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Research Article

Trudinger-Moser Embedding on the Hyperbolic Space

Yunyan Yang and Xiaobao Zhu

Department of Mathematics, Renmin University of China, Beijing 100872, China

Correspondence should be addressed to Xiaobao Zhu; zhuxiaobao@ruc.edu.cn

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Let $(\mathbb{H}^n, g)$ be the hyperbolic space of dimension n. By our previous work (Theorem 2.3 of (Yang (2012))), for any $0 < \alpha < \alpha_n$, there exists a constant $\tau > 0$ depending only on n and α such that $\sup_{u\in W^{1,n}(\mathbb{H}^n),\|u\|_{1,\tau}\le 1}\int_{\mathbb{H}^n}(e^{\alpha|u|^{n/(n-1)}} - \sum_{k=0}^{n-2}\alpha^k|u|^{nk/(n-1)}/k!)dv_g < \infty$, where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$, and $\|u\|_{1,\tau} = \|\nabla_g u\|_{L^n(\mathbb{H}^n)} + \tau\|u\|_{L^n(\mathbb{H}^n)}$. In this note we shall improve the above mentioned inequality. Particularly, we show that, for any $0 < \alpha < \alpha_n$ and any $\tau > 0$, the above mentioned inequality holds with the definition of $\|u\|_{1,\tau}$ replaced by $(\int_{\mathbb{H}^n}(|\nabla_g u|^n + \tau|u|^n)dv_g)^{1/n}$. We solve this problem by gluing local uniform estimates.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. The classical Trudinger-Moser inequality [1–3] says

$$\sup_{u\in W_0^{1,n}(\Omega),\|u\|_{W_0^{1,n}(\Omega)}\le 1}\int_\Omega e^{\alpha_n|u|^{n/(n-1)}}dx \le C|\Omega| \tag{1}$$

for some constant C depending only on n, where $W_0^{1,n}(\Omega)$ is the usual Sobolev space and $|\Omega|$ denotes the Lebesgue measure of Ω. In the case where Ω is an unbounded domain of $\mathbb{R}^n$, the above integral is infinite, but it was shown by Cao [4], Panda [5], and do Ó [6] that for any $\tau > 0$ and any $\alpha < \alpha_n$ there holds

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\le 1}\int_{\mathbb{R}^n}\left(e^{\alpha|u|^{n/(n-1)}} - \sum_{k=0}^{n-2}\frac{\alpha^k|u|^{nk/(n-1)}}{k!}\right)dx < \infty. \tag{2}$$

Later Ruf [7], Li and Ruf [8], and Adimurthi and Yang [9] obtained (2) in the critical case $\alpha = \alpha_n$.

The study of Trudinger-Moser inequalities on compact Riemannian manifolds can be traced back to Aubin [10], Cherrier [11, 12], and Fontana [13]. A particular case is as follows. Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. Then there holds

$$\sup_{\int_M|\nabla_g u|^n dv_g\le 1,\int_M u\,dv_g=0}\int_M e^{\alpha_n|u|^{n/(n-1)}}dv_g < \infty. \tag{3}$$

In view of (2), it is natural to consider extension of (3) on complete noncompact Riemannian manifolds. In [14] we obtained the following results. Let (M, g) be a complete noncompact Riemannian manifold. If the Trudinger-Moser inequality holds on it, then there holds $\inf_{x\in M}\mathrm{vol}_g(B_1(x)) > 0$. If the Ricci curvature has lower bound, say $\mathrm{Ric}_g(M) \ge -K$, the injectivity radius has a positive lower bound i_0 then for any $\alpha < \alpha_n$ there exists a constant $\tau > 0$ depending only on α, n, K, and i_0 such that

$$\sup_{(\int_M|\nabla u|^n dv_g)^{1/n}+\tau(\int_M|u|^n dv_g)^{1/n}\le 1}\int_M\left(e^{\alpha|u|^{n/(n-1)}} - \sum_{k=0}^{n-2}\frac{\alpha^k|u|^{nk/(n-1)}}{k!}\right)dv_g < \infty. \tag{4}$$

Since τ depends on α, (4) is weaker than (2) when (M, g) is replaced by $\mathbb{R}^n$. Moreover, the condition that $\mathrm{Ric}_g(M)$ has lower bound is not necessary for the validity of the Trudinger-Moser inequality.

In this note, we will continue to study (4) in whole $\mathbb{H}^n$ by gluing local uniform estimates. Particularly, we have the following.

