

Research Article

Exponential Stability Analysis for Genetic Regulatory Networks with Both Time-Varying and Continuous Distributed Delays

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The global exponential stability is investigated for genetic regulatory networks with time-varying delays and continuous distributed delays. By choosing an appropriate Lyapunov-Krasovskii functional, new conditions of delay-dependent stability are obtained in the form of linear matrix inequality (LMI). The lower bound of derivatives of time-varying delay is first taken into account in genetic networks stability analysis, and the main results with less conservatism are established by interactive convex combination method to estimate the upper bound of derivative function of the Lyapunov-Krasovskii functional. In addition, two numerical examples are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

From the late 20th century to the early 21st century, more than a decade's time, life science, especially molecular biology science, had great surprising changes. The research of genetic regulatory networks has become an important area in the molecular biology science and received great attention. There are plenty of results [1, 2]. Genetic regulatory networks can be seen as biochemically dynamical systems, and it is natural to simulate them by using dynamical system model. There are a variety of models that have been proposed but mainly are the Boolean network model (discrete model) [3–6] and the differential equation model (continuous model) [7, 8]. In the Boolean models, each gene's activity is expressed with ON or OFF, and each gene's state is described by the Boolean function of other related genes' states. In differential equation model, the variables delineate the concentrations of gene products, such as mRNAs and proteins, which are continuous values. According to a large number of biological experiments, we know that gene expression is usually continuously variable, so it is

more reasonable to describe genetic regulation networks with differential equation model than with the Boolean network model.

Gene expression is a complex process regulation by the stimulation and inhibition of protein including transcription, translation, and posttranslation processes, and a large number of reactions and reacting species participate in this process. There are fast reaction and slow reaction in real genetic regulatory systems. The fast reaction includes dimerization, binding reaction, and phosphorylation, and the slow reaction contains transcription, translation, and translocation or the finite switching speed of amplifiers. Due to the slow reaction, time delays exist in genetic regulatory networks. In [9], there is a biochemistry experiment on mice which has proved that there exists a time lag of about 15 min in the peaks between the mRNA molecules and the proteins of the gene *Hes1*.

The emergence of the time delays will influence the genetic regulatory networks' dynamic behaviors, which cause the researchers' interest. The stability is one of the very important dynamic characteristics. Hence, it is necessary

to consider the stability of genetic regulatory networks with time delay; see [10–16]. In [11], random time delays are taken into account, and some stability criteria for the uncertain delayed genetic networks with SUM regulatory logic where each transcription factor acts additively to regulate a gene were obtained; asymptotical stability criteria were proposed for genetic regulatory networks with interval time-varying delays and nonlinear disturbance in [12]. In [14], authors gave some stochastic asymptotic stability sufficient conditions for the uncertain stochastic genetic regulatory networks with both mixed time-varying delays by constructing the Lyapunov functional and employing stochastic analysis methods. In [15, 16], the authors studied genetic regulatory networks with constant delay.

Motivated by the above discussions, we analyze the exponential stability of genetic regulatory networks with time-varying delays and continuous distributed delays. It is worth mentioning that the asymptotical stability of genetic regulatory networks is studied in most literature; see [17, 18]. But the exponential stability of our research is of better stability than asymptotical stability. Literature [11] discusses the exponential stability of genetic regulatory networks with random time delays, which are essentially constant delays. Our paper considers the system with interval time-varying delays and continuous distributed delays, which is more reasonable than literature [11]. Literature [19] studies robust exponential stability for stochastic genetic regulatory networks with time-varying delays, whose derivatives' upper bound is less than 1. In our models, the derivatives' upper bound of interval time-varying delays has no limit of less than 1. And we study the lower bound of derivatives of time-varying delay to systems stability effect for the first time. In the theorem for evidence, convex combination and interactive convex combination method were adopted, which have less conservatism.

This paper is organized as follows. In Section 2, model description and some assumptions are given. In Section 3, several sufficient results are obtained to check the exponential stability for genetic regulatory networks with time-varying delays and continuous distributed delays. Some numerical examples are given to demonstrate the effectiveness of our analysis in Section 4. Finally, conclusions are drawn in Section 5.

