) + P_2(\varphi) - Q_1(\varphi) + Q_2(\varphi) + P_3(\varphi)],\\
\phi^{-+}(t_1,t_2) &= \frac{1}{4}[-\varphi(t_1,t_2) + P_1(\varphi) - P_2(\varphi) + Q_1(\varphi) - Q_2(\varphi) + P_3(\varphi)],\\
\phi^{--}(t_1,t_2) &= \frac{1}{4}[\varphi(t_1,t_2) - P_1(\varphi) - P_2(\varphi) - Q_1(\varphi) - Q_2(\varphi) + P_3(\varphi)],
\end{aligned} \tag{22}$$

*where $(t_1,t_2) \in \partial\Omega_1 \times \partial\Omega_2$, $P_3(\varphi) = 4\phi(t_1,t_2)$.*

## 4. The Boundary Value Problem for Bihypermonogenic Functions

In this section, we consider the boundary value problem.

*Definition 9.* Let $\Omega_1 \times \Omega_2$ and $H(\partial\Omega_1 \times \partial\Omega_2, \beta)$ be as before. We want to find a bihypermonogenic function $\phi(x,y)$ defined in $\mathbb{R}_+^{m+1} \times \mathbb{R}_+^{k+1}/\partial\Omega_1 \times \partial\Omega_2$, which is continuous to $\partial\Omega_1 \times \partial\Omega_2$ and $\phi^{+-}(x,\infty) = \phi^{-+}(\infty,y) = \phi^{--}(\infty,\infty) = 0$ and satisfies the nonlinear boundary condition

$$\begin{aligned}
&A(t_1,t_2)\phi^{++}(t_1,t_2) + B(t_1,t_2)\phi^{+-}(t_1,t_2)\\
&\quad + C(t_1,t_2)\phi^{-+}(t_1,t_2) + D(t_1,t_2)\phi^{--}(t_1,t_2)\\
&\quad = g(t_1,t_2) f(t_1,t_2,\phi^{++}(t_1,t_2),\phi^{+-}(t_1,t_2),\phi^{-+}(t_1,t_2),\\
&\qquad \phi^{--}(t_1,t_2)),
\end{aligned} \tag{23}$$

in which $A(t_1,t_2), B(t_1,t_2), C(t_1,t_2), D(t_1,t_2), g(t_1,t_2) \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$ and $f$ are known functions. The above boundary value problem is called Problem R.

From Theorem 8, we can transform the boundary condition of Problem R into an integral equation

$$F\varphi = \varphi, \tag{24}$$

where

$$\begin{aligned} &F\varphi \\ &\quad = (A+B)\left(\varphi + P_1\varphi + P_2\varphi + Q_1\varphi + Q_2\varphi + P_3\varphi\right) \\ &\qquad + (C+D)\left(-\varphi + P_1\varphi - P_2\varphi + Q_1\varphi - Q_2\varphi + P_3\varphi\right) \\ &\qquad + (B+D)\left(2\varphi - 2P_1\varphi - 2Q_1\varphi\right) + (1-4B)\varphi - 4gf. \end{aligned} \tag{25}$$

**Theorem 10** (see [9]). *Let $\Omega, \partial\Omega \subset R_+^{l+1}$, and let $H(\partial\Omega, \beta)$ be the set of Hölder continuous functions on $\partial\Omega$ with the index $\beta$. For $\varphi \in H(\partial\Omega, \beta)$ and*

$$\theta\varphi = \phi\varphi - \frac{\varphi}{2},$$

$$\begin{aligned} \phi\varphi = \frac{2^{l-1}x_l^{l-1}}{\omega_{l+1}} \Big[ \int_{\partial\Omega} & E_l(t,x)\, d\sigma_0(t)\, \varphi(t) \\ & - F_l(t,x)\, \widehat{d\sigma_0(t)}\widehat{\varphi(t)} \Big], \end{aligned} \tag{26}$$

