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Research Article

New Oscillatory Behavior of Third-Order Nonlinear Delay Dynamic Equations on Time Scales

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A class of third-order nonlinear delay dynamic equations on time scales is studied. By using the generalized Riccati transformation and the inequality technique, four new sufficient conditions which ensure that every solution is oscillatory or converges to zero are established. The results obtained essentially improve earlier ones. Some examples are considered to illustrate the main results.

1. Introduction

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the studies by Bohner and Saker [1] and Erbe et al. [2, 3]. And there are some results dealing with oscillatory behavior of second-order delay dynamic equations on time scales [4–10]. However, there are few results dealing with the oscillation of the solutions of third-order delay dynamic equations on time scales, we refer the reader to the papers [11–14].

In this paper, we consider new oscillatory behavior of all solutions of the third-order nonlinear delay dynamic equation

$$\left(r_2(t)\left[\left(r_1(t)\,x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha}\right)^{\Delta} + q(t)\,f(x[\tau(t)]) = 0, \qquad t \in \mathbb{T}, \quad t \geq t_0, \tag{1}$$

where $\alpha \geq 1$ is the ratio of two positive odd integers.

Throughout this paper, we will assume the following hypotheses.

(H_1) $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above, and $t_0 \in \mathbb{T}$ with $t_0 > 0$, we define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$.

(H_2) $r_1(t)$, $r_2(t)$, $q(t)$ are positive and real-valued rd-continuous functions defined on $\mathbb{T}$, and $r_1(t)$, $r_2(t)$ satisfy

$$\int_{t_0}^{\infty} \frac{1}{r_1(s)}\Delta s = \infty, \qquad \int_{t_0}^{\infty} \left(\frac{1}{r_2(s)}\right)^{1/\alpha}\Delta s = \infty. \tag{2}$$

(H_3) $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing and differentiable function, such that

$$\tau(t) \leq t, \qquad \lim_{t\to\infty} \tau(t) = \infty, \qquad \tau(\mathbb{T}) = \mathbb{T}. \tag{3}$$

(H_4) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists some positive constant L such that $f(x)/x^{\alpha} \geq L$ for all $x \neq 0$.

By a solution of (1), we mean a nontrivial function $x(t)$ satisfying (1) which has the properties $x(t) \in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ for $T_x \geq t_0$, and $r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha} \in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$. Our attention is restricted to those solutions of (1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$, where C_{rd} is the space of rd-continuous functions. A solution $x(t)$ of (1) is said to be oscillatory on $[T_x,\infty)_{\mathbb{T}}$ if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

If $\alpha = 1$, $\tau(t) = t$, then (1) simplifies to the third-order nonlinear dynamic equation

$$\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]\right)^{\Delta} + q(t)f(x(t)) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{4}$$

If, furthermore, $r_1(t) = r_2(t) = 1$, $f(x) = x$, $\tau(t) = t$, then (1) reduces to the third-order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + q(t)x(t) = 0, \quad t \in \mathbb{T}, \ t \geq t_0. \tag{5}$$

If, in addition, $\alpha = 1$, then (1) reduces to the nonlinear delay dynamic equation

$$\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]\right)^{\Delta} + q(t)f(x[\tau(t)]) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{6}$$

In 2005, Erbe et al. [11] considered the general third-order nonlinear dynamic equation (4). By employing the generalized Riccati transformation techniques, they established some sufficient conditions which ensure that every solution of (4) is oscillatory or converges to zero. In 2007, Erbe et al. [12] studied the third-order linear dynamic equation (5), and they obtained Hille and Nehari type oscillation criteria for (5). In 2011, Han et al. [13] extended and improved the results of [12], meanwhile obtaining some oscillatory criteria for (6). In 2014, Gao et al. [14] extended some results of [12, 13] to (1). On this basis, we continue to discuss the oscillation of solutions of (1). By using the generalized Riccati transformation and the inequality technique, we obtain some new sufficient conditions which guarantee that every solution of (1) is oscillatory or converges to zero. Our results will improve some results that have been established in [11–14].

Throughout this paper, we will make use of the following product and quotient rules:

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),\end{aligned} \tag{7}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))} \quad \text{if } gg^{\sigma} \neq 0. \tag{8}$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\Delta t = f(c) - f(b). \tag{9}$$

The integration by parts formula reads

$$\int_b^c f^{\Delta}(t)g(t)\Delta t = f(c)g(c) - f(b)g(b) - \int_b^c f^{\sigma}(t)g^{\Delta}(t)\Delta t, \tag{10}$$

and improper integrals are defined in the usual way by

$$\int_b^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_b^t f(s)\Delta s. \tag{11}$$

For more details, see [15, 16].

2. Several Lemmas

In this section we present several lemmas that will be needed in the proofs of our results in Section 3.

Lemma 1. *Assume that $x(t)$ is an eventually positive solution of* (1), *then there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that either*

$$\text{(I)} \quad x(t) > 0, \quad x^{\Delta}(t) > 0, \quad \left(r_1(t)x^{\Delta}(t)\right)^{\Delta} > 0,$$
$$\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha}\right)^{\Delta} < 0, \quad t \in [T, \infty)_{\mathbb{T}}, \tag{12}$$

or

$$\text{(II)} \quad x(t) > 0, \quad x^{\Delta}(t) < 0, \quad \left(r_1(t)x^{\Delta}(t)\right)^{\Delta} > 0,$$
$$\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha}\right)^{\Delta} < 0, \quad t \in [T, \infty)_{\mathbb{T}}. \tag{13}$$

Proof. Assume that $x(t)$ is an eventually positive solution of (1), then there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. From (1); we obtain

$$\begin{aligned}\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha}\right)^{\Delta} &= -q(t)f(x(\tau(t))) \\ &\leq -Lq(t)x^{\alpha}[\tau(t)] < 0.\end{aligned} \tag{14}$$

Hence, $r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha}$ is decreasing and therefore eventually of one sign, so $(r_1(t)x^{\Delta}(t))^{\Delta}$ is either eventually positive or eventually negative. We assert that $(r_1(t)x^{\Delta}(t))^{\Delta} > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$.

If there exists $t_1 \in [T, \infty)_{\mathbb{T}}$ such that $(r_1(t_1)x^{\Delta}(t_1))^{\Delta} < 0$, we get

$$r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha} \leq r_2(t_1)\left[\left(r_1(t_1)x^{\Delta}(t_1)\right)^{\Delta}\right]^{\alpha} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{15}$$

Let $M = -r_2(t_1)[(r_1(t_1)x^{\Delta}(t_1))^{\Delta}]^{\alpha} > 0$, then

$$\left(r_1(t)x^{\Delta}(t)\right)^{\Delta} \leq -M^{1/\alpha}\frac{1}{\left(r_2(t)\right)^{1/\alpha}}. \tag{16}$$

Integrating (16) from t_1 to t $(t \in [t_1, \infty)_{\mathbb{T}})$ provides

$$\begin{aligned}r_1(t)x^{\Delta}(t) &\leq r_1(t_1)x^{\Delta}(t_1) - M^{1/\alpha} \\ &\quad \times \int_{t_1}^t \frac{1}{\left(r_2(s)\right)^{1/\alpha}}\Delta s \longrightarrow -\infty, \quad t \longrightarrow +\infty.\end{aligned} \tag{17}$$