Notations. Throughout this paper, \mathbf{R} , \mathbf{R}^n , and $\mathbf{R}^{n \times m}$ denote, respectively, the set of all real numbers, real n -dimensional space, and real $n \times m$ -dimensional space. \mathbf{Z}_+ denote the set of all positive integers. $\|\cdot\|$ denote the Euclidean norms in \mathbf{R}^n . I and 0 denote, respectively, the identity matrix and the zero matrix with appropriate dimension. For a vector or matrix A , A^T denotes its transpose. For a square matrix A , $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum eigenvalue and minimum eigenvalue of matrix A , respectively, and $\text{sym}(A)$ is used to represent $A + A^T$. For simplicity, in symmetric block matrices, we often use $*$ to represent the term that is induced by symmetry.

2. Problem Formulation and Some Preliminaries

In this paper, we are devoted to studying the stability to an autoregulatory genetic network with time delays described by the following delay differential equations:

$$\dot{m}_i(t) = -a_i m_i(t) + \omega_i(p_1(t - \sigma(t)), p_n(t - \sigma(t))), \quad i = 1, \dots, n, \quad (1)$$

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t - \tau(t)), \quad i = 1, \dots, n,$$

where $m_i(t)$'s and $p_i(t)$'s are the concentrations of mRNAs and proteins, respectively; a_i 's and c_i 's are the degradation rates of mRNAs and proteins, respectively; d_i 's are the translation rates of proteins; $\omega_i(\cdot)$'s are the regulatory functions of mRNAs, being generally monotonic to each argument; and $\sigma(t)$ and $\tau(t)$ are time-varying delays.

Assumption 1. $\sigma(t)$ and $\tau(t)$ are the time-varying delay satisfying

$$\begin{aligned} 0 \leq \sigma_1 \leq \sigma(t) \leq \sigma_2, & \quad \sigma_3 \leq \dot{\sigma}(t) \leq \sigma_4 < \infty, \\ 0 \leq \tau_1 \leq \tau(t) \leq \tau_2, & \quad \tau_3 \leq \dot{\tau}(t) \leq \tau_4 < \infty, \end{aligned} \quad (2)$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \tau_1, \tau_2, \tau_3, \tau_4$ are some constants.

Remark 2. In this paper, the lower bound of derivatives of time-varying delay is considered for the first time in the research of genetic regulatory networks. When information on lower bound of time-varying delay's derivatives can be measured, our results are better than the previous works.

In genetic regulatory networks, some genes can be activated by one of a few different transcription factors ("OR" logic), and others can be activated by two or more transcription factors which must be bounded at the same time ("AND" logic). In this paper, we take a model of genetic regulatory networks where each transcription factor acts additively to regulate the i th gene ("SUM" logic) [20]. The regulatory function takes the form $\omega_i(p_1(t), \dots, p_n(t)) = \sum_{j=1}^n \omega_{ij}(p_j(t))$, and $\omega_{ij}(p_j(t))$ is a monotonic function with the following Hill form [21]:

$$\omega_{ij}(p_j(t)) = \begin{cases} \alpha_{ij} \frac{(p_j(t)/\beta_j)^{H_j}}{1 + (p_j(t)/\beta_j)^{H_j}} & \text{if transcription factor } j \text{ is an} \\ & \text{activator of gene } i, \\ 1 & \\ \alpha_{ij} \frac{1}{1 + (p_j(t)/\beta_j)^{H_j}} & \text{if transcription factor } j \text{ is a} \\ & \text{repressor of gene } i, \end{cases} \quad (3)$$

where H_j is the Hill coefficient, β_j is a positive constant, and α_{ij} is the constant transcriptional rate of j th transcriptional factor to i th gene.

Therefore, (1) can be rewritten as

$$\dot{m}_i(t) = -a_i m_i(t) + \sum_{j=1}^n b_{ij} f_j(p_j(t - \sigma(t))) + e_i, \quad (4)$$

$$i = 1, \dots, n,$$

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t - \tau(t)), \quad i = 1, \dots, n,$$

where $f_j(x) = (x/\beta_j)^{H_j}/(1 + (x/\beta_j)^{H_j})$ is monotonically increasing function; e_i 's are basal rate defined by $e_i = \sum_{j \in U_k} \alpha_{ij}$ with $U_k = \{j \mid \text{the } j\text{th transcription factor being a repressor of the } k\text{th gene, } j = 1, \dots, n\}$; and matrix $B = (b_{ij}) \in \mathbf{R}^{n \times n}$ is defined as

$$b_{ij} = \begin{cases} \alpha_{ij}, & \text{if transcription factor } j \text{ is an activator} \\ & \text{of gene } i, \\ 0, & \text{if there is no link from node } j \text{ to node } i, \\ -\alpha_{ij}, & \text{if transcription factor } j \text{ is a repressor} \\ & \text{of gene } i. \end{cases} \quad (5)$$

