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Research Article

Traveling Wave Solutions of the Kadomtsev-Petviashvili-Benjamin-Bona-Mahony Equation

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We use bifurcation method of dynamical systems to study exact traveling wave solutions of a nonlinear evolution equation. We obtain exact explicit expressions of bell-shaped solitary wave solutions involving more free parameters, and some existing results are corrected and improved. Also, we get some new exact periodic wave solutions in parameter forms of the Jacobian elliptic function. Further, we find that the bell-shaped waves are limits of the periodic waves in some sense. The results imply that we can deduce bell-shaped waves from periodic waves for some nonlinear evolution equations.

1. Introduction

The Benjamin-Bona-Mahony (BBM) equation [1]

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1}$$

was proposed as the model for propagation of long waves where nonlinear dispersion is incorporated. The Kadomtsev-Petviashvili (KP) equation [2]

$$(u_t + auu_x + u_{xxx})_x + u_{yy} = 0 \tag{2}$$

was given as the generalization of the KdV equation. In addition, both BBM and KdV equations can be used to describe long wavelength in liquids, fluids, and so forth. Combining the two equations, the Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation

$$\left(u_t + u_x - a\left(u^2\right)_x - bu_{xxt}\right)_x + ku_{yy} = 0 \tag{3}$$

was presented in [3] for further study. Some methods are developed and applied to find exact solutions of nonlinear evolution equations because exact solutions play an important role in the comprehension of nonlinear phenomena. For instance, extended tanh method, extended mapping method with symbol computation, and bifurcation method of dynamical systems are employed to study (3) [4–6], and some solitary wave solutions and triangle periodic wave solutions were obtained.

However, there is no method which can be applied to all nonlinear evolution equations. The research on the solutions of the KP-BBM equation now appears insufficient. Further studies are necessary for the traveling wave solutions of the KP-BBM equation. The purpose of this paper is to apply the bifurcation method [7–10] of dynamical systems to continue to seek traveling waves of (3). Firstly, we obtain bell-shaped solitary wave solutions involving more free parameters, and some results in [6] are corrected and improved. Then, we get some new periodic wave solutions in parameter forms of Jacobian elliptic function, and numerical simulation verifies the validity of these periodic solutions. The periodic wave solutions obtained in this paper are different from those in [5]. Furthermore, we find an interesting relationship between the bell-shaped waves and periodic waves; that is, the bell-shaped waves are limits of the periodic waves in forms of Jacobian elliptic function as modulus approaches 1.

This paper is organized as follows. First, we draw the bifurcation phase portraits of planar system according to the KP-BBM equation in Section 2. Second, bell-shaped solitary wave solutions to the equation under consideration are presented in Section 3. Third, periodic solitary waves are given in the forms of Jacobian elliptic function and numerical simulation is done. Finally, the relationship between the bell-shaped solitary waves and periodic waves is proved in Section 4.

2. Bifurcation Phase Portraits of System (6)

Suppose (3) possesses traveling wave solutions in the form $u(x,t)=\varphi(\xi)$, $\xi = x + ry - ct$, where c is the wave speed and r is a real constant. Substituting $u(x,t)=\varphi(\xi)$, $\xi = x+ry-ct$ into (3) admits to the following ODE:

$$\left(1+kr^2-c\right)\varphi'' - a\left(\varphi^2\right)'' + bc\varphi^{(4)} = 0, \tag{4}$$

where the derivative is for variable ξ. Integrating (4) twice with respect to ξ and letting the first integral constant take value zero, it follows that

$$\left(1+kr^2-c\right)\varphi - a\varphi^2 + bc\varphi'' = g, \tag{5}$$

where g is the second integral constant.

Equation (5) is equivalent to the following two-dimensional system:

$$\frac{d\varphi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{a\varphi^2 - \left(1+kr^2-c\right)\varphi + g}{bc}. \tag{6}$$

It is obvious that system (6) has the first integral

$$H(\varphi, y) = \frac{1}{2}y^2 - \frac{a}{3bc}\varphi^3 + \frac{1+kr^2-c}{2bc}\varphi^2 - \frac{g}{bc}\varphi = h, \tag{7}$$

where h is the constant of integration.

Define $\Delta = (1+kr^2-c)^2 - 4ag$. When $\Delta > 0$, there are two equilibrium points $(\varphi_1, 0)$ and $(\varphi_2, 0)$ of (6) on φ-axis, where $\varphi_1 = ((1+kr^2-c)-\sqrt{\Delta})/2a$, $\varphi_2 = ((1+kr^2-c)+\sqrt{\Delta})/2a$. The Hamiltonian H of $(\varphi_1,0)$ and $(\varphi_2,0)$ is denoted by $h_1 = H(\varphi_1, 0)$ and $h_2 = H(\varphi_2, 0)$.

In the case of $1+kr^2-c<0$ and $1+kr^2-c>0$, the bifurcation phase portraits of system (6) see Figures 1 and 2 in [6], respectively, in which there are some homoclinic and periodic orbits of system (6). For our purpose, we redraw the homolinic and periodic orbits in this paper(see Figures 1–3).

3. Exact Explicit Expressions of Solitary Wave Solutions

In this section, we discuss bell-shaped wave solutions under $g=0$ and $g \neq 0$, respectively.