Theorem 1. *Let $(\mathbb{H}^n, g)$ be an n-dimensional hyperbolic space, $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Then for any $\alpha < \alpha_n$, any $\tau > 0$, and any $u \in W^{1,n}(\mathbb{H}^n)$ satisfying $\int_{\mathbb{H}^n}(|\nabla_g u|^n+\tau|u|^n)dv_g \leq 1$, there exists some constant β depending only on n and τ such that*

$$\int_{\mathbb{H}^n}\left(e^{\alpha|u|^{n/(n-1)}} - \sum_{k=0}^{n-2}\frac{\alpha^k|u|^{nk/(n-1)}}{k!}\right)dv_g \leq \beta. \quad (5)$$

The proof of Theorem 1 is based on local uniform estimates (Lemma 2 below). This idea comes from [14] and can also be used in other cases [15, 16].

We remark that critical case of (5) was studied by Adimurthi and Tintarev [17], Mancini and Sandeep [18], and Mancini et al. ([19]) via different methods.

The remaining part of this note is organized as follows. In Section 2 we derive local uniform Trudinger-Moser inequalities; in Section 3, Theorem 1 is proved.

2. Local Estimates

To get (5), we need the following uniform local estimates which is an analogy of ([15], Lemma 4.1) or ([16], Lemma 1), and it is of its own interest.

Lemma 2. *For any $p \in \mathbb{H}^n$, any $R > 0$, and any $u \in W_0^{1,n}(B_R(p))$ with $\int_{B_R(p)}|\nabla_g u|^n dv_g \leq 1$, there exists some constant C_n depending only on n such that*

$$\int_{B_R(p)}\left(e^{\alpha_n|u|^{n/(n-1)}} - \sum_{k=0}^{n-2}\frac{\alpha_n^k|u|^{nk/(n-1)}}{k!}\right)dv_g \leq C_n(\sinh R)^n\int_{B_R(p)}\left|\nabla_g u\right|^n dv_g, \quad (6)$$

where $B_R(p)$ denotes the geodesic ball of $(\mathbb{H}^n, g)$ which is centered at p with radius R.

Proof. It is well known (see, e.g., [20], II.5, Theorem 1) that there exists a homomorphism $\varphi : \mathbb{H}^n \to D = \{x \in \mathbb{R}^n : |x| < 1\}$ such that $\varphi(p) = 0$, that in these coordinates the Riemannian metric g can be represented by

$$g(x) = \frac{4}{\left(1-|x|^2\right)^2}g_0(x), \quad (7)$$

where $g_0(x) = \sum_{i=1}^n(dx^i)^2$ is the standard Euclidean metric on $\mathbb{R}^n$, and that

$$\varphi(B_R(p)) = \mathbb{B}_{\tanh R/2}(0), \quad (8)$$

where $\mathbb{B}_r(0) \subset \mathbb{R}^n$ denotes a ball centered at 0 with radius r. Moreover, the corresponding polar coordinates $(r,\theta) \in [0,\infty)\times\mathbb{S}^{n-1}$ read

$$g = dr^2 + (\sinh r)^2 d\theta^2, \quad (9)$$

where $d\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$.

Denote $f = 2/(1-|x|^2)$; then $g = f^2 g_0$, $|\nabla_g u| = f^{-1}|\nabla_{g_0}(u\circ\varphi^{-1})|$, and $dv_g = f^n dv_{g_0}$. Calculating directly, we have

$$\int_{B_R(0)}\left|\nabla_g u\right|^n dv_g = \int_{\mathbb{B}_{\tanh R/2}(0)}\left|\nabla_{g_0}\left(u\circ\varphi^{-1}\right)\right|^n dv_{g_0}. \quad (10)$$

Since $u \in W_0^{1,n}(B_R(p))$, we have $u\circ\varphi^{-1} \in W_0^{1,n}(\mathbb{B}_{\tanh R/2}(0))$. Noting that $\int_{B_R(p)}|\nabla_g u|^n dv_g \leq 1$, we have by (10)

$$\int_{\mathbb{B}_{\tanh R/2}(0)}\left|\nabla_{g_0}\left(u\circ\varphi^{-1}\right)\right|^n dv_{g_0} \leq 1. \quad (11)$$