In the compact matrix form, (4) can be rewritten as

$$\dot{m}(t) = -Am(t) + Bf(p(t - \sigma(t))) + E, \quad (6)$$

$$\dot{p}(t) = -CP(t) + Dm(t - \tau(t)),$$

where

$$m(t) = [m_1(t), \dots, m_n(t)]^T, \quad (7)$$

$$p(t) = [p_1(t), \dots, p_n(t)]^T,$$

$$f(p(t)) = [f_1(p_1(t)), \dots, f_n(p_n(t))]^T,$$

$$A = \text{diag}(a_1, \dots, a_n), \quad I = [e_1, \dots, e_n]^T,$$

$$C = \text{diag}(c_1, \dots, c_n), \quad D = \text{diag}(d_1, \dots, d_n).$$

In order to get the stability results, the following assumption is necessarily imposed on (6).

Assumption 3. $f_i : \mathbf{R} \rightarrow \mathbf{R}, i = 1, \dots, n$, are monotonically increasing functions with saturation and moreover satisfy

$$m_i^- \leq \frac{f_i(a) - f_i(b)}{a - b} \leq m_i^+, \quad \forall a, b \in \mathbf{R}, \quad i = 1, \dots, n, \quad (8)$$

where m_i^- and m_i^+ are constants.

Remark 4. In the most existing literature, (8) was strengthened to $0 \leq ((f_i(a) - f_i(b))/(a - b)) \leq l_i$, for all $a, b \in \mathbf{R}$, where l_i 's are positive constant. Therefore, Assumption 1 is somewhat general and, in fact, similar to that of [12]; see the discussion below.

The vectors m^*, p^* are said to be an equilibrium point of system (6), if they satisfy

$$0 = -Am^* + Bf(p^*(t - \sigma(t))) + E, \quad (9)$$

$$0 = -CP^* + Dm^*(t - \tau(t)).$$

Let $x(t) = m(t) - m^*$ and let $y(t) = p(t) - p^*$; we get

$$\dot{x}(t) = -Ax(t) + Bg(y(t - \sigma(t))), \quad (10)$$

$$\dot{y}(t) = -Cy(t) + Dx(t - \tau(t)),$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T, y(t) = [y_1(t), \dots, y_n(t)]^T, g(y(t)) = [g(y_1(t)), \dots, g(y_n(t))]^T$, and $g_i(y_i(t)) = f_i(y_i(t) + p_i^*) - f_i(p_i^*)$.

By the definition of $g_i(\cdot)$, it satisfies sector condition:

$$m_i^- \leq \frac{g_i(a)}{a} \leq m_i^+, \quad i = 1, \dots, n, \quad (11)$$

which implies that

$$\frac{g_i(a) - m_i^- a}{a} \geq 0, \quad \frac{m_i^+ a - g_i(a)}{a} \geq 0. \quad (12)$$

Let $M_0 = \text{diag}(m_1^-, \dots, m_n^-), M_1 = \text{diag}(m_1^+, \dots, m_n^+)$, and $m = \max\{|m_1^-|, \dots, |m_n^-|, |m_1^+|, \dots, |m_n^+|\}$.

The initial condition of system (10) is assumed to be

$$x(t) = \varphi_1(t), \quad y(t) = \psi_1(t), \quad -\rho \leq t \leq 0, \quad (13)$$

$$\rho = \max\{\sigma_2, \tau_2\}.$$

Based on system (10), we also consider the genetic regulatory networks with continuous distributed delays:

$$\dot{x}(t) = -Ax(t) + Bg(y(t - \sigma(t))) + W \int_{t-\vartheta(t)}^t g(y(s)) ds, \quad (14)$$

$$\dot{y}(t) = -Cy(t) + Dx(t - \tau(t)),$$

where $0 \leq \vartheta(t) \leq \vartheta_1$.

For completeness, we recall the following definition and lemmas.

Definition 5. System (10) or (14) is said to be globally exponentially stable, if there exist constants $\lambda > 0$ and $M \geq 1$, such that, for any initial value z_{t_0} ,

$$\|z(t)\| \leq M \|z_{t_0}\|_{C^1} e^{-\lambda(t-t_0)} \quad (15)$$

hold, for all $t \geq 0$, where $z(t) = [x(t), y(t)]^T$ and $\|z(t)\|_{C^1} = \sup_{-\rho \leq \theta \leq 0} \{\|z(t + \theta)\|, \|\dot{z}(t + \theta)\|\}$.