*where $E_l(t,x), F_l(t,x)$ are as before, then $\theta\varphi$ is a hypermonogenic function with*

$$\|\theta\varphi\|_\beta \le J_1 \|\varphi\|_\beta, \tag{27}$$

*where $J_1$ is a constant independent of $\varphi$.*

**Lemma 11** (see [6]). *Let $E_m(u,x)$, $E_m(u,t_1)$ be the same as in Theorem 2. If $u \in \overline{\Omega_1}$, $x \in \overline{\Omega_2}$, $t_1 \in \overline{\Omega_2}$, then there exists a constant $M$ such that*

$$\begin{aligned} &\left|E_m(u,x) - E_m(u,t_1)\right| \\ &\quad \le M\left[\sum_{i=0}^{m-1}\left|\frac{u-t_1}{u-x}\right|^i \left|\frac{x-t_1}{u-x}\right| + \left|x-t_1\right|\right]\left|u-t_1\right|^{-m}. \end{aligned} \tag{28}$$

**Lemma 12.** *If $\varphi(t_1,t_2) \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$, then*

$$\begin{aligned} &\left\|\varphi \pm \left(P_i\varphi + Q_i\varphi\right)\right\|_\beta \le J_2\|\varphi\|_\beta, \\ &\left\|P_i\varphi + Q_i\varphi\right\|_\beta \le J_2\|\varphi\|_\beta, \quad i = 1,2, \end{aligned} \tag{29}$$

*where $J_2$ is a positive constant.*

*Proof.* Using the following equations:

$$\begin{aligned} P_1\varphi + Q_1\varphi &= 2\lambda_1 \int_{\partial\Omega_1} E_m(u,t_1)\, d\sigma_m(u)\, \varphi(\mu,t_2) \\ &\quad - 2\lambda_1 \int_{\partial\Omega_1} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\widehat{\varphi(u,t_2)}, \end{aligned}$$

$$\begin{aligned} P_2\varphi + Q_2\varphi &= 2\lambda_2 \int_{\partial\Omega_2} \varphi(t_1,v)\, d\sigma_k(v)\, E_k(v,t_2) \\ &\quad - 2\lambda_2 \int_{\partial\Omega_2} \widehat{\varphi(t_1,v)}\widehat{d\sigma_k(v)} F_k(v,t_2) \end{aligned} \tag{30}$$

and Theorem 10, we can obtain the result. □

**Theorem 13.** *Suppose the boundaries $\partial\Omega_1$, $\partial\Omega_2$ of $\Omega_1$, $\Omega_2$ be differentiable, oriented, compact Liapunov surfaces. If $\varphi(t_1,t_2) \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$, then*

$$\left\|\left(P_2\varphi + Q_2\varphi\right) \pm P_3\varphi\right\|_\beta \le J_3\|\varphi\|_\beta, \tag{31}$$

*where $J_3$ is a positive constant which is independent of $\varphi$.*

*Proof.* From (20), it follows that

$$P_2\varphi + Q_2\varphi - P_3\varphi = \varphi - 4\sum_{i=1}^{4} X_i(t_1,t_2) - \left(P_1\varphi + Q_1\varphi\right). \tag{32}$$

Moreover, based on Lemma 12 we only need to prove $\|\sum_{i=1}^4 X_i(t_1,t_2)\|_\beta \le J_4\|\varphi\|_\beta$. It is easy to prove $|\sum_{i=1}^4 X_i(t_1,t_2)| \le B_1\|\varphi\|_\beta$. We rewrite $\psi_i(u,v)$ as $\psi_i^0(t_1,t_2)$, $(i = 1,2,3,4)$. Now we consider $H(\sum_{i=1}^4 X_i(t_1,t_2), \partial\Omega_1 \times \partial\Omega_2, \beta)$ and write $\delta = |(t_1,t_2) - (t_1',t_2')| = \sqrt{\delta_1^2 + \delta_2^2}$ for any $(t_1,t_2)$, $(t_1',t_2') \in \partial\Omega_1 \times \partial\Omega_2$ and denote by $\rho_{01}$, $\rho_{02}$, $\rho_{01}'$, $\rho_{02}'$ the projections of $|\mu - t_1|$, $|v - t_2|$, $|\mu - t_1'|$, $|v - t_2'|$ on the tangent plane of $t_1$, $t_2$, $t_1'$, $t_2'$, respectively. Moreover we construct spheres $O_i(t_i, 3\delta_i)$ with the center at $t_i$ and radius $3\delta_i$, where $6\delta_i < d_i$, $\delta_i < 1$, $i = 1,2$, where $d_i$ is a constant as in [5]. Denote by $\partial\Omega_{i1}$, $\partial\Omega_{i2}$ the part of $\partial\Omega_i$ lying inside the sphere $O_i$ and its surplus part, respectively, and set