3.1. The Case $g=0$. System (6) can be rewritten as

$$\frac{d\varphi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{a\varphi^2 - \left(1+kr^2-c\right)\varphi}{bc}. \tag{8}$$

The first integral of (8) is

$$H(\varphi, y) = \frac{1}{2}y^2 - \frac{a}{3bc}\varphi^3 + \frac{1+kr^2-c}{2bc}\varphi^2 = h. \tag{9}$$

When $(1+kr^2-c)/bc < 0$, there are two homoclinic orbits Γ_1 and Γ_2 (see Figures 1(a) and 1(b)). In φ-y plane, Γ_1 and Γ_2 can be described by

$$y^2 = \frac{2a}{3bc}\varphi^3 - \frac{1+kr^2-c}{bc}\varphi^2, \qquad \varphi \in (0, \varphi^*) \quad \text{or} \quad \varphi \in (\varphi^*, 0), \tag{10}$$

where $\varphi^* = 3(1+kr^2-c)/2a$. That is,

$$y = \pm\sqrt{\frac{2a}{3bc}\varphi^3 - \frac{1+kr^2-c}{bc}\varphi^2}. \tag{11}$$

Substituting (11) into $d\varphi/d\xi = y$ and integrating along homoclinc orbits Γ_1 and Γ_2, respectively, we get

$$\int_{\varphi}^{\varphi^*} \frac{ds}{\sqrt{(2a/3bc)s^3 - ((1+kr^2-c)/bc)s^2}} = \pm|\xi|. \tag{12}$$

Completing the above integration, it follows that

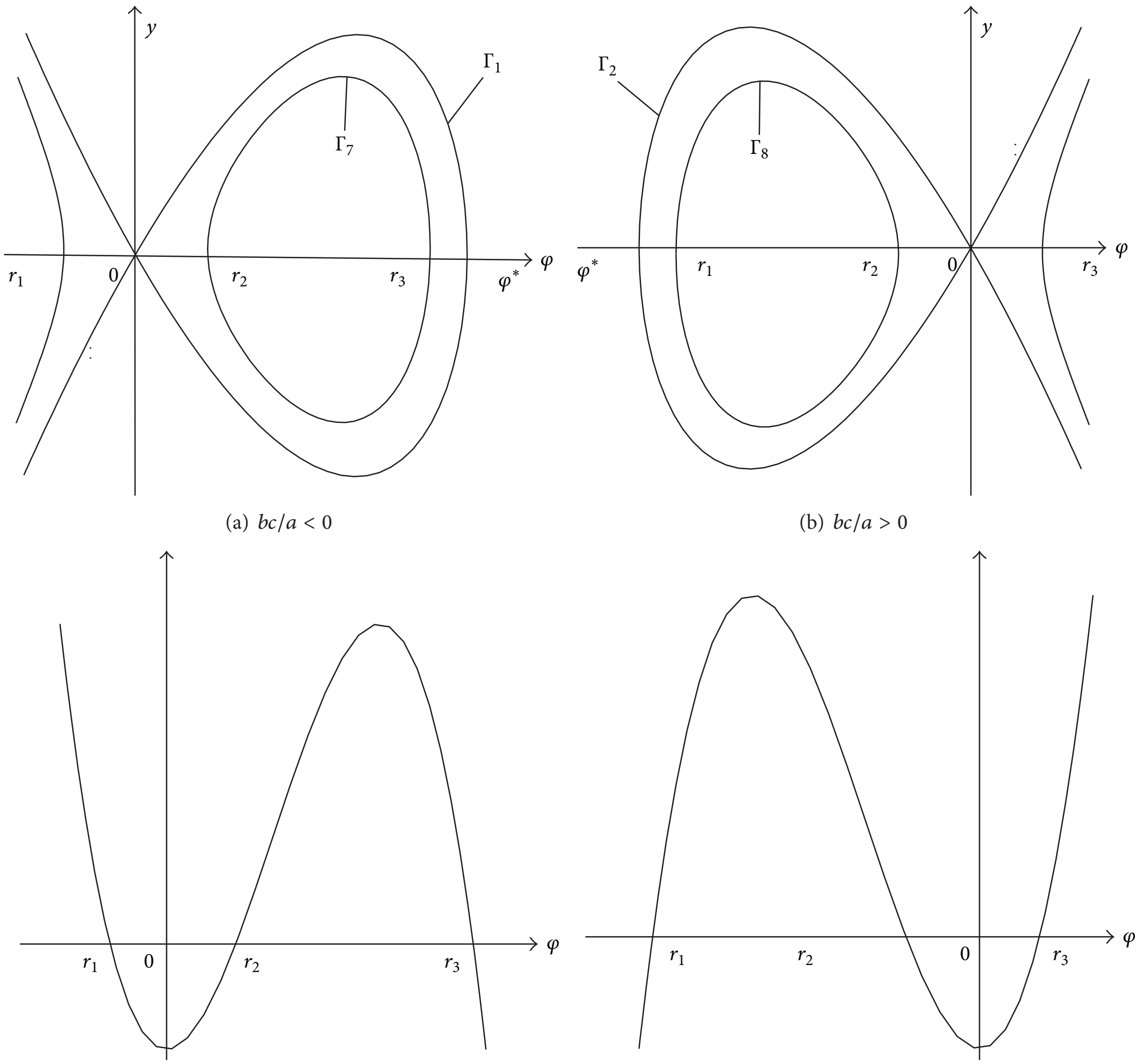
$$u_1(x,y,t) = \frac{3\left(1+kr^2-c\right)}{a\left[1+\cosh\left(\sqrt{-((1+kr^2-c)/bc)}\,(x+ry-ct)\right)\right]}. \tag{13}$$

Remark 1. $u_1(x,y,t)$ is a bright soliton solution when $bc/a < 0$ and a dark soliton solution when $bc/a > 0$. If the real number r in (13) takes value 1, then solution (13) is the same to solution (1.2) in [6]. Solution (1.1) in [6] is not a real solution of the KP-BBM equation; it is obvious that solution (1.1) tends to infinite as $\xi \to 0$, and it does not satisfy the KP-BBM equation (3).