Lemma 6 (see [22]). For any positive definite matrix $M \in \mathbf{R}^{n \times n}$, there exists a scalar $q > 0$ and a vector-valued function $\omega : [0, q] \rightarrow \mathbf{R}^n$ such that

$$\left(\int_0^q \omega(s) ds \right)^T M \left(\int_0^q \omega(s) ds \right) \leq q \int_0^q \omega^T(s) M \omega(s) ds. \quad (16)$$

Lemma 7 (see [23]). Let $h_1, \dots, h_N : \mathbf{R}^m \rightarrow \mathbf{R}$ take positive values in an open subset \mathbf{D} of \mathbf{R}^m . Then, the reciprocally convex combination of h_i over \mathbf{D} satisfies

$$\min_{\{\alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} h_i(\eta) = \sum_i h_i(\eta) + \max_{k_{i,j}(\eta)_{i \neq j}} \sum_i k_{i,j}(\eta) \quad (17)$$

subject to

$$\left\{ k_{i,j} : \mathbf{R}^m \mapsto \mathbf{R}, k_{j,i}(\eta) = k_{i,j}(\eta), \right. \\ \left. \begin{bmatrix} h_i(\eta) & k_{i,j}(\eta) \\ k_{i,j}(\eta) & h_j(\eta) \end{bmatrix} \geq 0 \right\}. \quad (18)$$

3. Main Results

In this section, several theorems are presented of genetic regulatory networks with both time-varying delays and continuous distributed delays. Firstly, a globally exponential stability result is developed for the genetic regulatory network with time-varying delays.

Theorem 8. For system (10) with Assumptions 1 and 3, the equilibrium point is globally exponentially stable (that is, there are two positive constants α and λ such that $\|z(t)\| \leq \alpha e^{-\lambda t} \|z(t_0)\|_{C_1}$, for all $t \geq t_0$) if there exist positive definite matrices $H_2, H_4, P_1, P_2, Q_i, i = 1, \dots, 7, W_i, i = 1, \dots, 4, \Gamma_1 = \text{diag}(\gamma_{11}, \dots, \gamma_{1n})$, and $\Gamma_2 = \text{diag}(\gamma_{21}, \gamma_{22}, \dots, \gamma_{2n})$, such that, for any appropriate dimensions constant matrices H_1, H_3, X, Y , the following LMIs hold:

$$\begin{pmatrix} W_2 & X \\ X & W_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} W_4 & Y \\ Y & W_4 \end{pmatrix} \geq 0, \quad (19)$$

$$\begin{aligned} \Omega_1 + \Omega_2 + O_3^T [e^{-\lambda \tau_2} Q_2 + e^{-\lambda \tau_1} Q_3] O_3 \\ + O_8^T [e^{-\lambda \sigma_2} Q_5 + e^{-\lambda \sigma_1} Q_6] O_8 < 0, \end{aligned} \quad (20)$$

where $\Omega_1 = \text{diag}(\lambda P_1 + Q_1 - e^{-\lambda \tau_1} W_1 - H_1^T A - A^T H_1, e^{-\lambda \tau_1} (Q_1 - Q_2) - e^{-\lambda \tau_1} W_1 - e^{-\lambda \tau_2} W_2, e^{-\lambda \tau_1} (1 - \tau_4) Q_3 - e^{-\lambda \tau_2} [(1 - \tau_3) Q_2 + 2W_2 - X^T - X], -e^{-\lambda \tau_2} W_2, \tau_1^2 W_1 + (\tau_2 - \tau_1)^2 W_2 - H_2^T - H_2, \lambda P_2 + Q_4 - e^{-\lambda \sigma_1} W_3 - 2M_1^T \Gamma_1 M_0 - H_3^T C - C^T H_3, -e^{-\lambda \sigma_1} (Q_4 - Q_5 + W_3) - e^{-\lambda \sigma_2} W_4, e^{-\lambda \sigma_1} (1 - \sigma_4) Q_6 - e^{-\lambda \sigma_2} [(1 - \sigma_3) Q_5 + 2W_4 - Y^T - Y] - M_1^T \Gamma_2 M_0, -e^{-\lambda \sigma_2} (Q_4 + W_4), \sigma_1^2 W_3 + (\sigma_2 - \sigma_1)^2 W_4 - H_4^T - H_4, Q_7 - 2\Gamma_1, -e^{-\lambda \sigma_2} Q_7 (1 - \sigma_d) - 2\Gamma_2), \Omega_2 = \text{sym}(O_1^T e^{-\lambda \tau_1} W_1 O_2 + O_1^T [P_1 - H_1^T - H_2^T A] O_5 + O_1^T H_1^T B O_{12} + O_2^T e^{-\lambda \tau_2} (W_2 - X) O_3 + O_2^T e^{-\lambda \tau_2} X O_4 + O_3^T e^{-\lambda \tau_2} (W_2 - X) O_4 + O_3^T H_3^T D O_6 + O_3^T H_4^T D O_{10} + O_5^T H_2^T B O_{12} + O_6^T e^{-\lambda \sigma_1} W_3 O_7 + O_6^T [P_2 - H_3^T - H_4^T C] O_{10} + O_6^T [\Gamma_1 M_0 + M_1^T \Gamma_1] O_{11} + O_7^T e^{-\lambda \sigma_2} (W_4 - Y) O_8 + O_7^T e^{-\lambda \sigma_2} Y O_9 + O_8^T e^{-\lambda \sigma_2} (W_4 - Y) O_9 + O_8^T [\Gamma_2 M_0 + M_1^T \Gamma_2] O_{12}), and $O_i = [0_{n \times (i-1)n}, I_{n \times n}, 0_{n \times (13-i)n}]$, $i = 1, \dots, 12$.$