The standard Trudinger-Moser inequality (1) implies

$$\begin{aligned}
&\int_{\mathbb{B}_{\tanh R/2}(0)}\left(e^{\alpha_n|u\circ\varphi^{-1}|^{n/(n-1)}} - \sum_{k=0}^{n-2}\frac{\alpha_n^k\left|u\circ\varphi^{-1}\right|^{nk/(n-1)}}{k!}\right)dv_{g_0}\\
&= \int_{\mathbb{B}_{\tanh R/2}(0)}\sum_{k=n-1}^{\infty}\frac{\alpha_n^k\left|u\circ\varphi^{-1}\right|^{nk/(n-1)}}{k!}dv_{g_0}\\
&\leq \int_{\mathbb{B}_{\tanh R/2}(0)}\sum_{k=n-1}^{\infty}\frac{\alpha_n^k\left|\left(u\circ\varphi^{-1}\right)/\left\|\nabla_{g_0}\left(u\circ\varphi^{-1}\right)\right\|_{L^n}\right|^{nk/(n-1)}}{k!}dv_{g_0}\\
&\quad\times\int_{\mathbb{B}_{\tanh R/2}(0)}\left|\nabla_{g_0}\left(u\circ\varphi^{-1}\right)\right|^n dv_{g_0}\\
&\leq C_n\left(\tanh\frac{R}{2}\right)^n\int_{\mathbb{B}_{\tanh R/2}(0)}\left|\nabla_{g_0}\left(u\circ\varphi^{-1}\right)\right|^n dv_{g_0},
\end{aligned} \quad (12)$$

where C_n is a constant depending only on n. This together with (10) immediately leads to

$$\begin{aligned}
&\int_{B_R(p)}\left(e^{\alpha_n|u|^{n/(n-1)}} - \sum_{k=0}^{n-2}\frac{\alpha_n^k|u|^{nk/(n-1)}}{k!}\right)dv_g\\
&= \int_{\mathbb{B}_{\tanh R/2}(0)}\left(e^{\alpha_n|u\circ\varphi^{-1}|^{n/(n-1)}} - \sum_{k=0}^{n-2}\frac{\alpha_n^k\left|u\circ\varphi^{-1}\right|^{nk/(n-1)}}{k!}\right)f^n dv_{g_0}\\
&\leq C_n\left(\frac{2\tanh R/2}{1-(\tanh R/2)^2}\right)^n\int_{\mathbb{B}_{\tanh R/2}(0)}\left|\nabla_{g_0}\left(u\circ\varphi^{-1}\right)\right|^n dv_{g_0}\\
&= C_n(\sinh R)^n\int_{B_R(p)}\left|\nabla_g u\right|^n dv_g.
\end{aligned} \quad (13)$$

This is exactly (6) and thus ends the proof of the lemma. $\square$

As a corollary of Lemma 2, the following estimates can be compared with (1).

Corollary 3. *For any $p \in \mathbb{H}^n$, any $R > 0$, and any $u \in W_0^{1,n}(B_R(p))$ with $\int_{B_R(p)} |\nabla_g u|^n dv_g \leq 1$, there exists some constant C depending only on n such that*

$$\frac{1}{Vol_g(B_R(p))} \int_{B_R(p)} e^{\alpha_n |u|^{n/(n-1)}} dv_g \leq C \frac{\sinh R}{R}. \qquad (14)$$

Proof. Since

$$\lim_{R \to 0+} \frac{\mathrm{Vol}_g(B_R(p))}{R(\sinh R)^{n-1}} = \lim_{R \to \infty} \frac{\mathrm{Vol}_g(B_R(p))}{R(\sinh R)^{n-1}} = 1, \qquad (15)$$

it follows from (13) that there exists some constant C depending only on n such that

$$\frac{1}{\mathrm{Vol}_g(B_R(p))} \int_{B_R(p)} \left(e^{\alpha_n |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{nk/(n-1)}}{k!} \right) dv_g \leq C \frac{\sinh R}{R}. \qquad (16)$$

In particular,

$$\int_{B_R(p)} |u|^n dv_g \leq C \frac{\sinh R}{R} \mathrm{Vol}_g(B_R(p)). \qquad (17)$$

Here and in the sequel we often denote various constants by the same C; the reader can easily distinguish them from the context. Noting that for any q, $0 \leq q \leq n$,

$$\int_{B_R(p)} |u|^q dv_g \leq \mathrm{Vol}_g(B_R(p)) + \int_{B_R(p)} |u|^n dv_g, \qquad (18)$$

we conclude

$$\int_{B_R(p)} \sum_{k=0}^{n-2} \frac{\alpha_n^k |u|^{nk/(n-1)}}{k!} dv_g \leq C \frac{\sinh R}{R} \mathrm{Vol}_g(B_R(p)). \qquad (19)$$

Combining (16) and (19), we obtain (14). □

3. Proof of Theorem 1

In this section, we will prove Theorem 1 by gluing local estimates (6).