When $(1+kr^2+c)/bc > 0$, there are two homoclinic orbits Γ_3 and Γ_4 (see Figures 2(a) and 2(b)). Similarly solitary wave solutions according to Γ_3 and Γ_4 are obtained as follows:

$$\begin{aligned} u_2(x,y,t) &= \left(\left(1+kr^2-c\right)\right. \\ &\quad \times \left.\left[-2+\cosh\left(\sqrt{\frac{1+kr^2-c}{bc}}\,(x+ry-ct)\right)\right]\right) \\ &\quad \times \left(a\left[1+\cosh\left(\sqrt{\frac{1+kr^2-c}{bc}}\,(x+ry-ct)\right)\right]\right)^{-1}. \end{aligned} \tag{14}$$

Remark 2. If the real number r in (14) takes value 1, then solution (14) is the same to solution (1.4) in [6]. Solution (1.5) in [6] is not a real solution of the KP-BBM equation; it is easy to verify that it does not satisfy the KP-BBM equation (3).



(a) $bc/a < 0$ (b) $bc/a > 0$

(c) (d)

Figure 1: The bifurcation phase portraits of system (6) with $g = 0$ and $(1 + kr^2 + c)/bc < 0$.

3.2. The Case $g \neq 0$. There are two homoclinic orbits Γ_5 and Γ_6 when $g \neq 0$ (see Figures 3(a) and 3(b)). Γ_5 and Γ_6 can be described by

$$y^2 = \frac{2a}{3bc}\varphi^3 - \frac{1 + kr^2 - c}{bc}\varphi^2 + \frac{2g}{bc}\varphi + 2h. \tag{15}$$

When $bc/a < 0$, the corresponding homoclinic orbit Γ_5 has a double zero point φ_1 and a zero point φ_3 on φ-axis (see Figure 3(a)), so (15) can be rewritten as

$$y^2 = \frac{2a}{3bc}(\varphi - \varphi_1)^2 (\varphi - \varphi_3); \tag{16}$$

that is,

$$y = \pm\sqrt{\frac{2a}{3bc}(\varphi - \varphi_1)^2 (\varphi - \varphi_3)}. \tag{17}$$

Substituting (17) into $d\varphi/d\xi = y$ and integrating along homoclinic orbits Γ_5, we get

$$\int_{\varphi}^{\varphi_3} \frac{ds}{\sqrt{(2a/3bc)(s - \varphi_1)^2 (s - \varphi_3)}} = |\xi|, \tag{18}$$

where $\varphi_1 = (1+kr^2-c-\sqrt{\Delta})/2a$ and $\varphi_3 = (1+kr^2-c+2\sqrt{\Delta})/2a$ if $a > 0$ and $bc < 0$; then, completing (18) we get the following solution:

$$u_3(x, y, t)$$

$$= \Bigg(\left(1 + kr^2 - c - \sqrt{\Delta}\right)$$

$$\times \cosh\sqrt{-\frac{\sqrt{\Delta}}{bc}}(x + ry - ct) + 1$$

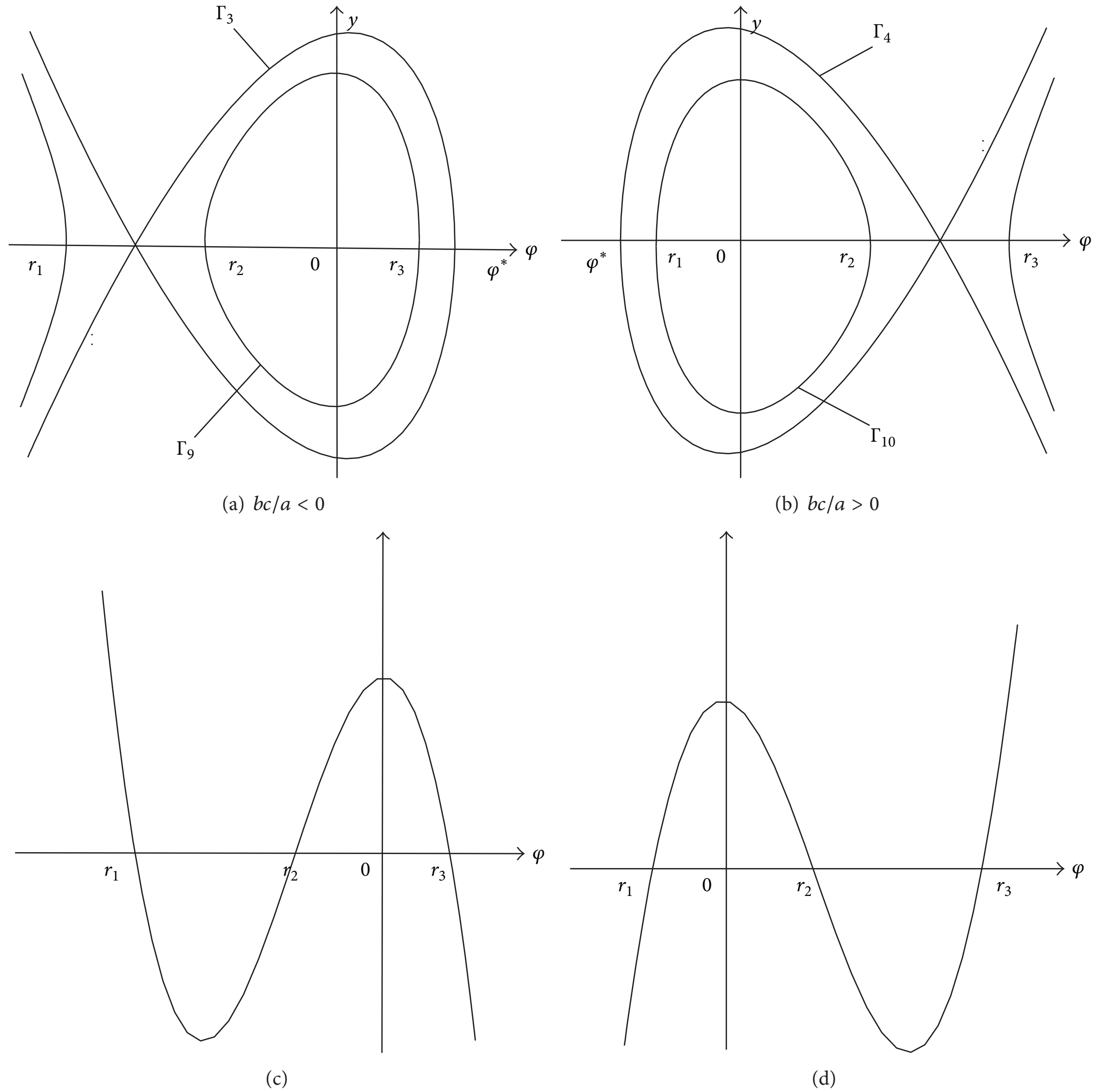


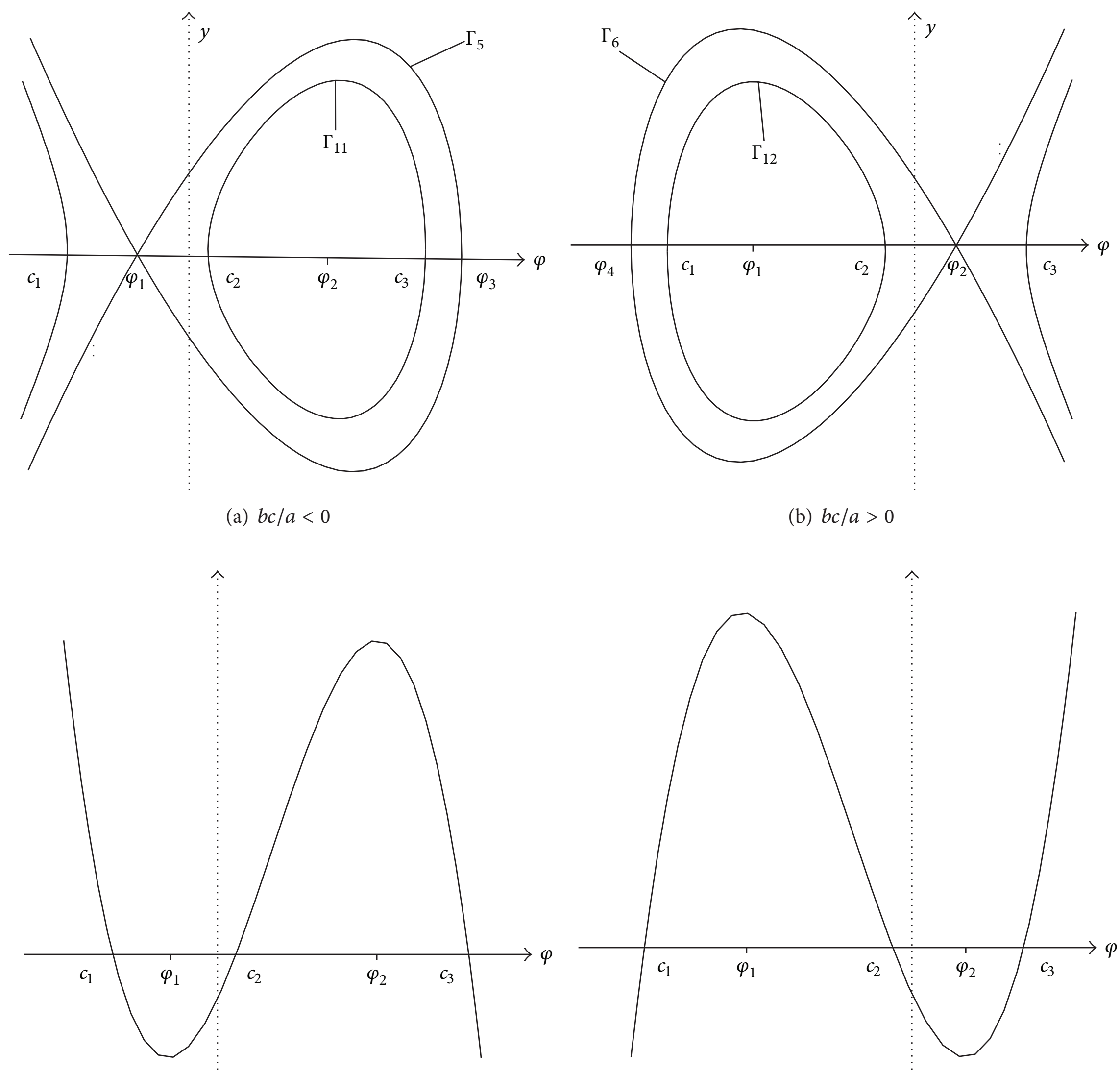
FIGURE 2: The bifurcation phase portraits of system (6) with $g = 0$ and $(1 + kr^2 + c)/bc > 0$.

$$\left.+kr^2 - c + 5\sqrt{\Delta}\right)$$
$$\times\left(2a\left(\cosh\sqrt{-\frac{\sqrt{\Delta}}{bc}}(x+ry-ct)+1\right)\right)^{-1}. \tag{19}$$