$$\begin{aligned} R\left(\partial\Omega_1 \times \partial\Omega_2\right) &= \sum_{i=1}^4 X_i(t_1,t_2) - \sum_{i=1}^4 X_i\left(t_1',t_2'\right) \\ &= \sum_{i=1}^4 \overline{X_i}\left(\partial\Omega_1 \times \partial\Omega_2\right) - \sum_{i=1}^4 \overline{\overline{X_i}}\left(\partial\Omega_1 \times \partial\Omega_2\right). \end{aligned} \tag{33}$$

From

$$\begin{aligned} &\left|\psi_i^0(t_1,t_2)\right| \le M\|\varphi\|_\beta \left|u - t_1\right|^\beta, \\ &\left|\psi_i^0(t_1,t_2)\right| \le M\|\varphi\|_\beta \left|v - t_2\right|^\beta, \\ &\left|\psi_i^0\left(t_1',t_2'\right)\right| \le M\|\varphi\|_\beta \left|u - t_1'\right|^\beta, \\ &\left|\psi_i^0\left(t_1',t_2'\right)\right| \le M\|\varphi\|_\beta \left|v - t_2'\right|^\beta, \\ &\qquad i = 1,2,3, \\ &\left|\psi_4^0(t_1,t_2)\right| \le M\|\varphi\|_\beta, \\ &\left|\psi_4^0\left(t_1',t_2'\right)\right| \le M\|\varphi\|_\beta, \end{aligned} \tag{34}$$

we obtain that

$$\left|E_m(u,t_1)\,d\sigma_m(u)\,\psi_1^0(t_1,t_2)\,d\sigma_k(v)\,E_k(v,t_2)\right| \le M\|\varphi\|_\beta \rho_{01}^{(\beta/2)-1} d\rho_{01}\rho_{02}^{(\beta/2)-1} d\rho_{02}, \tag{35}$$

$$\left|E_m(u,t_1)\,d\sigma_m(u)\,\psi_2^0(t_1,t_2)\,\widetilde{d\sigma_k(v)}F_k(v,t_2)\right| \le M\|\varphi\|_\beta \rho_{01}^{\beta-1} d\rho_{01} d\rho_{02}, \tag{36}$$

$$\left|F_m(u,t_1)\,\widehat{d\sigma_m(u)}\psi_3^0(t_1,t_2)\,d\sigma_k(v)\,E_k(v,t_2)\right| \le M\|\varphi\|_\beta d\rho_{01}\rho_{02}^{\beta-1} d\rho_{02}, \tag{37}$$

$$\left|F_m(u,t_1)\,\widehat{d\sigma_m(u)}\psi_4^0(t_1,t_2)\,\widetilde{d\sigma_k(v)}F_k(v,t_2)\right| \le M\|\varphi\|_\beta d\rho_{01} d\rho_{02}. \tag{38}$$

Thus we have

$$\begin{aligned}&\left|R(\partial\Omega_{11}\times\partial\Omega_{21})\right| \\ &\quad\le \sum_{i=1}^{4}\left|\overline{X_i}(\partial\Omega_{11}\times\partial\Omega_{21})\right| + \sum_{i=1}^{4}\left|\overline{\overline{X_i}}(\partial\Omega_{11}\times\partial\Omega_{21})\right| \\ &\quad\le B_2\|\varphi\|_\beta\left|(t_1,t_2)-(t_1',t_2')\right|^\beta.\end{aligned} \tag{39}$$