Proof. Based on system (10), we construct the following Lyapunov-Krasovskii functional:

$$V(x(t), y(t)) = V_1(x(t), y(t)) + V_2(x(t), y(t)) + V_3(x(t), y(t)), \quad (21)$$

where

$$\begin{aligned} V_1(x(t), y(t)) &= x^T(t) P_1 x(t) + y^T(t) P_2 y(t), \\ V_2(x(t), y(t)) &= \int_{t-\tau_1}^t e^{\lambda(s-t)} x^T(s) Q_1 x(s) ds \\ &\quad + \int_{t-\tau(t)}^{t-\tau_1} e^{\lambda(s-t)} x^T(s) Q_2 x(s) ds \\ &\quad + \int_{t-\tau_2}^{t-\tau(t)} e^{\lambda(s-t)} x^T(s) Q_3 x(s) ds \\ &\quad + \int_{t-\sigma_1}^t e^{\lambda(s-t)} y^T(s) Q_4 y(s) ds \\ &\quad + \int_{t-\sigma(t)}^{t-\sigma_1} e^{\lambda(s-t)} y^T(s) Q_5 y(s) ds \\ &\quad + \int_{t-\sigma_2}^{t-\sigma(t)} e^{\lambda(s-t)} y^T(s) Q_6 y(s) ds \\ &\quad + \int_{t-\sigma(t)}^t e^{\lambda(s-t)} g^T(y(s)) Q_7 g(y(s)) ds, \end{aligned}$$

$$\begin{aligned} V_3(x(t), y(t)) &= \int_{-\tau_1}^0 \int_{t+\theta}^t \tau_1 e^{\lambda(s-t)} \dot{x}^T(s) W_1 \dot{x}(s) ds d\theta \\ &\quad + \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t (\tau_2 - \tau_1) e^{\lambda(s-t)} \dot{x}^T(s) W_2 \dot{x}(s) ds d\theta \\ &\quad + \int_{-\sigma_1}^0 \int_{t+\theta}^t \sigma_1 e^{\lambda(s-t)} \dot{y}^T(s) W_3 \dot{y}(s) ds d\theta \\ &\quad + \int_{-\sigma_2}^{-\sigma_1} \int_{t+\theta}^t (\sigma_2 - \sigma_1) e^{\lambda(s-t)} \dot{y}^T(s) W_4 \dot{y}(s) ds d\theta. \end{aligned} \quad (22)$$

Taking the derivatives of V_i , $i = 1, 2, 3$, we have

$$\begin{aligned} \dot{V}_1(x(t), y(t)) &= 2x^T(t) P_1 \dot{x}(t) + 2y^T(t) P_2 \dot{y}(t), \\ \dot{V}_2(x(t), y(t)) &= -\lambda \int_{t-\tau_1}^t e^{\lambda(s-t)} x^T(s) Q_1 x(s) ds \\ &\quad - \lambda \int_{t-\tau(t)}^{t-\tau_1} e^{\lambda(s-t)} x^T(s) Q_2 x(s) ds \\ &\quad - \lambda \int_{t-\tau_2}^{t-\tau(t)} e^{\lambda(s-t)} x^T(s) Q_3 x(s) ds \end{aligned}$$