In (18), $\varphi_1 = (1 + kr^2 - c + \sqrt{\Delta})/2a$ and $\varphi_3 = (1 + kr^2 - c - 2\sqrt{\Delta})/2a$ if $a < 0$ and $bc > 0$; then, completing (18) we get the following solution:

$$u_4(x, y, t)$$
$$= \left(\left(1 + kr^2 - c + \sqrt{\Delta}\right)\right.$$
$$\left.\times\cosh\sqrt{\frac{\sqrt{\Delta}}{bc}}(x+ry-ct) + 1 + kr^2 - c - 5\sqrt{\Delta}\right)$$
$$\times\left(2a\left(\cosh\sqrt{\frac{\sqrt{\Delta}}{bc}}(x+ry-ct)+1\right)\right)^{-1}. \tag{20}$$

Remark 3. If the real number r in (19) and (20) takes value 1, then solutions (19) and (20) are the same to solutions (1.6) and (1.8) in [6]. Solutions (1.7) and (1.9) in [6] are not real solutions of the KP-BBM equation, and it is easy to verify that they do not satisfy the KP-BBM equation (3).



(a) $bc/a < 0$ (b) $bc/a > 0$

(c) (d)

Figure 3: The bifurcation phase portraits of system (6) with $g \neq 0$.

When $bc/a > 0$, the corresponding homoclinic orbit Γ_6 has a double zero point φ_2 and a zero point φ_4 on φ-axis (see Figure 3(b)), so (15) can be rewritten as

$$y^2 = \frac{2a}{3bc}(\varphi - \varphi_2)^2 (\varphi - \varphi_4); \tag{21}$$

that is,

$$y = \pm\sqrt{\frac{2a}{3bc}(\varphi - \varphi_2)^2 (\varphi - \varphi_4)}. \tag{22}$$

Substituting (22) into $d\varphi/d\xi = y$ and integrating along homoclinic orbits Γ_6, we get

$$\int_{\varphi_4}^{\varphi} \frac{ds}{\sqrt{(2a/3bc)(s - \varphi_2)^2 (s - \varphi_4)}} = |\xi|, \tag{23}$$

where $\varphi_2 = (1+kr^2-c-\sqrt{\Delta})/2a$ and $\varphi_4 = (1+kr^2-c+2\sqrt{\Delta})/2a$ if $a < 0$ and $bc < 0$; then, completing (23) we get the solution $u_3(x, y, t)$. In (23), $\varphi_2 = (1 + kr^2 - c + \sqrt{\Delta})/2a$ and $\varphi_4 = (1 + kr^2 - c - 2\sqrt{\Delta})/2a$ if $a > 0$ and $bc > 0$; then, completing (23) we get the solution $u_4(x, y, t)$.

4. Periodic Wave Solutions

So as to explain our work conveniently, in this section the Jacobian elliptic function $\mathrm{sn}(l, m)$ with modulus m will be expressed by $\mathrm{sn}l$. We discuss the periodic wave solutions under conditions $g = 0$ and $g \neq 0$, respectively.

4.1. The Case $g = 0$. When $(1 + kr^2 + c)/bc < 0$, system (6) has periodic orbits Γ_7 and Γ_8 (see Figures 1(a) and 1(b)). Their expressions are (9) on φ-y plane, where $h_1 < h < h_2$ (or $h_2 < h < h_1$). Let

$$f_1(\varphi) = \frac{2a}{3bc}\varphi^3 - \frac{1+kr^2-c}{bc}\varphi^2 + 2h; \tag{24}$$

then, we have the following results.

Claim 1. In the case of $g = 0$, $(1+kr^2+c)/bc < 0$ and $h_1 < h < h_2$ (or $h_2 < h < h_1$); then, the function $f_1(\varphi)$ must have three different real zero points.

Proof. Since $f_1(\varphi)$ is a cubic polynomial about φ, we can use the Shengjin Theorem [11] to distinguish its solutions. We only discuss the case $bc/a < 0$, and the case $bc/a > 0$ is the same. Under the above conditions,

$$\begin{aligned} h_1 &= H(\varphi_1, 0) = H(0,0) = 0, \\ h_2 &= H(\varphi_2, 0) = H\left(\frac{1+kr^2-c}{a}, 0\right) = \frac{(1+kr^2-c)^3}{6a^2bc} < 0. \end{aligned} \tag{25}$$

So $(1+kr^2-c)/(6a^2bc) < h < 0$. In $f_1(\varphi)$ the coefficients $2a/3bc < 0$, $(1+kr^2-c)/bc < 0$, By Shengjing Theorem [11], it follows that the function $f_1(\varphi)$ has three different real zero points. Let $A = ((1+kr^2-c)/bc)^2$, $B = -9(2a/3bc)\cdot 2h$, and $C = 3((1+kr^2-c)/bc)\cdot 2h$; then, $B^2-4AC = 144(a/bc)^2h[h-((1+kr^2-c)^3/6a^2bc)] < 0$. □

Let $r_1 < r_2 < r_3$ be three different real zero points of $f_1(\varphi)$. Then Claim 1 means that (9) has three intersection points $(r_1, 0)$, $(r_2, 0)$, and $(r_3, 0)$ on φ-axis. Therefore, (9) can be rewritten as

$$y^2 = \frac{2a}{3bc}(\varphi-r_1)(\varphi-r_2)(\varphi-r_3), \tag{26}$$

where $r_1 < 0 < r_2 < \varphi < r_3$ when $bc/a < 0$ and $r_1 < \varphi < r_2 < 0 < r_3$ when $bc/a > 0$.