Noting that $|v-t_2'|\ge 2\delta_2$, $|v-t_2|\ge 3\delta_2>0$ on $\partial\Omega_{22}$, we have

$$\left|R(\partial\Omega_{11}\times\partial\Omega_{22})\right| \le B_3\|\varphi\|_\beta\left|(t_1,t_2)-(t_1',t_2')\right|^\beta. \tag{40}$$

Similarly, we can obtain $|R(\partial\Omega_{12}\times\partial\Omega_{21})| \le B_4\|\varphi\|_\beta|(t_1,t_2)-(t_1',t_2')|^\beta$.

Next we want to prove

$$\left|R(\partial\Omega_{12}\times\partial\Omega_{22})\right| \le C_4\|\varphi\|_\beta\left|(t_1,t_2)-(t_1',t_2')\right|^\beta. \tag{41}$$

According to (33), we have

$$\begin{aligned}&\overline{X_1}(\partial\Omega_{12}\times\partial\Omega_{22}) - \overline{\overline{X_1}}(\partial\Omega_{12}\times\partial\Omega_{22}) \\ &\quad= \lambda\int_{\partial\Omega_{12}\times\partial\Omega_{22}}\left[E_m(u,t_1)-E_m(m,t_1')\right]d\sigma_m(u)\,\psi_1^0(t_1,t_2) \\ &\qquad\qquad\times d\sigma_k(v)\,E_k(v,t_2) \\ &\qquad+\lambda\int_{\partial\Omega_{12}\times\partial\Omega_{22}} E_m(u,t_1')\,d\sigma_m(u)\,\psi_1^0(t_1,t_2)\,d\sigma_k(v) \\ &\qquad\qquad\times\left[E_k(v,t_2)-E_k(v,t_2')\right] \\ &\qquad+\lambda\int_{\partial\Omega_{12}\times\partial\Omega_{22}} E_m(u,t_1')\,d\sigma_m(u) \\ &\qquad\qquad\times\left[\psi_1^0(t_1,t_2)-\psi_1^0(t_1',t_2')\right] \\ &\qquad\qquad\times d\sigma_k(v)\,E_k(v,t_2') \\ &\quad= E_1+E_2+E_3.\end{aligned} \tag{42}$$

Similarly, we can deal with

$$\begin{aligned}\overline{X_2}(\partial\Omega_{12}\times\partial\Omega_{22}) - \overline{\overline{X_2}}(\partial\Omega_{12}\times\partial\Omega_{22}) &= F_1+F_2+F_3, \\ \overline{X_3}(\partial\Omega_{12}\times\partial\Omega_{22}) - \overline{\overline{X_3}}(\partial\Omega_{12}\times\partial\Omega_{22}) &= G_1+G_2+G_3, \\ \overline{X_4}(\partial\Omega_{12}\times\partial\Omega_{22}) - \overline{\overline{X_4}}(\partial\Omega_{12}\times\partial\Omega_{22}) &= H_1+H_2+H_3.\end{aligned} \tag{43}$$

From Lemma 11 and (35) and by $|v-t_2| \ge 3\delta_2 > 0$, $|v-t_2'|\ge 2\delta_2$, we obtain $|E_2| \le C_1\|\varphi\|_\beta|(t_1,t_2)-(t_1',t_2')|^\beta$. Similarly, we can get the inequality estimation for $E_1$ and $G_2$. By (36) and (37), $|v-t_2|\ge 3\delta_2>0$, $|v-t_2'|\ge 2\delta_2$, we have $|F_2|\le C_2\|\varphi\|_\beta|(t_1,t_2)-(t_1',t_2')|^\beta$. Similarly, we can obtain the inequality estimation for $F_1$, $G_1$, $H_1$, and $H_2$.