Then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r_1(t)x^{\Delta}(t) \leq r_1(t_2)x^{\Delta}(t_2) < 0$. Similarly, $t \in [t_2, \infty)_{\mathbb{T}}$, we obtain

$$x(t) \leq x(t_2) + r_1(t_2)x^{\Delta}(t_2)\int_{t_2}^t \frac{1}{r_1(s)}\Delta s \longrightarrow -\infty, \quad t \longrightarrow +\infty, \tag{18}$$

which contradicts with $x(t) > 0$. So $(r_1(t)x^{\Delta}(t))^{\Delta} > 0$; this implies that $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. This completes the proof. □

Lemma 2 (see [17]). *Assume that (H_3) and the following conditions hold:*

(I) $u(t) \in C^2_{rd}(I, \mathbb{R})$, *where* $I = [t^*, +\infty)_{\mathbb{T}}$ *for some* $t^* > 0$;

(II) $u(t) > 0$, $u^{\Delta}(t) > 0$, $u^{\Delta\Delta}(t) \leq 0$ *for* $t \geq t^*$.

Then, for each $k \in (0,1)$, *there exists a constant* $t_k \in \mathbb{T}$, $t_k \geq t^*$ *such that*

$$u(\sigma(t)) \leq \frac{\sigma(t) u(\tau(t))}{k\tau(t)} \quad \text{for } t \geq t_k. \tag{19}$$

Lemma 3 (see [15]). *If x is differentiable, then*

$$(x^{\gamma})^{\Delta} = \gamma x^{\Delta} \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1} dh. \tag{20}$$

Lemma 4 (see [12]). *Assume that u satisfies*

$$u(t) > 0, \qquad u^{\Delta}(t) > 0, \qquad u^{\Delta\Delta}(t) > 0, \\ u^{\Delta\Delta\Delta}(t) \leq 0, \quad t \in [T, \infty)_{\mathbb{T}}. \tag{21}$$

Then

$$\liminf_{t\to\infty} \frac{tu(t)}{h_2(t,t_0) u^{\Delta}(t)} \geq 1, \tag{22}$$

where the Taylor monomials $h_n(t,s)_{n=0}^{+\infty}$ are defined recursively by

$$h_0(t,s) = 1, \qquad h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, \\ t, s \in \mathbb{T}, \quad n \geq 1. \tag{23}$$

Lemma 5 (see [18]). *Assume that X and Y are nonnegative real numbers. Then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda - 1) Y^{\lambda}, \quad \forall \lambda > 1, \tag{24}$$

where the equality holds if and only if $X = Y$.

Lemma 6. *Assume that $x(t)$ is an eventually positive solution of* (1) *which satisfies case (II) in Lemma 1, if either*

$$\int_{t_0}^{\infty} q(s)\Delta s = \infty \tag{25}$$

or

$$\int_{t_0}^{\infty} q(s)\Delta s < \infty, \\ \int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} q(u)\Delta u\right]^{1/\alpha} \Delta s \Delta t = \infty. \tag{26}$$

Then $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is an eventually positive solution of (1) which satisfies case (II) in Lemma 1. Then $x(t)$ is decreasing and $\lim_{t\to\infty} x(t) = l \geq 0$. If $l > 0$; it is easy to see that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x[\tau(t)] \geq x(t) \geq l > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. From (14),

$$\left(r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha}\right)^{\Delta} \leq -Lq(t)x^{\alpha}[\tau(t)] \leq -Ll^{\alpha}q(t). \tag{27}$$

If (25) holds, then integrating (27) from t_1 to t ($t \in [t_1, \infty)_{\mathbb{T}}$), we get

$$r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha} \leq r_2(t_1)\left[\left(r_1(t_1)x^{\Delta}(t_1)\right)^{\Delta}\right]^{\alpha} - Ll^{\alpha} \\ \times \int_{t_1}^{t} q(s)\Delta s \longrightarrow -\infty \\ (t \longrightarrow +\infty). \tag{28}$$

This is contrary to $(r_1(t)x^{\Delta}(t))^{\Delta} > 0$.

If (26) holds, then integrating (1) from t to ∞, we get

$$-r_2(t)\left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha} \\ \leq -L\int_t^{\infty} q(s)x^{\alpha}[\tau(s)]\Delta s \\ \leq -Ll^{\alpha}\int_t^{\infty} q(s)\Delta s, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{29}$$

and hence,

$$-\left(r_1(t)x^{\Delta}(t)\right)^{\Delta} \leq -l\left[\frac{1}{r_2(t)}\int_t^{\infty} Lq(s)\Delta s\right]^{1/\alpha}. \tag{30}$$

Again, integrating this inequality from t to ∞, we obtain

$$r_1(t)x^{\Delta}(t) \leq -lL^{1/\alpha}\int_t^{\infty}\left[\frac{1}{r_2(s)}\int_s^{\infty} q(u)\Delta u\right]^{1/\alpha}\Delta s. \tag{31}$$

Finally, integrating the last inequality from T to t, we get

$$x(t) - x(T) \leq -lL^{1/\alpha}\int_T^t \frac{1}{r_1(s)} \\ \times \int_s^{\infty}\left[\frac{1}{r_2(u)}\int_u^{\infty} q(v)\Delta v\right]^{1/\alpha}\Delta u \Delta s. \tag{32}$$

Hence by (26), we obtain $\lim_{t\to\infty} x(t) = -\infty$, which contradicts $x(t) > 0$. Thus, we get $l = 0$. This completes the proof. □

Lemma 7 (see [19]). *Let $a, b \in \mathbb{T}$. Then for positive rd-continuous functions $f, g : [a,b] \to \mathbb{R}$, one has*

$$\int_a^b |f(s)g(s)|\Delta s \leq \left(\int_a^b |f(s)|^p \Delta s\right)^{1/p}\left(\int_a^b |g(s)|^q \Delta s\right)^{1/q}, \tag{33}$$

where $p > 1$ and $1/p + 1/q = 1$.

3. Main Results

New we state and prove the main results of this paper.

Theorem 8. *Assume that* (H_1)–(H_4), (26), *and* $r_2^{\Delta}(t) \geq 0$ *hold. Furthermore, suppose that there exists a positive function* $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *with* $\delta^{\Delta}(t) \geq 0$, *and for all sufficiently large* T, *there exists* $T_0 > T$, *such that*

$$\limsup_{t\to\infty} \int_{T_0}^{t} \left\{ Q(s) - \frac{r_2(s)\delta(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}} \left[\frac{\delta^{\Delta}(s)}{\delta(s)}\right]^{\alpha+1} \left[\frac{\sigma(s)}{\tau(s)}\right]^{\alpha^2} \right\} \Delta s = \infty, \tag{34}$$

where $Q(t) = Lq(t)\delta(\sigma(t))(kh_2(\tau(t),t_0)/(2r_1(\tau(t))\sigma(t)))^{\alpha}$. *Then every solution* $x(t)$ *of* (1) *is either oscillatory or converges to zero.*

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$, such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [T,\infty)_{\mathbb{T}}$. In the case when $x(t)$ is eventually negative, the proof is similar. By Lemma 1, we see that $x(t)$ satisfies either case (I) or case (II).