$$\begin{aligned}
 & -\lambda \int_{t-\sigma_1}^t e^{\lambda(s-t)} y^T(s) Q_4 y(s) ds \\
 & -\lambda \int_{t-\sigma(t)}^{t-\sigma_1} e^{\lambda(s-t)} y^T(s) Q_5 y(s) ds \\
 & -\lambda \int_{t-\sigma_2}^{t-\sigma(t)} e^{\lambda(s-t)} y^T(s) Q_6 y(s) ds \\
 & -\lambda \int_{t-\sigma(t)}^t e^{\lambda(s-t)} g^T(y(s)) Q_7 g(y(s)) ds \\
 & + x^T(t) Q_1 x(t) - e^{-\lambda\tau_1} x^T(t-\tau_1) Q_1 x(t-\tau_1) \\
 & + e^{-\lambda\tau_1} x^T(t-\tau_1) Q_2 x(t-\tau_1) \\
 & - e^{-\lambda\tau_2} x^T(t-\tau(t)) Q_2 x(t-\tau(t)) (1-\dot{\tau}(t)) \\
 & - e^{-\lambda\tau_2} x^T(t-\tau_2) Q_3 x(t-\tau_2) \\
 & + e^{-\lambda\tau_1} x^T(t-\tau(t)) Q_3 x(t-\tau(t)) (1-\dot{\tau}(t)) \\
 & + y^T(t) Q_4 y(t) - e^{-\lambda\sigma_1} y^T(t-\sigma_1) Q_4 y(t-\sigma_1) \\
 & + e^{-\lambda\sigma_1} y^T(t-\sigma_1) Q_5 y(t-\sigma_1) \\
 & - e^{-\lambda\sigma_2} y^T(t-\sigma(t)) Q_5 y(t-\sigma(t)) (1-\dot{\sigma}(t)) \\
 & - e^{-\lambda\sigma_2} y^T(t-\sigma_2) Q_6 y(t-\sigma_2) \\
 & + e^{-\lambda\sigma_1} y^T(t-\sigma(t)) Q_6 y(t-\sigma(t)) (1-\dot{\sigma}(t)) \\
 & + g^T(y(t)) Q_7 g(y(t)) - e^{-\lambda\sigma_2} g^T \\
 & \times (y(t-\sigma(t))) Q_7 g(y(t-\sigma(t))) (1-\dot{\sigma}(t)),
 \end{aligned}$$

$$\begin{aligned}
 & \dot{V}_3(x(t), y(t)) \\
 & = -\lambda \int_{-\tau_1}^0 \int_{t+\theta}^t \tau_1 e^{\lambda(s-t)} \dot{x}^T(s) W_1 \dot{x}(s) ds d\theta \\
 & -\lambda \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^t (\tau_2 - \tau_1) e^{\lambda(s-t)} \dot{x}^T(s) W_2 \dot{x}(s) ds d\theta \\
 & -\lambda \int_{-\sigma_1}^0 \int_{t+\theta}^t \sigma_1 e^{\lambda(s-t)} \dot{y}^T(s) W_3 \dot{y}(s) ds d\theta \\
 & -\lambda \int_{-\sigma_2}^{-\sigma_1} \int_{t+\theta}^t (\sigma_2 - \sigma_1) e^{\lambda(s-t)} \dot{y}^T(s) W_4 \dot{y}(s) ds d\theta \\
 & + \tau_1^2 \dot{x}^T(t) W_1 \dot{x}(t) - \int_{t-\tau_1}^t \tau_1 e^{\lambda(\theta-t)} \dot{x}^T(\theta) W_1 \dot{x}(\theta) d\theta \\
 & + (\tau_2 - \tau_1)^2 \dot{x}^T(t) W_2 \dot{x}(t) \\
 & - \int_{t-\tau_2}^{t-\tau_1} (\tau_2 - \tau_1) e^{\lambda(\theta-t)} \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & + \sigma_1^2 \dot{y}^T(t) W_3 \dot{y}(t) - \int_{t-\sigma_1}^t \sigma_1 e^{\lambda(\theta-t)} \dot{y}^T(\theta) W_3 \dot{y}(\theta) d\theta
 \end{aligned}$$

$$\begin{aligned}
 & + (\sigma_2 - \sigma_1)^2 \dot{y}^T(t) W_4 \dot{y}(t) \\
 & - \int_{t-\sigma_2}^{t-\sigma_1} (\sigma_2 - \sigma_1) e^{\lambda(\theta-t)} \dot{y}^T(\theta) W_4 \dot{y}(\theta) d\theta.
 \end{aligned}
 \tag{23}$$