Since

$$\begin{aligned}&E_3+F_3+G_3+H_3 \\ &\quad= \frac{1}{2}\lambda_1\int_{\partial\Omega_{12}} E_m(u,t_1')\,d\sigma_m(u)\left[\varphi(u,t_2')-\varphi(u,t_2)\right] \\ &\qquad-\frac{1}{2}\lambda_1\int_{\partial\Omega_{12}} F_m(u,t_1')\,\widehat{d\sigma_m(u)}\left[\widehat{\varphi(u,t_2')}-\widehat{\varphi(u,t_2)}\right] \\ &\qquad+\frac{1}{2}\lambda_2\int_{\partial\Omega_{22}}\left[\varphi(t_1',v)-\varphi(t_1,v)\right]d\sigma_k(v)\,E_k(v,t_2') \\ &\qquad-\frac{1}{2}\lambda_2\int_{\partial\Omega_{22}}\left[\widetilde{\varphi(t_1',v)}-\widetilde{\varphi(t_1,v)}\right]\widetilde{d\sigma_k(v)}F_k(v,t_2') \\ &\qquad+\frac{1}{4}\varphi(t_1,t_2)-\frac{1}{4}\varphi(t_1',t_2'),\end{aligned} \tag{44}$$

by (35)–(38), we have $|E_3+F_3+G_3+H_3| \le C_3\|\varphi\|_\beta|(t_1,t_2)-(t_1',t_2')|^\beta$.

Summarizing the above discussion shows that

$$\left|R(\partial\Omega_{12}\times\partial\Omega_{22})\right| \le C_4\|\varphi\|_\beta\left|(t_1,t_2)-(t_1',t_2')\right|^\beta. \tag{45}$$

Thus we infer

$$\left|R(\partial\Omega_1\times\partial\Omega_2)\right| \le C_5\|\varphi\|_\beta\left|(t_1,t_2)-(t_1',t_2')\right|^\beta. \tag{46}$$

Hence

$$\left\|\sum_{i=1}^{4}X_i(t_1,t_2)\right\|_\beta \le J_4\|\varphi\|_\beta. \tag{47}$$

This completes the proof. □

**Corollary 14.** *If $\varphi(t_1,t_2) \in H(\partial\Omega_1 \times \partial\Omega_2,\beta)$, then*

$$\begin{aligned} &\|\phi^{++}(t_1,t_2)\|_\beta \le J_5\|\varphi\|_\beta, \qquad \|\phi^{+-}(t_1,t_2)\|_\beta \le J_5\|\varphi\|_\beta,\\ &\|\phi^{-+}(t_1,t_2)\|_\beta \le J_5\|\varphi\|_\beta, \qquad \|\phi^{--}(t_1,t_2)\|_\beta \le J_5\|\varphi\|_\beta. \end{aligned} \tag{48}$$

**Theorem 15.** *Let $A(x,y), B(x,y), C(x,y), D(x,y), g(x,y) \in H(\partial\Omega_1 \times \partial\Omega_2,\beta)$; then the function $f(t_1,t_2,\phi^1,\phi^2,\phi^3,\phi^4)$ is a Hölder continuous function for $(t_1,t_2) \in \partial\Omega_1 \times \partial\Omega_2$ and satisfies the Lipschitz-condition for $\phi^1,\phi^2,\phi^3,\phi^4$ and any $(t_1,t_2)$, namely,*

$$\begin{aligned} &\left|f\left(t_{11},t_{21},\phi_1^1,\phi_1^2,\phi_1^3,\phi_1^4\right) - f\left(t_{12},t_{22},\phi_2^1,\phi_2^2,\phi_2^3,\phi_2^4\right)\right|\\ &\quad \le J_6\left|(t_{11},t_{21}) - (t_{12},t_{22})\right|^\beta + J_7\left|\phi_1^1 - \phi_2^1\right|\\ &\qquad + J_8\left|\phi_1^2 - \phi_2^2\right| + J_9\left|\phi_1^3 - \phi_2^3\right| + J_{10}\left|\phi_1^4 - \phi_2^4\right|, \end{aligned} \tag{49}$$