If case (I) holds, then $x^{\Delta}(t) > 0$, $t \in [T,\infty)_{\mathbb{T}}$. Define the function $W(t)$ by

$$W(t) = \delta(t) r_2(t) \left(\frac{\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}}{r_1(t)x^{\Delta}(t)} \right)^{\alpha}, \quad t \in [T,\infty)_{\mathbb{T}}. \tag{35}$$

Then $W(t) > 0$. By the product rule (7) and the quotient rule (8), we have

$$\begin{aligned} W^{\Delta}(t) = {} & \frac{\delta^{\Delta}(t)}{\delta(t)} W(t) - \delta(\sigma(t)) \frac{q(t) f(x(\tau(t)))}{\left[r_1(\sigma(t)) x^{\Delta}(\sigma(t))\right]^{\alpha}} \\ & - \delta(\sigma(t)) r_2(t) \left[\frac{\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}}{r_1(t)x^{\Delta}(t)} \right]^{\alpha} \\ & \times \frac{\left[\left(r_1(t)x^{\Delta}(t)\right)^{\alpha}\right]^{\Delta}}{\left[r_1(\sigma(t)) x^{\Delta}(\sigma(t))\right]^{\alpha}}. \end{aligned} \tag{36}$$

Let $u(t) = r_1(t)x^{\Delta}(t)$, from case (I) in Lemma 1, we get $u(t) > 0$, $u^{\Delta}(t) > 0$. In view of that

$$\begin{aligned} \left[r_2(t)\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\Delta} &= \left[r_2(t)u^{\Delta}(t)\right]^{\Delta} \\ &= r_2^{\Delta}(t) u^{\Delta}(\sigma(t)) + r_2(t) u^{\Delta\Delta}(t) \\ &< 0 \end{aligned} \tag{37}$$

and $r_2^{\Delta}(t) \geq 0$, it is not difficult to see that $u^{\Delta\Delta}(t) < 0$. Thus, by Lemma 2, for every $k \in (0,1)$, there exists $t_1 \in [T,\infty)_{\mathbb{T}}$ with $t_1 \geq \max\{t_k, T\}$, such that $u(\sigma(t)) \leq \sigma(t)u(\tau(t))/k\tau(t) \leq \sigma(t)u(t)/k\tau(t)$ for all $t \in [t_1,\infty)_{\mathbb{T}}$; this implies that

$$\begin{aligned} r_1(\sigma(t)) x^{\Delta}(\sigma(t)) &\leq \frac{\sigma(t) r_1(\tau(t)) x^{\Delta}(\tau(t))}{k\tau(t)} \\ &\leq \frac{\sigma(t) r_1(t) x^{\Delta}(t)}{k\tau(t)}. \end{aligned} \tag{38}$$

By Lemma 3, we get $[(u(t))^{\alpha}]^{\Delta} \geq \alpha u^{\Delta}(t) \int_0^1 [hu + (1-h)u]^{\alpha-1} dh = \alpha(u(t))^{\alpha-1}u^{\Delta}(t)$, that is,

$$\left[\left(r_1(t)x^{\Delta}(t)\right)^{\alpha}\right]^{\Delta} \geq \alpha\left(r_1(t)x^{\Delta}(t)\right)^{\alpha-1}\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}. \tag{39}$$

Using (38) and (39),

$$\begin{aligned} W^{\Delta}(t) \leq {} & \frac{\delta^{\Delta}(t)}{\delta(t)} W(t) - \frac{Lq(t)x^{\alpha}(\tau(t))\delta(\sigma(t))}{\left[r_1(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\alpha}} \\ & - \frac{\alpha\delta(\sigma(t)) r_2(t) \left[\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\alpha+1}}{\left(r_1(t)x^{\Delta}(t)\right)\left(r_1(\sigma(t))x^{\Delta}(\sigma(t))\right)^{\alpha}} \\ \leq {} & \frac{\delta^{\Delta}(t)}{\delta(t)} W(t) - \frac{Lq(t)x^{\alpha}(\tau(t))\delta(\sigma(t))}{\left[r_1(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\alpha}} \\ & - \frac{\alpha k^{\alpha}\tau^{\alpha}(t) r_2(t)\delta(\sigma(t))}{\sigma^{\alpha}(t)} \left[\frac{\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}}{r_1(t)x^{\Delta}(t)} \right]^{\alpha+1} \\ \leq {} & \frac{\delta^{\Delta}(t)}{\delta(t)} W(t) - \frac{Lq(t)x^{\alpha}(\tau(t))\delta(\sigma(t))}{\left[r_1(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\alpha}} \\ & - \frac{\alpha k^{\alpha}\tau^{\alpha}(t) r_2(t)\delta(\sigma(t)) W^{1+1/\alpha}(t)}{\sigma^{\alpha}(t)\left(\delta(t)r_2(t)\right)^{1+1/\alpha}} \\ \leq {} & \frac{\delta^{\Delta}(t)}{\delta(t)} W(t) - \frac{Lq(t)x^{\alpha}(\tau(t))\delta(\sigma(t))}{\left[r_1(\sigma(t))x^{\Delta}(\sigma(t))\right]^{\alpha}} \\ & - \frac{\alpha k^{\alpha}\tau^{\alpha}(t) W^{1+1/\alpha}(t)}{\sigma^{\alpha}(t)\left(\delta(t)r_2(t)\right)^{1/\alpha}}. \end{aligned} \tag{40}$$

Let $v(t) = \int_T^t r_1(s)x^{\Delta}(s)\Delta s$ for all $t \in (T,\infty)_{\mathbb{T}}$; it is easy to see that $v(t) > 0$, $v^{\Delta}(t) > 0$, $v^{\Delta\Delta}(t) > 0$, $v^{\Delta\Delta\Delta}(t) \leq 0$. Thus, by Lemma 4, there exists $t_{1/2} \in [T,\infty)_{\mathbb{T}}$ such that $tv(t)/(h_2(t,t_0)v^{\Delta}(t)) \geq 1/2$ for all $t \in [t_{1/2},\infty)_{\mathbb{T}}$. Then, we get

$$\frac{\int_T^t r_1(s)x^{\Delta}(s)\Delta s}{r_1(t)x^{\Delta}(t)} \geq \frac{h_2(t,t_0)}{2t}, \quad t \in [t_{1/2},\infty)_{\mathbb{T}}. \tag{41}$$

From $\int_T^t r_1(s)x^\Delta(s)\Delta s = r_1(t)x(t) - r_1(T)x(T) - \int_T^t r_1^\Delta(s)x^\Delta(\sigma(s))\Delta s$, we obtain $r_1(t)x(t) \geq \int_T^t r_1(s)x^\Delta(s)\Delta s$. By (41), we get

$$\frac{x(t)}{x^\Delta(t)} = \frac{r_1(t)x(t)}{r_1(t)x^\Delta(t)} \geq \frac{\int_T^t r_1(s)x^\Delta(s)\Delta s}{r_1(t)x^\Delta(t)} \geq \frac{h_2(t,t_0)}{2t}, \quad t \in [t_{1/2},\infty)_{\mathbb{T}}. \tag{42}$$