From Lemma 6, we have

$$\begin{aligned}
 & - \int_{t-\tau_1}^t \tau_1 e^{\lambda(\theta-t)} \dot{x}^T(\theta) W_1 \dot{x}(\theta) d\theta \\
 & \leq -e^{-\lambda\tau_1} [x(t) - x(t-\tau_1)]^T W_1 [x(t) - x(t-\tau_1)], \\
 & - \int_{t-\sigma_1}^t \sigma_1 e^{\lambda(\theta-t)} \dot{y}^T(\theta) W_3 \dot{y}(\theta) d\theta \\
 & \leq -e^{-\lambda\sigma_1} [y(t) - y(t-\sigma_1)]^T W_3 [y(t) - y(t-\sigma_1)].
 \end{aligned}
 \tag{24}$$

Meanwhile,

$$\begin{aligned}
 & - \int_{t-\tau_2}^{t-\tau_1} (\tau_2 - \tau_1) e^{\lambda(\theta-t)} \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & \leq -e^{-\lambda\tau_2} \left[\int_{t-\tau_2}^{t-\tau(t)} (\tau_2 - \tau_1) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \right. \\
 & \quad \left. + \int_{t-\tau(t)}^{t-\tau_1} (\tau_2 - \tau_1) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \right] \\
 & = -e^{-\lambda\tau_2} \left[\int_{t-\tau_2}^{t-\tau(t)} (\tau_2 - \tau(t)) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \right. \\
 & \quad + \int_{t-\tau(t)}^{t-\tau_1} (\tau(t) - \tau_1) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & \quad + \int_{t-\tau_2}^{t-\tau(t)} (\tau(t) - \tau_1) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & \quad \left. + \int_{t-\tau(t)}^{t-\tau_1} (\tau_2 - \tau(t)) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \right] \\
 & = -e^{-\lambda\tau_2} \left[\int_{t-\tau_2}^{t-\tau(t)} (\tau_2 - \tau(t)) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \right. \\
 & \quad + \int_{t-\tau(t)}^{t-\tau_1} (\tau(t) - \tau_1) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & \quad + \frac{\tau(t) - \tau_1}{\tau_2 - \tau(t)} \\
 & \quad \times \int_{t-\tau_2}^{t-\tau(t)} (\tau_2 - \tau(t)) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & \quad + \frac{\tau_2 - \tau(t)}{\tau(t) - \tau_1} \\
 & \quad \left. \times \int_{t-\tau(t)}^{t-\tau_1} (\tau(t) - \tau_1) \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \right].
 \end{aligned}
 \tag{25}$$

By Lemma 6, we obtain that

$$\begin{aligned}
 & - \int_{t-\tau_2}^{t-\tau_1} (\tau_2 - \tau_1) e^{\lambda(\theta-t)} \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & \leq -e^{-\lambda\tau_2} \left\{ \frac{\tau_2 - \tau_1}{\tau_2 - \tau(t)} [x(t - \tau(t)) - x(t - \tau_2)]^T \right. \\
 & \quad \times W_2 [x(t - \tau(t)) - x(t - \tau_2)] \\
 & \quad + \frac{\tau_2 - \tau_1}{\tau(t) - \tau_1} [x(t - \tau_1) - x(t - \tau(t))]^T \\
 & \quad \left. \times W_2 [x(t - \tau_1) - x(t - \tau(t))] \right\}. \tag{26}
 \end{aligned}$$

And by Lemma 7, we get that

$$\begin{aligned}
 & - \int_{t-\tau_2}^{t-\tau_1} (\tau_2 - \tau_1) e^{\lambda(\theta-t)} \dot{x}^T(\theta) W_2 \dot{x}(\theta) d\theta \\
 & \leq -e^{-\lambda\tau_2} \left\{ [x(t - \tau_1) - x(t - \tau(t))]^T \right. \\
 & \quad \times W_2 [x(t - \tau_1) - x(t - \tau(t))] \\
 & \quad + [x(t - \tau(t)) - x(t - \tau_2)]^T \\
 & \quad \times W_2 [x(t - \tau(t)) - x(t - \tau_2)] \\
 & \quad + 2[x(t - \tau_1) - x(t - \tau(t))]^T \\
 & \quad \left. \times X [x(t - \tau(t)) - x(t - \tau_2)] \right\}. \tag{27}
 \end{aligned}$$