*where $J_i$ $(i = 6,\dots,10)$ is a positive constant and has nothing to do with $t_{1j}, t_{2j}, \phi_j^1,\dots,\phi_j^4$, $j = 1,2$. If $f(0,0,0,0,0,0) = 0$, $\|A+B\|_\beta < \varepsilon$, $\|C+D\|_\beta < \varepsilon$, $\|D+B\|_\beta < \varepsilon$, $\|1-4B\|_\beta < \varepsilon$, $0<\varepsilon<1$, $0<\mu=\varepsilon J_0(2J_5+J_2+1)<1$, $\|g\|_\beta<\delta$, $0<\delta<M(1-\mu)/4\cdot J_0(J_{13}+J_{14}M)$, then the problem R has at least one solution, where $M(\|\varphi\|_\beta < M)$, $J_{13}, J_{14}$ are both positive constants satisfying $\|f\|_\beta \le J_{13} + J_{14}\|\varphi\|_\beta$.*

*Proof.* Suppose that $T = \{\varphi \mid \varphi \in H(\partial\Omega_1 \times \partial\Omega_2,\beta), \|\varphi\|_\beta < M\}$ be denoted a subset of $C(\partial\Omega_1 \times \partial\Omega_2)$. From Theorem 13, Corollary 14, and (38), we obtain $C(f,\partial\Omega_1 \times \partial\Omega_2) \le J_{11} + J_{12}\|\varphi\|_\beta$. Similarly, we can get $\|f\|_\beta \le J_{13} + J_{14}\|\varphi\|_\beta$. Hence, by (12) and $F\varphi = \varphi$, $\|F\varphi\|_\beta \le M$ is derived. This shows that the operator $F$ is the mapping of $T \to T$.

Next we prove the $F$ is a continuous mapping.

Suppose that the sequence of functions $\{\varphi_n\} \in T$ uniformly converges to a function $\varphi(t_1,t_2), (t_1,t_2) \in \partial\Omega_1 \times \partial\Omega_2$; thus for arbitrary $\varepsilon > 0$ and if $n$ is large enough, then $|(P_i+Q_i)\varphi_n - (P_i+Q_i)\varphi| < \varepsilon$, $(i=1,2)$.

Now we consider $P_3\varphi_n - P_3\varphi$ by

$$\begin{aligned} &P_3\varphi_n - P_3\varphi\\ &\quad = \sum_{i=1}^{2} A_i(\partial\Omega_1 \times \partial\Omega_2) + \sum_{i=1}^{2} B_i(\partial\Omega_1 \times \partial\Omega_2)\\ &\qquad + \sum_{i=1}^{2} C_i(\partial\Omega_1 \times \partial\Omega_2) + \sum_{i=1}^{2} D_i(\partial\Omega_1 \times \partial\Omega_2) + \sum_{i=1}^{3} E_i, \end{aligned} \tag{50}$$

where

$$\begin{aligned} &A_1(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = 4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\, d\sigma_m(u)\, \psi_{1n}(u,v)\, d\sigma_k(v) \times E_k(v,t_2),\\ &A_2(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = -4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\, d\sigma_m(u)\, \psi_1(u,v)\, d\sigma_k(v) \times E_k(v,t_2),\\ &B_1(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = -4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\, d\sigma_m(u)\, \psi_{2n}(u,v)\, \widehat{d\sigma_k(v)} \times F_k(v,t_2),\\ &B_2(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = 4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} E_m(u,t_1)\, d\sigma_m(u)\, \psi_2(u,v)\, \widehat{d\sigma_k(v)} \times F_k(v,t_2)\\ &C_1(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = -4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\psi_{3n}(u,v)\, d\sigma_k(v) \times E_k(v,t_2),\\ &C_2(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = 4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\psi_3(u,v)\, d\sigma_k(v) \times E_k(v,t_2),\\ &D_1(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = 4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\psi_{4n}\widehat{d\sigma_k(v)}F_k(v,t_2),\\ &D_2(\partial\Omega_1 \times \partial\Omega_2)\\ &\quad = -4\lambda \int_{\partial\Omega_1\times\partial\Omega_2} F_m(u,t_1)\, \widehat{d\sigma_m(u)}\psi_4\widehat{d\sigma_k(v)}F_k(v,t_2),\\ &E_1 = \varphi(t_1,t_2) - \varphi_n(t_1,t_2),\\ &E_2 = (P_1+Q_1)\varphi_n - (P_1+Q_1)\varphi,\\ &E_3 = (P_2+Q_2)\varphi_n - (P_2+Q_2)\varphi. \end{aligned} \tag{51}$$