Therefore, from (38) and (42), there exists $t_2 \in [t_0,\infty)_{\mathbb{T}}$ with $t_2 \geq \max\{t_1, t_{1/2}\}$ such that

$$\frac{x(\tau(t))}{r_1(\sigma(t))x^\Delta(\sigma(t))} \geq \frac{kh_2(\tau(t),t_0)}{2r_1(\tau(t))\sigma(t)}, \quad t \in [t_2,\infty)_{\mathbb{T}}. \tag{43}$$

Using (43), we get

$$W^\Delta(t) \leq -\frac{Lk^\alpha q(t)\delta(\sigma(t))h_2^\alpha(\tau(t),t_0)}{2^\alpha[r_1(\tau(t))\sigma(t)]^\alpha} + \frac{\delta^\Delta(t)}{\delta(t)}W(t) - \frac{\alpha k^\alpha \tau^\alpha(t)}{\sigma^\alpha(t)(\delta(t)r_2(t))^{1/\alpha}}W^{1+1/\alpha}(t), \quad t \in [t_2,\infty)_{\mathbb{T}}, \tag{44}$$

that is,

$$Q(t) \leq -W^\Delta(t) + \frac{\delta^\Delta(t)}{\delta(t)}W(t) - \frac{\alpha k^\alpha \tau^\alpha(t)}{\sigma^\alpha(t)(\delta(t)r_2(t))^{1/\alpha}}W^{1+1/\alpha}(t). \tag{45}$$

Now, set

$$X^\lambda = \frac{\alpha k^\alpha \tau^\alpha(t)}{\sigma^\alpha(t)(\delta(t)r_2(t))^{1/\alpha}}W^\lambda(t), \qquad Y^{\lambda-1} = \frac{\delta^\Delta(t)\left[\sigma^\alpha(t)(\delta(t)r_2(t))^{1/\alpha}\right]^{1/\lambda}}{\lambda\delta(t)[\alpha k^\alpha\tau^\alpha(t)]^{1/\lambda}}, \tag{46}$$

where $\lambda = (\alpha+1)/\alpha > 1$, $X \geq 0$ and $Y \geq 0$. Using the equality (24), we obtain

$$\frac{\delta^\Delta(t)}{\delta(t)}W(t) - \frac{\alpha k^\alpha \tau^\alpha(t)}{\sigma^\alpha(t)(\delta(t)r_2(t))^{1/\alpha}}W^{1+1/\alpha}(t) \leq \frac{r_2(t)\delta(t)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}}\left[\frac{\delta^\Delta(t)}{\delta(t)}\right]^{\alpha+1}\left[\frac{\sigma(t)}{\tau(t)}\right]^{\alpha^2}. \tag{47}$$

From (47), we obtain

$$Q(t) \leq -W^\Delta(t) + \frac{r_2(t)\delta(t)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}}\left[\frac{\delta^\Delta(t)}{\delta(t)}\right]^{\alpha+1}\left[\frac{\sigma(t)}{\tau(t)}\right]^{\alpha^2}. \tag{48}$$

Integrating (48) from T_0 to t, we get

$$\int_{T_0}^t Q(s)\Delta s \leq W(T_0) - W(t) + \int_{T_0}^t \frac{r_2(s)\delta(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}}\left[\frac{\delta^\Delta(s)}{\delta(s)}\right]^{\alpha+1}\left[\frac{\sigma(s)}{\tau(s)}\right]^{\alpha^2}\Delta s, \tag{49}$$

consequently,

$$\int_{T_0}^t \left\{Q(s) - \frac{r_2(s)\delta(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}}\left[\frac{\delta^\Delta(s)}{\delta(s)}\right]^{\alpha+1}\left[\frac{\sigma(s)}{\tau(s)}\right]^{\alpha^2}\right\}\Delta s \leq W(T_0) - W(t) \leq W(T_0). \tag{50}$$

This is contrary to (34).

If case (II) holds, from (26), we get $\lim_{t\to\infty}x(t) = 0$. This completes the proof. □

Theorem 9. *Assume that* (H_1)–(H_4), (26), *and* $r_2^\Delta(t) \geq 0$ *hold. Furthermore, suppose that there exist functions* $H, h \in C_{rd}(\mathbb{D},\mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,s) : t \geq s \geq T\}$ *such that*

$$H(t,t) = 0, \quad t \geq T; \qquad H(t,s) > 0, \quad t > s \geq T, \tag{51}$$

and H *has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *on* $\mathbb{D}$ *with respect to the second variable and satisfies, for all sufficiently large* $T \in [t_0,\infty)_{\mathbb{T}}$, *that there exists* $T_0 > T$, *such that*

$$H^{\Delta_s}(\sigma(t),s) + \frac{\delta^\Delta(s)}{\delta(s)}H(\sigma(t),s) = -\frac{h(t,s)}{\delta(s)}H^{\alpha/(\alpha+1)}(\sigma(t),s) \quad \textit{for } (t,s) \in \mathbb{D}, \tag{52}$$

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),T_0)} \times \int_{T_0}^{\sigma(t)}\left[H(\sigma(t),s)Q(s) - \frac{h_-^{\alpha+1}(t,s)r_2(s)\sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}\tau^{\alpha^2}(s)\delta^\alpha(s)}\right]\Delta s = \infty, \tag{53}$$

where $\delta(t)$ *and* $Q(t)$ *are defined in Theorem 8.* $h_-(t,s) = \max\{0, -h(t,s)\}, h_+(t,s) = \max\{0, h(t,s)\}$. *Then every solution* $x(t)$ *of* (1) *is either oscillatory or converges to zero.*

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [T,\infty)_{\mathbb{T}}$. In the case when $x(t)$ is eventually negative, the proof is similar. By Lemma 1, we see that $x(t)$ satisfies either case (I) or case (II). If case (I) holds, we proceed as in the proof of Theorem 8 and get (45). In

(45), replace t by s and multiply both sides by $H(\sigma(t), s)$ and integrate with respect to s from T_0 to $\sigma(t)$, $t \geq T_0$; we get

$$\begin{aligned}\int_{T_0}^{\sigma(t)} & H(\sigma(t), s) Q(s) \Delta s \\ & \leq -\int_{T_0}^{\sigma(t)} H(\sigma(t), s) W^{\Delta}(s) \Delta s \\ & \quad + \int_{T_0}^{\sigma(t)} H(\sigma(t), s) \frac{\delta^{\Delta}(s)}{\delta(s)} W(s) \Delta s \\ & \quad - \int_{T_0}^{\sigma(t)} H(\sigma(t), s) \times \frac{\alpha k^{\alpha} \tau^{\alpha}(s)}{\sigma^{\alpha}(s)(\delta(s) r_2(s))^{1/\alpha}} \\ & \qquad \times W^{1+1/\alpha}(s) \Delta s.\end{aligned} \tag{54}$$