When $bc/a < 0$, the orbit Γ_7 is according to a periodic solution of (6) and its expression is given by

$$y = \pm\sqrt{\frac{2a}{3bc}(\varphi-r_1)(\varphi-r_2)(\varphi-r_3)}, \quad (r_1 < r_2 \le \varphi \le r_3). \tag{27}$$

Substituting (27) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating along orbit Γ_7, we get

$$\begin{aligned} &\int_{\varphi}^{r_3} \frac{\mathrm{d}s}{\sqrt{(r_3-s)(s-r_1)(s-r_2)}} \\ &\quad = \sqrt{-\frac{2a}{3bc}}|\xi|, \quad (r_1 < r_2 \le \varphi < r_3). \end{aligned} \tag{28}$$

By formula (236) in [12], we have

$$g_1 \operatorname{sn}^{-1}(\sin\psi_1, m_5) = \sqrt{-\frac{2a}{3bc}}|\xi|, \tag{29}$$

where $g_1 = 2/\sqrt{r_3-r_1}$, $\sin\psi_1 = \sqrt{(r_3-\varphi)/(r_3-r_2)}$, and $m_5 = \sqrt{(r_3-r_2)/(r_3-r_1)}$. Solving (29), we get

$$\varphi = r_3 - (r_3-r_2)\operatorname{sn}^2\sqrt{-\frac{a(r_3-r_1)}{6bc}}\xi. \tag{30}$$

That is,

$$u_5(x,y,t) = r_3 - (r_3-r_2)\operatorname{sn}^2\sqrt{-\frac{a(r_3-r_1)}{6bc}}(x+ry-ct), \tag{31}$$

where the modulus of sn is $m_5 = \sqrt{(r_3-r_2)/(r_3-r_1)}$.

Similarly, when $bc/a > 0$, the expression Γ_8 is

$$y = \pm\sqrt{\frac{2a}{3bc}(\varphi-r_1)(\varphi-r_2)(\varphi-r_3)}, \quad (r_1 \le \varphi \le r_2 < r_3). \tag{32}$$

Substituting (32) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating along orbit Γ_8, we have

$$\begin{aligned} &\int_{r_1}^{\varphi} \frac{\mathrm{d}s}{\sqrt{(r_3-s)(r_2-s)(s-r_1)}} \\ &\quad = \sqrt{\frac{2a}{3bc}}|\xi|, \quad (r_1 < \varphi \le r_2 < r_3). \end{aligned} \tag{33}$$

The according periodic solution of Γ_8 can be obtained as

$$u_6(x,y,t) = r_1 + (r_2-r_1)\operatorname{sn}^2\sqrt{\frac{a(r_3-r_1)}{6bc}}(x+ry-ct), \tag{34}$$

where the modulus of sn is $m_6 = \sqrt{(r_2-r_1)/(r_3-r_1)}$.

When $(1+kr^2+c)/bc > 0$, system (6) has periodic orbits Γ_9 and Γ_{10} (see Figure 2). Their expressions are (9), where $h_1 < h < h_2$ (or $h_2 < h < h_1$). Similarly, we can get the according periodic solutions of Γ_9 and Γ_{10} as u_5 and u_6.

To verify validity of the periodic wave solutions, we take $a = b = k = r = 1$, $c = -1$, and $h = -9/4$ to make the conditions in Claim 1 satisfied. By simple calculation, we get that $r_1 = -1.09808$, $r_2 = 1.5$, and $r_3 = 4.09808$. The specific periodic wave solution is

$$u(x,y,t) = 4.090808 - 2.590808\operatorname{sn}^2\sqrt{\frac{5.19616}{6}}(x+y+t), \tag{35}$$

where the modulus of sn is $\sqrt{2}/2$.

4.2. The Case $g \neq 0$. System (6) has periodic orbits Γ_{11} and Γ_{12} (see Figure 3). Their expressions are (7) on the φ-y plane, where $h_1 < h < h_2$ (or $h_1 < h < h_2$). Let

$$f_2(\varphi) = \frac{2a}{3bc}\varphi^3 - \frac{1+kr^2-c}{bc}\varphi^2 + \frac{2g}{bc}\varphi + 2h; \tag{36}$$

then, we have the following results about $f_2(\varphi)$.

Claim 2. If $g \neq 0$ and $h_1 < h < h_2$ (or $h_2 < h < h_1$), then the function $f_2(\varphi)$ must have three different real zero points.

Proof. We only prove the case $bc/a < 0$, and the case $bc/a > 0$ is the same. Under the above conditions,

$$\begin{aligned} h_1 &= H(\varphi_1, 0) = -\frac{a}{3bc}\varphi_1^3 + \frac{1+kr^2-c}{2bc}\varphi_1^2 - \frac{g}{bc}\varphi_1 \\ &= -\frac{1}{2}f_2(\varphi_1) + h, \\ h_2 &= H(\varphi_2, 0) = -\frac{a}{3bc}\varphi_2^3 + \frac{1+kr^2-c}{2bc}\varphi_2^2 - \frac{g}{bc}\varphi_2 \\ &= -\frac{1}{2}f_2(\varphi_2) + h. \end{aligned} \tag{37}$$

So $f_2(\varphi_1) \cdot f_2(\varphi_2) = 4(h-h_1)(h-h_2) < 0$. For $f_2(\varphi)$, we have $f_2(-\infty) > 0$, $f_2(\varphi_1) < 0$, $f_2(\varphi_2) > 0$, and $f_2(+\infty) < 0$. Again, $f_2'(\varphi) = (2a/bc)(\varphi - \varphi_1)(\varphi - \varphi_2)$, which is monotonous in the intervals $(-\infty, \varphi_1)$, (φ_1, φ_2), and $(\varphi_2, +\infty)$. By zero point theorem of continuous function, there must be one real zero point of $f_2(\varphi)$ that lies in each of the three intervals, proving the claim. □

Let $c_1 < c_2 < c_3$ be three different real zero points of $f_2(\varphi)$. Then Claim 2 means that (7) has three intersection points on φ-axis denoted by $(c_1, 0)$, $(c_2, 0)$, and $(c_3, 0)$. Then (7) can be rewritten as

$$y^2 = \frac{2a}{3bc}(\varphi - c_1)(\varphi - c_2)(\varphi - c_3), \tag{38}$$

where $c_1 < \varphi_1 < c_2 < \varphi_2 < c_3$.