Suppose $6\delta < d_i$, $i = 1, 2, \delta > 0, O((t_1, t_2), 3\delta)$ is the $3\delta$-neighborhood of $(t_1, t_2)$ with the center at point $(t_1, t_2) \in \partial\Omega_1 \times \partial\Omega_2$ and the radius $3\delta$, $\partial\Omega_{i1} \times \partial\Omega_{i2}$ is as above; then

$$\begin{aligned} &\square_j(\partial\Omega_1 \times \partial\Omega_2) \\ &\quad = \square_j(\partial\Omega_{11} \times \partial\Omega_{21}) + \square_j(\partial\Omega_{11} \times \partial\Omega_{22}) \\ &\qquad + \square_j(\partial\Omega_{12} \times \partial\Omega_{21}) + \square_j(\partial\Omega_{12} \times \partial\Omega_{22}), \\ &\qquad\qquad (\square = A, B, C, D,\ j = 1, 2). \end{aligned} \tag{52}$$

By (35), we can obtain that

$$\begin{aligned} \left|A_1(\partial\Omega_{11} \times \partial\Omega_{21})\right| &\le J_{15}\delta^{\beta} \le J_{16}\delta^{\beta/2}, \\ \left|A_1(\partial\Omega_{12} \times \partial\Omega_{21})\right| &\le J_{17}\delta^{\beta/2}, \\ \left|A_1(\partial\Omega_{11} \times \partial\Omega_{22})\right| &\le J_{18}\delta^{\beta/2}. \end{aligned} \tag{53}$$

Similarly, we can get the inequality estimations for $A_2(\partial\Omega_{11} \times \partial\Omega_{21})$, $A_2(\partial\Omega_{12} \times \partial\Omega_{21})$, and $A_2(\partial\Omega_{11} \times \partial\Omega_{22})$. By (35), (36), and (37), we can obtain the similar inequality estimations for $B_i(\partial\Omega_{11}\times\partial\Omega_{21})$, $B_i(\partial\Omega_{12}\times\partial\Omega_{21})$, $B_i(\partial\Omega_{11}\times\partial\Omega_{22})$, $C_i(\partial\Omega_{11} \times \partial\Omega_{21})$, $C_i(\partial\Omega_{12} \times \partial\Omega_{21})$, $C_i(\partial\Omega_{11} \times \partial\Omega_{22})$, $D_i(\partial\Omega_{11} \times \partial\Omega_{21})$, $D_i(\partial\Omega_{12} \times \partial\Omega_{21})$, and $D_i(\partial\Omega_{11} \times \partial\Omega_{22})$, $i = 1, 2$, respectively.