Integrating by parts using (51) and (52), we obtain

$$\begin{aligned}\int_{T_0}^{\sigma(t)} & H(\sigma(t), s) Q(s) \Delta s \\ & \leq H(\sigma(t), T_0) W(T_0) + \int_{T_0}^{\sigma(t)} H^{\Delta_s}(\sigma(t), s) W(s) \Delta s \\ & \quad + \int_{T_0}^{\sigma(t)} \frac{H(\sigma(t), s) \delta^{\Delta}(s)}{\delta(s)} W(s) \Delta s \\ & \quad - \int_{T_0}^{\sigma(t)} \frac{\alpha k^{\alpha} H(\sigma(t), s) \tau^{\alpha}(s)}{\sigma^{\alpha}(s)(\delta(s) r_2(s))^{1/\alpha}} W^{1+1/\alpha}(s) \Delta s \\ & \leq H(\sigma(t), T_0) W(T_0) \\ & \quad + \int_{T_0}^{\sigma(t)} \left[-\frac{h(t,s) H^{\alpha/(1+\alpha)}(\sigma(t), s)}{\delta(s)} W(s) \right. \\ & \qquad \left. - \frac{\alpha k^{\alpha} H(\sigma(t), s) \tau^{\alpha}(s)}{\sigma^{\alpha}(s)(\delta(s) r_2(s))^{1/\alpha}} W^{1+1/\alpha}(s) \right] \Delta s,\end{aligned} \tag{55}$$

and so

$$\begin{aligned}\int_{T_0}^{\sigma(t)} & H(\sigma(t), s) Q(s) \Delta s \\ & \leq H(\sigma(t), T_0) W(T_0) \\ & \quad + \int_{T_0}^{\sigma(t)} \left[\frac{h_{-}(t,s) H^{\alpha/(1+\alpha)}(\sigma(t), s)}{\delta(s)} W(s) \right. \\ & \qquad \left. - \frac{\alpha k^{\alpha} H(\sigma(t), s) \tau^{\alpha}(s)}{\sigma^{\alpha}(s)(\delta(s) r_2(s))^{1/\alpha}} W^{1+1/\alpha}(s) \right] \Delta s.\end{aligned} \tag{56}$$

Now, set

$$\begin{aligned} X^{\lambda} &= \frac{\alpha k^{\alpha} H(\sigma(t), s) \tau^{\alpha}(s)}{\sigma^{\alpha}(s)(\delta(s) r_2(s))^{1/\alpha}} W^{\lambda}(s), \\ Y^{\lambda-1} &= \frac{h_{-}(t,s) \left[\sigma^{\alpha}(s)(\delta(s) r_2(s))^{1/\alpha} \right]^{1/\lambda}}{\lambda \delta(s) [\alpha k^{\alpha} \tau^{\alpha}(s)]^{1/\lambda}},\end{aligned} \tag{57}$$

where $\lambda = (\alpha+1)/\alpha > 1$, $X \geq 0$ and $Y \geq 0$. Using inequality (24), we obtain

$$\begin{aligned}\frac{h_{-}(t,s) H^{1/\lambda}(\sigma(t), s)}{\delta(s)} W(s) - \frac{\alpha k^{\alpha} H(\sigma(t), s) \tau^{\alpha}(s)}{\sigma^{\alpha}(s)(\delta(s) r_2(s))^{1/\alpha}} W^{\lambda}(s) \\ \leq \frac{h_{-}^{\alpha+1}(t,s) r_2(s) \sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1} k^{\alpha^2} \tau^{\alpha^2}(s) \delta^{\alpha}(s)}.\end{aligned} \tag{58}$$

Combining (56) and (58), we get

$$\begin{aligned}\frac{1}{H(\sigma(t), T_0)} & \\ \times \int_{T_0}^{\sigma(t)} & \left[H(\sigma(t), s) Q(s) - \frac{h_{-}^{\alpha+1}(t,s) r_2(s) \sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1} k^{\alpha^2} \tau^{\alpha^2}(s) \delta^{\alpha}(s)} \right] \Delta s \\ \leq W(T_0), &\end{aligned} \tag{59}$$

which contradicts (53).

If case (II) holds, from (26), we get $\lim_{t \to \infty} x(t) = 0$. This completes the proof. □

If (53) is not held, then we get the following result.

Theorem 10. *Assume that* (H_1)–(H_4), (26), *and* $r_2^{\Delta}(t) \geq 0$ *hold. Furthermore, suppose that there exist functions* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,s) : t \geq s \geq T\}$, *such that* (51) *holds, H has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *on* $\mathbb{D}$ *with respect to the second variable and satisfies* (52). *Assume that*

$$0 < \inf_{s \geq T_0} \left[\liminf_{t \to \infty} \frac{H(\sigma(t), s)}{H(\sigma(t), T_0)} \right] \leq \infty, \quad T_0 \in [t_0, \infty)_{\mathbb{T}}, \tag{60}$$

$$\limsup_{t \to \infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} \frac{h_{-}^{\alpha+1}(t,s) r_2(s) \sigma^{\alpha^2}(s)}{\tau^{\alpha^2}(s) \delta^{\alpha}(s)} \Delta s < \infty, \tag{61}$$

and a real rd-continuous function $\Psi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ *such that*

$$\int_T^{\infty} \frac{\tau^{\alpha}(s)}{\sigma^{\alpha}(s)\left(\delta(s) r_2(s)\right)^{1/\alpha}} \Psi_+^{1+1/\alpha}(s)\,\Delta s = \infty, \tag{62}$$

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(\sigma(t), T_0)} \\ &\quad \times \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t,s) r_2(s) \sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1} k^{\alpha^2} \tau^{\alpha^2}(s) \delta^{\alpha}(s)} \right] \Delta s \\ &\geq \Psi(T_0) \end{aligned} \tag{63}$$

for $T_0 \in (T, \infty)_{\mathbb{T}}$, *where* $\delta(t)$ *and* $Q(t)$ *are defined in Theorem 8,* $\Psi_+(t) = \max\{0, \Psi(t)\}$. *Then every solution* $x(t)$ *of* (1) *is either oscillatory or converges to zero.*

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. In the case when $x(t)$ is eventually negative, the proof is similar. By Lemma 1, we see that $x(t)$ satisfies either case (I) or case (II).

If case (I) holds, proceeding as in the proof of Theorem 9, we get that (56) and (58) hold. Then we conclude that

$$\begin{aligned} &\frac{1}{H(\sigma(t), T_0)} \\ &\quad \times \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t,s) r_2(s) \sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1} k^{\alpha^2} \tau^{\alpha^2}(s) \delta^{\alpha}(s)} \right] \Delta s \\ &\leq W(T_0). \end{aligned} \tag{64}$$

From (63), we obtain

$$\Psi(T_0) \leq W(T_0), \qquad T_0 \in (T, \infty)_{\mathbb{T}}, \tag{65}$$

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s)\,\Delta s \geq \Psi(T_0). \tag{66}$$

By (56), we get

$$\begin{aligned} &\frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s)\,\Delta s \\ &\quad \leq W(T_0) + \frac{1}{H(\sigma(t), T_0)} \\ &\qquad \times \int_{T_0}^{\sigma(t)} \frac{h_-(t,s) H^{\alpha/(1+\alpha)}(\sigma(t), s)}{\delta(s)} W(s)\,\Delta s \\ &\qquad - \frac{1}{H(\sigma(t), T_0)} \\ &\qquad \times \int_{T_0}^{\sigma(t)} \frac{\alpha k^{\alpha} H(\sigma(t), s) \tau^{\alpha}(s)}{\sigma^{\alpha}(s)\left(\delta(s) r_2(s)\right)^{1/\alpha}} W^{1+1/\alpha}(s)\,\Delta s. \end{aligned} \tag{67}$$