When $bc/a < 0$, the expression of periodic orbit Γ_{11} is

$$y = \pm\sqrt{\frac{2a}{3bc}(\varphi - c_1)(\varphi - c_2)(\varphi - c_3)}, \quad (c_1 < c_2 \leq \varphi \leq c_3). \tag{39}$$

When $bc/a > 0$, the expression of periodic orbit Γ_{12} is

$$y = \pm\sqrt{\frac{2a}{3bc}(\varphi - c_1)(\varphi - c_2)(\varphi - c_3)}, \quad (c_1 \leq \varphi \leq c_2 < c_3). \tag{40}$$

Substituting (39) and (40) into $d\varphi/d\xi = y$ and integrating along orbits Γ_{11} and Γ_{12}, respectively, it is the same to the proceeding for solving u_5 and u_6 and we can get the according periodic solutions of Γ_{11} and Γ_{12} as follows:

$$\begin{aligned} u_7(x, y, t) &= c_3 - (c_3 - c_2)\,\mathrm{sn}^2\sqrt{-\frac{a(c_3 - c_1)}{6bc}}(x + ry - ct), \\ u_8(x, y, t) &= c_1 + (c_2 - c_1)\,\mathrm{sn}^2\sqrt{\frac{a(c_3 - c_1)}{6bc}}(x + ry - ct), \end{aligned} \tag{41}$$

where the moduli for sn are $m_7 = \sqrt{(c_3 - c_2)/(c_3 - c_1)}$ and $m_8 = \sqrt{(c_2 - c_1)/(c_3 - c_1)}$ in (41).

For example, we take $a = b = k = r = 1$, $c = -1$, $g = 2$, and $h = 3/2$ such that the conditions in Claim 2 are satisfied. By simple calculation, we get that $c_1 = 0.663975$, $c_2 = 1.5$, and $c_3 = 2.36603$. The according periodic wave solution is

$$u(x, y, t) = 2.36603 - 0.86603\,\mathrm{sn}^2\sqrt{\frac{1.702055}{6}}(x + y + t), \tag{42}$$

where the modulus of sn is $\sqrt{0.86603/1.702055}$.

5. Relationship between Bell-Shaped Waves and Periodic Waves

In Sections 3 and 4, the bell-shaped solitary wave and periodic wave solutions are obtained. Via further study, we find that there exists an interesting relationship between these two kinds of solutions; that is, the bell-shaped solutions are limits of the periodic waves in some sense. The results are detailed as follows.

Proposition 4. *Let u_i $(i = 1, 2, \ldots, 8)$ be solutions of* (3), *let k, r, a, b, c, and g be parameters in* (5), *and let m_i $(i = 5, 6, 7, 8)$ be modulus of the Jacobian elliptic function sn; then, one has the following.*

Case 1. When $g = 0$ and $(1 + kr^2 - c)/bc < 0$, for modulus $m_i \to 1$ $(i = 5, 6)$, the periodic waves u_5 and u_6 degenerate bell-shaped wave u_1.

Case 2. When $g = 0$ and $(1 + kr^2 - c)/bc > 0$, for modulus $m_i \to 1$ $(i = 5, 6)$, the periodic waves u_5 and u_6 degenerate bell-shaped wave u_2.

Case 3. When $g \neq 0$ and $bc/a < 0$, for modulus $m_7 \to 1$, the periodic wave u_7 degenerates bell-shaped wave u_3.

Case 4. When $g \neq 0$ and $bc/a > 0$, for modulus $m_8 \to 1$, the periodic wave u_8 degenerates bell-shaped wave u_4.

Here, we only prove Cases 1 and 3 for simplicity. The remaining cases are the same. In the following proofs, we use the property of elliptic function that sn $\to$ tanh when the modulus $m \to 1$ [5, 13].

Proof of Case 1. When $m_5 = \sqrt{(r_3 - r_2)/(r_3 - r_1)} \to 1$, it means $r_1 = r_2$ and sn = tanh; then, we calculate

$$r_1 = r_2 = 0, \qquad r_3 = \frac{3(1 + kr^2 - c)}{2a}. \tag{43}$$

Substituting r_i $(i = 1, 2, 3)$ into u_5 admits to u_1 as follows:

$$\begin{aligned} &u_5(x, y, t) \\ &= r_3 - (r_3 - r_2)\,\mathrm{sn}^2\sqrt{-\frac{a(r_3 - r_1)}{6bc}}(x + ry - ct) \\ &= \frac{3(1 + kr^2 - c)}{2a} - \frac{3(1 + kr^2 - c)}{2a} \end{aligned}$$

$$\times \tanh^2 \sqrt{-\frac{\left(1+kr^2-c\right)}{4bc}}\left(x+ry-ct\right)$$
$$=\frac{3\left(1+kr^2-c\right)}{2a}$$
$$\times\left[1-\tanh^2\sqrt{-\frac{\left(1+kr^2-c\right)}{4bc}}\left(x+ry-ct\right)\right]$$
$$=\frac{3\left(1+kr^2-c\right)}{2a}$$
$$\times\frac{1}{\cosh^2\sqrt{-\left(\left(1+kr^2-c\right)/4bc\right)}\left(x+ry-ct\right)}$$
$$=\frac{3\left(1+kr^2-c\right)}{2a}$$
$$\times\frac{1}{(1/2)\left[\cosh\sqrt{-\left(\left(1+kr^2-c\right)/bc\right)}\left(x+ry-ct\right)+1\right]}$$
$$=\frac{3\left(1+kr^2-c\right)}{a\left[1+\cosh\sqrt{-\left(\left(1+kr^2-c\right)/bc\right)}\left(x+ry-ct\right)\right]}$$
$$=u_1\left(x,y,t\right). \tag{44}$$