Proof of Theorem 1. Let R be a positive real number which will be determined later. By ([21], Lemma 1.6) we can find a sequence of points $\{x_i\}_{i=1}^\infty \subset \mathbb{H}^n$ such that $\cup_{i=1}^\infty B_{R/2}(x_i) = \mathbb{H}^n$, that $B_{R/4}(x_i) \cap B_{R/4}(x_j) = \varnothing$ for any $i \neq j$, and that for any $x \in \mathbb{H}^n$, x belongs to at most N balls $B_R(x_i)$, where N depends only on n. Let ϕ_i be the cut-off function satisfies the following conditions: (i) $\phi_i \in C_0^\infty(B_R(x_i))$; (ii) $0 \leq \phi_i \leq 1$ on $B_R(x_i)$ and $\phi_i \equiv 1$ on $B_{R/2}(x_i)$; (iii) $|\nabla_g \phi_i(x)| \leq 4/R$. Let $\tau > 0$ be fixed. For any $u \in W^{1,n}(\mathbb{H}^n)$ satisfying

$$\int_{\mathbb{H}^n} \left(|\nabla_g u|^n + \tau |u|^n \right) dv_g \leq 1, \qquad (20)$$

we have $\phi_i u \in W_0^{1,n}(B_R(x_i))$. For any $\epsilon > 0$, using an elementary inequality $ab \leq \epsilon a^2 + (1/(4\epsilon))b^2$, we find some constant C depending only on n and ϵ such that

$$\begin{aligned} &\int_{B_R(x_i)} \left|\nabla_g(\phi_i u)\right|^n dv_g \\ &\leq (1+\epsilon) \int_{B_R(x_i)} \phi_i^n |\nabla_g u|^n dv_g + C \int_{B_R(x_i)} |\nabla_g \phi_i|^n |u|^n dv_g \\ &\leq (1+\epsilon) \int_{B_R(x_i)} |\nabla_g u|^n dv_g + \frac{4^n C}{R^n} \int_{B_R(x_i)} |u|^n dv_g \\ &\leq (1+\epsilon) \int_{B_R(x_i)} \left(|\nabla_g u|^n + \tau |u|^n \right) dv_g, \end{aligned} \qquad (21)$$

where in the last inequality we choose a sufficiently large R to make sure $4^n C/R^n \leq (1+\epsilon)\tau$. Let $\alpha_\epsilon = \alpha_n/(1+\epsilon)^{1/(n-1)}$ and $\widetilde{\phi_i u} = \phi_i u/(1+\epsilon)^{1/n}$. Noting that $\widetilde{\phi_i u} \in W_0^{1,n}(B_R(x_i))$, we have by (21) and Lemma 2

$$\begin{aligned} &\int_{B_{R/2}(x_i)} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ &\leq \int_{B_R(x_i)} \left(e^{\alpha_\epsilon |\phi_i u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |\phi_i u|^{nk/(n-1)}}{k!} \right) dv_g \\ &= \int_{B_R(x_i)} \left(e^{\alpha_n |\widetilde{\phi_i u}|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_n^k |\widetilde{\phi_i u}|^{nk/(n-1)}}{k!} \right) dv_g \\ &\leq C_n (\sinh R)^n \int_{B_R(x_i)} \left|\nabla_g(\widetilde{\phi_i u})\right|^n dv_g \\ &\leq C (\sinh R)^n \int_{B_R(x_i)} \left(|\nabla_g u|^n + \tau |u|^n \right) dv_g, \end{aligned} \qquad (22)$$

where C is a constant depending only on n and τ. By the choice of $\{x_i\}_{i=1}^\infty$ and (22), we have

$$\begin{aligned} &\int_{\mathbb{H}^n} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ &\leq \int_{\cup_{i=1}^\infty B_{R/2}(x_i)} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ &\leq \sum_{i=1}^\infty \int_{B_{R/2}(x_i)} \left(e^{\alpha_\epsilon |u|^{n/(n-1)}} - \sum_{k=0}^{n-2} \frac{\alpha_\epsilon^k |u|^{nk/(n-1)}}{k!} \right) dv_g \\ &\leq \sum_{i=1}^\infty C (\sinh R)^n \int_{B_R(x_i)} \left(|\nabla_g u|^n + \tau |u|^n \right) dv_g \\ &\leq CN (\sinh R)^n \int_{\mathbb{H}^n} \left(|\nabla_g u|^n + \tau |u|^n \right) dv_g \\ &\leq CN (\sinh R)^n \end{aligned} \qquad (23)$$

for some constant C depending only on n and τ. For any $\alpha < \alpha_n$, we can choose $\epsilon > 0$ sufficiently small such that $\alpha < \alpha_\epsilon$. This ends the proof of Theorem 1. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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