We denote

$$\begin{aligned} A(t) &= \frac{1}{H(\sigma(t), T_0)} \\ &\quad \times \int_{T_0}^{\sigma(t)} \frac{h_-(t,s) H^{\alpha/(1+\alpha)}(\sigma(t), s)}{\delta(s)} W(s)\,\Delta s, \\ B(t) &= \frac{1}{H(\sigma(t), T_0)} \\ &\quad \times \int_{T_0}^{\sigma(t)} \frac{\alpha k^{\alpha} H(\sigma(t), s) \tau^{\alpha}(s)}{\sigma^{\alpha}(s)\left(\delta(s) r_2(s)\right)^{1/\alpha}} W^{1+1/\alpha}(s)\,\Delta s, \end{aligned} \tag{68}$$

meanwhile noting that (63), we obtain

$$\liminf_{t\to\infty} [B(t) - A(t)] \leq W(T_0) - \Psi(T_0) < \infty. \tag{69}$$

Now we assert that

$$\int_T^{\infty} \frac{\tau^{\alpha}(s)}{\sigma^{\alpha}(s)\left(\delta(s) r_2(s)\right)^{1/\alpha}} W^{1+1/\alpha}(s)\,\Delta s < \infty \tag{70}$$

holds. Suppose to the contrary that

$$\int_T^{\infty} \frac{\tau^{\alpha}(s)}{\sigma^{\alpha}(s)\left(\delta(s) r_2(s)\right)^{1/\alpha}} W^{1+1/\alpha}(s)\,\Delta s = \infty, \tag{71}$$

by (60), there exists a constant $\varepsilon > 0$ such that

$$\inf_{s \geq T_0} \left[\liminf_{t\to\infty} \frac{H(\sigma(t), s)}{H(\sigma(t), T_0)} \right] > \varepsilon > 0, \tag{72}$$

from (71); there exists $T_1 \in [T_0, \infty)_{\mathbb{T}}$ for arbitrary real number $M > 0$ such that

$$\int_{T_1}^{\sigma(t)} \frac{\tau^{\alpha}(s)}{\sigma^{\alpha}(s)\left(\delta(s) r_2(s)\right)^{1/\alpha}} W^{1+1/\alpha}(s)\,\Delta s \geq \frac{M}{\alpha k^{\alpha} \varepsilon}, \quad \text{for } t \in [T_1, \infty)_{\mathbb{T}}. \tag{73}$$

By (10), we obtain

$$\begin{aligned}
B(t) &= \frac{1}{H(\sigma(t),T_0)} \\
&\quad \times \int_{T_0}^{\sigma(t)} \left\{ \alpha k^\alpha H(\sigma(t),s) \right. \\
&\quad \times \left(\int_{T_0}^{s} \frac{\tau^\alpha(u)}{\sigma^\alpha(u)(\delta(u) r_2(u))^{1/\alpha}} \right. \\
&\quad \left. \left. \times W^{1+1/\alpha}(u)\Delta u \right)^{\Delta_s} \right\} \Delta s \\
&= \frac{1}{H(\sigma(t),T_0)} \\
&\quad \times \int_{T_0}^{\sigma(t)} \left\{ -\alpha k^\alpha H^{\Delta_s}(\sigma(t),\sigma(s)) \right. \\
&\quad \times \int_{T_0}^{\sigma(s)} \frac{\tau^\alpha(u)}{\sigma^\alpha(u)(\delta(u) r_2(u))^{1/\alpha}} \\
&\quad \left. \times W^{1+1/\alpha}(u)\Delta u \right\} \Delta s \\
&\geq \frac{1}{H(\sigma(t),T_0)} \\
&\quad \times \int_{T_1}^{\sigma(t)} \left\{ -\alpha k^\alpha H^{\Delta_s}(\sigma(t),s) \right. \\
&\quad \times \int_{T_0}^{\sigma(s)} \frac{\tau^\alpha(u)}{\sigma^\alpha(u)(\delta(u) r_2(u))^{1/\alpha}} \\
&\quad \left. \times W^{1+1/\alpha}(u)\Delta u \right\} \Delta s \\
&\geq \frac{1}{H(\sigma(t),T_0)} \int_{T_1}^{\sigma(t)} -\alpha k^\alpha H^{\Delta_s}(\sigma(t),s) \frac{M}{\alpha k^\alpha \varepsilon} \Delta s \\
&= \frac{M}{\varepsilon} \frac{H(\sigma(t),T_1)}{H(\sigma(t),T_0)}.
\end{aligned} \tag{74}$$

From (72), there exists $T_2 \in [T_1,\infty)_{\mathbb{T}}$, we get $H(\sigma(t), T_1)/H(\sigma(t),T_0) \geq \varepsilon$ for $t \in [T_2,\infty)_{\mathbb{T}}$, so that $B(t) \geq M$. Since M is arbitrary, we obtain

$$\lim_{t\to\infty} B(t) = \infty. \tag{75}$$

Selecting a sequence $\{T_n\}_{n=1}^{\infty}$: $T_n \in [T_0,\infty)_{\mathbb{T}}$ with $\lim_{n\to\infty} T_n = \infty$ satisfying

$$\lim_{n\to\infty} [B(T_n) - A(T_n)] = \liminf_{t\to\infty} [B(t) - A(t)] < \infty, \tag{76}$$

then there exists a constant $M_0 > 0$ such that

$$B(T_n) - A(T_n) \leq M_0 \tag{77}$$

for sufficiently large positive integer n. From (75), we can easily see

$$\lim_{n\to\infty} B(T_n) = \infty; \tag{78}$$

(77) implies that

$$\lim_{n\to\infty} A(T_n) = \infty. \tag{79}$$

From (77) and (78), we obtain

$$\frac{A(T_n)}{B(T_n)} - 1 \geq -\frac{M_0}{B(T_n)} > -\frac{M_0}{2M_0} = -\frac{1}{2}, \tag{80}$$

that is,

$$\frac{A(T_n)}{B(T_n)} > \frac{1}{2} \tag{81}$$

for sufficiently large positive integer n, which together with (79) implies

$$\lim_{n\to\infty} \frac{[A(T_n)]^{\alpha+1}}{[B(T_n)]^{\alpha}} = \lim_{n\to\infty} \left[\frac{A(T_n)}{B(T_n)}\right]^{\alpha} A(T_n) = \infty. \tag{82}$$