When $m_6=\sqrt{(r_2-r_1)/(r_3-r_1)}\rightarrow 1$, it means $r_2=r_3$; then, we calculate $r_2=r_3=0$ and $r_1=3(1+kr^2-c)/2a$, and substituting r_i $(i=1,2,3)$ into u_6 we get $u_6=u_1$. □

Proof of Case 3. When $m_7=\sqrt{(c_3-c_2)/(c_3-c_1)}\rightarrow 1$, it means $c_1=c_2$ and sn = tanh; then, we calculate $c_1=c_2=(1+kr^2-c-\sqrt{\Delta})/2a$ and $c_3=(1+kr^2-c+2\sqrt{\Delta})/2a$, and substituting c_i $(i=1,2,3)$ into u_7 admits to u_3 as follows:

$$u_7\left(x,y,t\right)$$
$$=c_3-\left(c_3-c_2\right)\mathrm{sn}^2\sqrt{-\frac{a\left(c_3-c_1\right)}{6bc}}\left(x+ry-ct\right)$$
$$=\frac{1+kr^2-c+2\sqrt{\Delta}}{2a}-\frac{3\sqrt{\Delta}}{2a}\tanh^2\sqrt{\frac{-\sqrt{\Delta}}{4bc}}\left(x+ry-ct\right)$$
$$=\Bigg(1+kr^2-c-\sqrt{\Delta}$$
$$+3\sqrt{\Delta}\left(1-\tanh^2\sqrt{\frac{-\sqrt{\Delta}}{4bc}}\left(x+ry-ct\right)\right)\Bigg)$$
$$\times\left(2a\right)^{-1}$$
$$=\frac{1+kr^2-c-\sqrt{\Delta}}{2a}$$
$$+\frac{3\sqrt{\Delta}}{a\left[1+\cosh\sqrt{-\sqrt{\Delta}/bc}\left(x+ry-ct\right)\right]}$$
$$=\Bigg(\left(1+kr^2-c-\sqrt{\Delta}\right)$$
$$\times\cosh\sqrt{-\frac{\sqrt{\Delta}}{bc}}\left(x+ry-ct\right)+1$$
$$+kr^2-c+5\sqrt{\Delta}\Bigg)$$
$$\times\left(2a\left[\cosh\sqrt{-\frac{\sqrt{\Delta}}{bc}}\left(x+ry-ct\right)+1\right]\right)^{-1}$$
$$=u_3\left(x,y,t\right). \tag{45}$$

The results provide a manner that we can get bell-shaped waves from periodic waves for some nonlinear development equations. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] T. B. Benjamin, J. L. Bona, and J. J. Mahony, "Model equations for long waves in nonlinear dispersive systems," *Philosophical Transactions of the Royal Society of London A. Mathematical and Physical Sciences*, vol. 272, no. 1220, pp. 47–78, 1972.

[2] B. B. Kadomtsev and V. I. Petviashvili, "On the stability of solitary waves in weakly dispersive media," *Soviet Physics—Doklady*, vol. 15, pp. 539–541, 1970.

[3] A.-M. Wazwaz, "Exact solutions of compact and noncompact structures for the KP-BBM equation," *Applied Mathematics and Computation*, vol. 169, no. 1, pp. 700–712, 2005.

[4] A.-M. Wazwaz, "The extended tanh method for new compact and noncompact solutions for the KP-BBM and the ZK-BBM equations," *Chaos, Solitons and Fractals*, vol. 38, no. 5, pp. 1505–1516, 2008.

[5] M. A. Abdou, "Exact periodic wave solutions to some nonlinear evolution equations," *International Journal of Nonlinear Science*, vol. 6, no. 2, pp. 145–153, 2008.

[6] M. Song, C. X. Yang, and B. G. Zhang, "Exact solitary wave solutions of the Kadomtsov-Petviashvili-Benjamin-Bona-Mahony equation," *Applied Mathematics and Computation*, vol. 217, no. 4, pp. 1334–1339, 2010.

[7] S.-N. Chow and J. K. Hale, *Method of Bifurcation Theory*, Springer, Berlin, Germany, 1982.

[8] J. B. Li and Z. R. Liu, "Smooth and non-smooth traveling waves in a nonlinearly dispersive equation," *Applied Mathematical Modelling*, vol. 25, no. 1, pp. 41–56, 2000.

[9] J. B. Li and Z. R. Liu, "Traveling wave solutions for a class of nonlinear dispersive equations," *Chinese Annals of Mathematics B*, vol. 23, no. 3, pp. 397–418, 2002.

[10] Z. R. Liu and Z. Y. Ouyang, "A note on solitary waves for modified forms of Camassa-Holm and Degasperis-Procesi equations," *Physics Letters A*, vol. 366, no. 4-5, pp. 377–381, 2007.

[11] F. Shengjin, "A new extracting formula and a new distinguishing means on the one variable cubic equation," *Natural Science Journal of Hainan Teacheres College*, vol. 2, pp. 91–98, 1989.

[12] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer, Berlin, Germany, 1971.

[13] S. Yu and L. Tian, "Nonsymmetrical kink solution of generalized KdV equation with variable coefficients," *International Journal of Nonlinear Science*, vol. 5, no. 1, pp. 71–78, 2008.