From

$$\begin{aligned} &A_1(\partial\Omega_{12} \times \partial\Omega_{22}) + A_2(\partial\Omega_{12} \times \partial\Omega_{22}) \\ &\quad = 4\lambda \int_{\partial\Omega_{12}\times\partial\Omega_{22}} E_m(u, t_1)\, d\sigma_m(u)\, W_1(u, v)\, d\sigma_k(v) \\ &\qquad \times E_k(v, t_2), \end{aligned} \tag{54}$$

where

$$\begin{aligned} &W_1(u, v) \\ &\quad = \{[\varphi_n(u, v) - \varphi(u, v)] - [\varphi_n(t_1, v) - \varphi(t_1, v)]\} \\ &\qquad + \{[\varphi_n(t_1, t_2) - \varphi(t_1, t_2)] - [\ \varphi_n(u, t_2) - \varphi(u, t_2)\}, \end{aligned} \tag{55}$$

since $|W_1(u, v)| \le 2\|\varphi_n - \varphi\|_{\beta}|u - t_1|^{\beta/2}|v - t_2|^{\beta/2}$ and from (35), we have

$$\left|A_1(\partial\Omega_{12} \times \partial\Omega_{22}) + A_2(\partial\Omega_{12} \times \partial\Omega_{22})\right| \le J_{19}\|\varphi_n - \varphi\|_{\beta}. \tag{56}$$

Similarly, we can get the inequality estimations for $B_1(\partial\Omega_{12} \times \partial\Omega_{22}) + B_2(\partial\Omega_{12}\times\partial\Omega_{22})$,$C_1(\partial\Omega_{12}\times\partial\Omega_{22}) + C_2(\partial\Omega_{12}\times\partial\Omega_{22})$.

From

$$\begin{aligned} &D_1(\partial\Omega_{12} \times \partial\Omega_{22}) + D_2(\partial\Omega_{12} \times \partial\Omega_{22}) \\ &\quad = 4\lambda \int_{\partial\Omega_{12}\times\partial\Omega_{22}} F_m(u, t_2)\, \widehat{d\sigma_m(u)} W_4(u, v)\, \widetilde{d\sigma_k(v)} F_k(v, t_2), \end{aligned} \tag{57}$$

where

$$\begin{aligned} &W_4(u, v) \\ &\quad = \left\{\left[\widetilde{\varphi_n(u, v)} - \widetilde{\varphi(u, v)}\right] - \left[\widetilde{\varphi_n(t_1, v)} - \widetilde{\varphi(t_1, v)}\right]\right\} \\ &\qquad + \left\{[\varphi_n(t_1, t_2) - \varphi(t_1, t_2)] - \left[\widehat{\varphi_n(u, t_2)} - \widehat{\varphi(u, t_2)}\right]\right\}, \end{aligned} \tag{58}$$

since $|W_4(u, v)| \le 4\|\varphi_n - \varphi\|_{\beta}$ and from (38), we obtain that

$$\left|D_1(\partial\Omega_{12} \times \partial\Omega_{22}) + D_2(\partial\Omega_{12} \times \partial\Omega_{22})\right| \le J_{21}\|\varphi_n - \varphi\|_{\beta}. \tag{59}$$

Summarizing the above discussion, we conclude $|P_3\varphi_n - P_3\varphi| \le J_{22}(\varepsilon + \delta^{\beta/2} + \|\varphi_n - \varphi\|_{\beta})$. Then for arbitrary $\varepsilon > 0$, we first choose a sufficiently small number $\delta$ and next select a sufficiently large positive integer $n$; we have

$$\left|P_3\varphi_n - P_3\varphi\right| < G\varepsilon, \tag{60}$$

where $G$ is a positive constant.

Finally, we can choose $n$ large enough such that $|F\varphi_n - F\varphi| < W\varepsilon$ ($W$ is a positive constant). Hence we can obtain $F : T \to T$ is a continuous mapping. According to Ascoli-Arzela Theorem, $T$ is a compact set in the space $C(\partial\Omega_1\times\partial\Omega_2)$. Based on the Schauder fixed point principle, there exists a function $\varphi \in H(\partial\Omega_1 \times \partial\Omega_2, \beta)$ satisfying the equation $F\varphi = \varphi$. □

**Corollary 16.** *If $f \equiv 1$ in Theorem 15, then the Problem R has the unique solution.*

*Proof.* This corollary is not difficult to verify by the contraction mapping principle when $f \equiv 1$ in Theorem 15. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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