On the other hand, by Lemma 7, we obtain

$$\begin{aligned}
&A(T_n) \\
&= \frac{1}{H(\sigma(T_n),T_0)} \\
&\quad \times \int_{T_0}^{\sigma(T_n)} \frac{h_-(T_n,s) H^{\alpha/(\alpha+1)}(\sigma(T_n),s)}{\delta(s)} W(s)\Delta s \\
&= \int_{T_0}^{\sigma(T_n)} \frac{h_-(T_n,s) H^{\alpha/(\alpha+1)}(\sigma(T_n),s)}{H(\sigma(T_n),T_0)\delta(s)} W(s)\Delta s \\
&= \int_{T_0}^{\sigma(T_n)} \left\{ \left[\frac{\alpha k^\alpha H(\sigma(T_n),s)\tau^\alpha(s)}{H(\sigma(T_n),T_0)} \right]^{\alpha/(\alpha+1)} \right. \\
&\quad \left. \times \frac{W(s)}{\sigma^{\alpha^2/(\alpha+1)}(s)(\delta(s) r_2(s))^{1/(\alpha+1)}} \right\} \\
&\quad \times \left\{ \left(h_-(T_n,s) H^{\alpha/(\alpha+1)}(\sigma(T_n),s) \right. \right. \\
&\quad \left. \times \sigma^{\alpha^2/(\alpha+1)}(s) r_2^{1/(\alpha+1)}(s) \right) \\
&\quad \times \left(H(\sigma(T_n),T_0)\delta^{\alpha/(\alpha+1)}(s) \right)^{-1} \\
&\quad \left. \times \left[\frac{\alpha k^\alpha H(\sigma(T_n),s)\tau^\alpha(s)}{H(\sigma(T_n),T_0)} \right]^{-\alpha/(\alpha+1)} \right\} \Delta s
\end{aligned}$$

$$\leq \left\{ \int_{T_0}^{\sigma(T_n)} \frac{\alpha k^\alpha H(\sigma(T_n), s)\tau^\alpha(s)}{H(\sigma(T_n), T_0)} \times \left[\frac{W(s)}{\sigma^{\alpha^2/(\alpha+1)}(s)(\delta(s) r_2(s))^{1/(\alpha+1)}} \right]^{(\alpha+1)/\alpha} \Delta s \right\}^{\alpha/(\alpha+1)}$$
$$\times \left\{ \int_{T_0}^{\sigma(T_n)} \frac{h_-^{\alpha+1}(T_n, s) H^\alpha(\sigma(T_n), s)\sigma^{\alpha^2}(s) r_2(s)}{H^{\alpha+1}(\sigma(T_n), T_0)\delta^\alpha(s)} \times \left[\frac{\alpha k^\alpha H(\sigma(T_n), s)\tau^\alpha(s)}{H(\sigma(T_n), T_0)} \right]^{-\alpha} \Delta s \right\}^{1/(\alpha+1)}$$
$$= [B(T_n)]^{\alpha/(\alpha+1)} \times \left\{ \frac{1}{\alpha^\alpha k^{\alpha^2} H(\sigma(T_n), T_0)} \times \int_{T_0}^{\sigma(T_n)} \frac{h_-^{\alpha+1}(T_n, s)\sigma^{\alpha^2}(s) r_2(s)}{\delta^\alpha(s)\tau^{\alpha^2}(s)} \Delta s \right\}^{1/(\alpha+1)}. \quad (83)$$

The above inequality shows that

$$\frac{[A(T_n)]^{\alpha+1}}{[B(T_n)]^\alpha} \leq \frac{1}{\alpha^\alpha k^{\alpha^2} H(\sigma(T_n), T_0)} \times \int_{T_0}^{\sigma(T_n)} \frac{h_-^{\alpha+1}(T_n, s)\sigma^{\alpha^2}(s) r_2(s)}{\delta^\alpha(s)\tau^{\alpha^2}(s)} \Delta s. \quad (84)$$

Hence, (82) implies

$$\lim_{n\to\infty} \frac{1}{H(\sigma(T_n), T_0)} \int_{T_0}^{\sigma(T_n)} \frac{h_-^{\alpha+1}(T_n, s) r_2(s)\sigma^{\alpha^2}(s)}{\tau^{\alpha^2}(s)\delta^\alpha(s)} \Delta s = \infty. \quad (85)$$

This contradicts (61). Therefore (70) holds. Noting $\Psi(T_0) \leq W(T_0)$ for $T_0 \in [T, \infty)_{\mathbb{T}}$, by using (70), we obtain

$$\int_T^\infty \frac{\tau^\alpha(s)}{\sigma^\alpha(s)(\delta(s) r_2(s))^{1/\alpha}} \Psi_+^{1+1/\alpha}(s)\Delta s \leq \int_T^\infty \frac{\tau^\alpha(s)}{\sigma^\alpha(s)(\delta(s) r_2(s))^{1/\alpha}} W^{1+1/\alpha}(s)\Delta s < \infty. \quad (86)$$

This contradicts (62). This completes the proof. □

If case (II) holds, from (26), we get $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Theorem 11. *Assume that (H_1)–(H_4), (26), (52), (60), (62), and $r_2^\Delta(t) \geq 0$ hold, where H, h, and δ are defined in Theorem 10. Furthermore suppose that there is a real rd-continuous function $\Psi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ such that*

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s)\Delta s < \infty, \quad (87)$$

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t), T_0)} \times \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t, s) r_2(s)\sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1} k^{\alpha^2}\tau^{\alpha^2}(s)\delta^\alpha(s)} \right] \Delta s \geq \Psi(T_0) \quad (88)$$

for $T_0 \in (T, \infty)_{\mathbb{T}}$, where $Q(t)$ is defined in Theorem 8, $\Psi_+(t) = \max\{0, \Psi(t)\}$. Then every solution $x(t)$ of (1) *is either oscillatory or converges to zero.*

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. In the case when $x(t)$ is eventually negative, the proof is similar. By Lemma 1, we see that $x(t)$ satisfies either case (I) or case (II).

If case (I) holds, proceeding as in the proof of Theorem 9, we get that (56) and (58) hold. We conclude that

$$\frac{1}{H(\sigma(t), T_0)} \times \int_{T_0}^{\sigma(t)} \left[H(\sigma(t), s) Q(s) - \frac{h_-^{\alpha+1}(t, s) r_2(s)\sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1} k^{\alpha^2}\tau^{\alpha^2}(s)\delta^\alpha(s)} \right] \Delta s \leq W(T_0). \quad (89)$$

From (88), we obtain

$$\Psi(T_0) \leq W(T_0), \qquad T_0 \in (T, \infty)_{\mathbb{T}}; \quad (90)$$

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s)\Delta s \geq \Psi(T_0), \quad (91)$$

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t), T_0)} \int_{T_0}^{\sigma(t)} H(\sigma(t), s) Q(s)\Delta s - \liminf_{t\to\infty} \frac{1}{H(\sigma(t), T_0)} \times \int_{T_0}^{\sigma(t)} \frac{h_-^{\alpha+1}(t, s) r_2(s)\sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1} k^{\alpha^2}\tau^{\alpha^2}(s)\delta^\alpha(s)} \Delta s$$

$$\begin{aligned}&\geq \liminf_{t\to\infty}\frac{1}{H(\sigma(t),T_0)}\\&\quad\times\int_{T_0}^{\sigma(t)}\left[H(\sigma(t),s)Q(s)\right.\\&\qquad\left.-\frac{h_-^{\alpha+1}(t,s)r_2(s)\sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}\tau^{\alpha^2}(s)\delta^\alpha(s)}\right]\Delta s\\&\geq \Psi(T_0).\end{aligned} \tag{92}$$

Using (87) and (92), we get

$$\liminf_{t\to\infty}\frac{1}{H(\sigma(t),T_0)}\int_{T_0}^{\sigma(t)}\frac{h_-^{\alpha+1}(t,s)r_2(s)\sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}\tau^{\alpha^2}(s)\delta^\alpha(s)}\Delta s<\infty. \tag{93}$$

Thus, there exists a sequence $\{T_n\}_{n=1}^{\infty}$: $T_n\in[T_0,\infty)_{\mathbb{T}}$ with $\lim_{n\to\infty}T_n=\infty$ such that

$$\lim_{n\to\infty}\frac{1}{H(\sigma(T_n),T_0)}\int_{T_0}^{\sigma(T_n)}\frac{h_-^{\alpha+1}(T_n,s)r_2(s)\sigma^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}\tau^{\alpha^2}(s)\delta^\alpha(s)}\Delta s<\infty. \tag{94}$$

We define $A(t)$ and $B(t)$ also, as in the proof of Theorem 10. From (56) and (91), we obtain

$$\begin{aligned}&\limsup_{t\to\infty}[B(t)-A(t)]\\&\quad\leq W(T_0)-\liminf_{t\to\infty}\frac{1}{H(\sigma(t),T_0)}\int_{T_0}^{\sigma(t)}H(\sigma(t),s)Q(s)\Delta s\\&\quad<\infty.\end{aligned} \tag{95}$$

For the above sequence $\{T_n\}_{n=1}^{\infty}$, we get

$$\lim_{n\to\infty}[B(T_n)-A(T_n)]\leq\limsup_{t\to\infty}[B(t)-A(t)]<\infty. \tag{96}$$

Similar to the proof of Theorem 10, we get (70). The rest proofs are the same as the Theorem 10, here omitted. This completes the proof. □

Remark 12. From Theorems 9, 10, and 11, we can obtain different sufficient conditions for the oscillation of (1) with different choices of the functions δ and H.

Remark 13. The theorems in this paper are new even for the cases of $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.

Example 14. Consider the third-order nonlinear delay dynamic equation

$$\begin{aligned}&\left(t^{2/3}\left[\left(\frac{1}{t}x^\Delta(t)\right)^\Delta\right]^{5/3}\right)^\Delta+\frac{1}{t^2}\left(x\left(\frac{t}{2}\right)\right)^{5/3}\left(1+x^2\left(\frac{t}{2}\right)\right)\\&\quad=0,\quad t\in\overline{2^{\mathbf{Z}}},\ t\geq t_0:=2.\end{aligned} \tag{97}$$

Here $r_1(t)=1/t$, $r_2(t)=t^{2/3}$, $q(t)=1/t^2$, $f(x)=x^{5/3}(1+x^2)$, $\tau(t)=t/2<t$, and $\alpha=5/3$.

Conditions (H_1)–(H_3) are clearly satisfied, and (H_4) holds with $L=1$. $r_2^\Delta(t)=((2t)^{2/3}-t^{2/3})/(2t-t)=(2^{2/3}-1)/t^{1/3}>0$, and

$$\begin{aligned}&\int_{t_0}^{\infty}q(s)\Delta s=\int_2^\infty\frac{1}{s^2}\Delta s=\left[-\frac{2}{s}\right]_2^\infty=1<\infty,\\&\int_{t_0}^{\infty}\frac{1}{r_1(t)}\int_t^\infty\left[\frac{1}{r_2(s)}\int_s^\infty q(u)\Delta u\right]^{1/\alpha}\Delta s\,\Delta t\\&\quad=\int_2^\infty\frac{1}{t}\int_t^\infty\left[\frac{1}{s^{2/3}}\int_s^\infty\frac{1}{u^2}\Delta u\right]^{3/5}\Delta s\,\Delta t\\&\quad=\int_2^\infty\frac{1}{t}\int_t^\infty\left[\frac{1}{s^{2/3}}\cdot\frac{2}{s}\right]^{3/5}\Delta s\,\Delta t\\&\quad=2^{3/5}\int_2^\infty\frac{1}{t}\int_t^\infty\frac{1}{s}\Delta s\,\Delta t=\infty,\end{aligned} \tag{98}$$

so (26) holds.

Let $\delta(t)=t^2$, then $\delta^\Delta(t)=3t\geq0$. Since

$$\begin{aligned}Q(t)&=Lq(t)\delta(\sigma(t))\left(\frac{kh_2(\tau(t),t_0)}{2r_1(\tau(t))\sigma(t)}\right)^\alpha\\&=1\cdot\frac{1}{t^2}\cdot(2t)^2\cdot\left(\frac{kh_2(\tau(t),t_0)}{2\cdot(2/t)\cdot2t}\right)^\alpha\\&=\frac{k^{5/3}}{8}(h_2(\tau(t),t_0))^{5/3}\\&\geq\frac{k^{5/3}}{8}h_2(\tau(t),t_0)\quad\text{as }t\longrightarrow\infty,\end{aligned} \tag{99}$$

$$\begin{aligned}&\frac{r_2(t)\delta(t)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}}\left[\frac{\delta^\Delta(t)}{\delta(t)}\right]^{\alpha+1}\left[\frac{\sigma(t)}{\tau(t)}\right]^{\alpha^2}\\&\quad=\frac{t^{2/3}\cdot t^2}{(8/3)^{8/3}\cdot k^{25/9}}\cdot\left(\frac{3t}{t^2}\right)^{8/3}\left(\frac{2t}{t/2}\right)^{25/9}\\&\quad=\left(\frac{9}{8}\right)^{8/3}\cdot4^{25/9}\cdot k^{-25/9},\end{aligned}$$

so that

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left\{Q(s)-\frac{r_2(s)\delta(s)}{(\alpha+1)^{\alpha+1}k^{\alpha^2}}\left[\frac{\delta^\Delta(s)}{\delta(s)}\right]^{\alpha+1}\left[\frac{\sigma(s)}{\tau(s)}\right]^{\alpha^2}\right\}\Delta s\\&\quad\geq\limsup_{t\to\infty}\int_2^t\left\{\frac{k^{5/3}}{8}h_2(\tau(s),2)-\left(\frac{9}{8}\right)^{8/3}\right.\\&\qquad\left.\cdot4^{25/9}\cdot k^{-25/9}\right\}\Delta s\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\to\infty} \left\{ \frac{k^{5/3}}{8} \int_2^t h_2(\tau(s),2)\,\Delta s - \left(\frac{9}{8}\right)^{8/3} \right. \\
&\qquad \left. \cdot 4^{25/9} \cdot k^{-25/9} \cdot \int_2^t \Delta s \right\} \\
&= \limsup_{t\to\infty} \left\{ \frac{k^{5/3}}{8} \int_2^t 2h_2(\tau(s),2)\,\tau^{\Delta}(s)\,\Delta s - \left(\frac{9}{8}\right)^{8/3} \right. \\
&\qquad \left. \cdot 4^{25/9} \cdot k^{-25/9} \cdot \int_2^t \Delta s \right\} \\
&= \limsup_{t\to\infty} \left\{ \frac{k^{5/3}}{4} \left[h_3(\tau(t),2) - h_3(1,2)\right] - \left(\frac{9}{8}\right)^{8/3} \right. \\
&\qquad \left. \cdot 4^{25/9} \cdot k^{-25/9} \cdot (t-2) \right\} \\
&= \limsup_{t\to\infty} \left\{ \frac{k^{5/3}}{4} \left[\frac{((t/2)-2)((t/2)-4)((t/2)-8)}{21} + 1 \right] \right. \\
&\qquad \left. - \left(\frac{9}{8}\right)^{8/3} \cdot 4^{25/9} \cdot k^{-25/9} \cdot (t-2) \right\} = \infty.
\end{aligned}
\tag{100}
$$

Then by Theorem 8, every solution $x(t)$ of (97) is either oscillatory or converges to zero. But the other known results cannot be applied in (97).

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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