Similar to (27),

$$\begin{aligned}
 & - \int_{t-\sigma_2}^{t-\sigma_1} e^{\lambda(\theta-t)} \dot{y}^T(\theta) W_4 \dot{y}(\theta) d\theta \\
 & \leq -e^{-\lambda\sigma_2} \left\{ [y(t - \sigma_1) - y(t - \sigma(t))]^T \right. \\
 & \quad \times W_4 [y(t - \sigma_1) - y(t - \sigma(t))] \\
 & \quad + [y(t - \sigma(t)) - y(t - \sigma_2)]^T \\
 & \quad \times W_4 [y(t - \sigma(t)) - y(t - \sigma_2)] \\
 & \quad + 2[y(t - \sigma_1) - y(t - \sigma(t))]^T \\
 & \quad \left. \times Y [y(t - \sigma(t)) - y(t - \sigma_2)] \right\}. \tag{28}
 \end{aligned}$$

By Assumption 3, for any $\Gamma_i = \text{diag}(\gamma_{i1}, \dots, \gamma_{in}) \geq 0, i = 1, 2$, the following inequality is true:

$$\begin{aligned}
 & -2 \sum_{i=1}^n \gamma_{i1} [g_i(y_i(t)) - m_i^+ y_i(t)] [g_i(y_i(t)) - m_i^- y_i(t)] \\
 & -2 \sum_{i=1}^n \gamma_{i2} [g_i(y_i(t - \tau(t))) - m_i^+ y_i(t - \tau(t))] \\
 & \quad \times [g_i(y_i(t - \tau(t))) - m_i^- y_i(t - \tau(t))] \geq 0. \tag{29}
 \end{aligned}$$

It can be rewritten as

$$\begin{aligned}
 & -2[g(y(t)) - M_1 y(t)]^T \Gamma_1 [g(y(t)) - M_0 y(t)] \\
 & -2[g(y(t - \sigma(t))) - M_1 y(t - \sigma(t))]^T \\
 & \quad \times \Gamma_2 [g(y(t - \sigma(t))) - M_0 y(t - \sigma(t))] \geq 0. \tag{30}
 \end{aligned}$$

For any constant matrices of appropriate dimensions $H_i, i = 1, \dots, 4$, and from (10), we can obtain that

$$\begin{aligned}
 0 & = 2 [x^T(t) H_1^T + \dot{x}^T(t) H_2^T] \\
 & \quad \times [-\dot{x}(t) - Ax(t) + Bg(y(t - \sigma(t)))] , \\
 0 & = 2 [y^T(t) H_3^T + \dot{y}^T(t) H_4^T] \\
 & \quad \times [-\dot{y}(t) - Cy(t) + Dx(t - \tau(t))]. \tag{31}
 \end{aligned}$$

Combining (21)–(31), we have

$$\begin{aligned}
 & \dot{V}(x(t), y(t)) + \lambda V(x(t), y(t)) \\
 & \leq \xi^T(t) \left\{ \Omega + O_3^T [e^{-\lambda\tau_2} (\dot{t}(t) - \tau_3) Q_2 \right. \\
 & \quad \left. + e^{-\lambda\tau_1} (\tau_4 - \dot{t}(t)) Q_3] O_3 \right. \\
 & \quad + O_8^T [e^{-\lambda\sigma_2} (\dot{\sigma}(t) - \sigma_3) Q_5 \\
 & \quad \left. + e^{-\lambda\sigma_1} (\sigma_4 - \dot{\sigma}(t)) Q_6] O_8 \right\} \xi(t), \tag{32}
 \end{aligned}$$

where

$$\begin{aligned}
 \xi^T(t) & = [x^T(t), x^T(t - \tau_1), x^T(t - \tau(t)), x^T(t - \tau_2), \\
 & \quad \dot{x}^T(t), y^T(t), y^T(t - \sigma_1), y^T(t - \sigma(t)), \\
 & \quad y^T(t - \sigma_2), \dot{y}^T(t), g^T(y(t)), g^T(y(t - \sigma(t)))] . \tag{33}
 \end{aligned}$$

From (19) and (20), we can see that

$$\dot{V}(x(t), y(t)) + \lambda V(x(t), y(t)) < 0, \tag{34}$$

for all nonzero $\xi(t)$. Integrating the above inequality (34) from t_0 to t gives

$$V(x(t), y(t)) \leq e^{-\lambda(t-t_0)} V(x(t_0), y(t_0)). \tag{35}$$

From (21), we know that

$$\begin{aligned}
 V(x(t), y(t)) & \geq V_1(x(t), y(t)) \\
 & \geq \min \{ \lambda_{\min}(P_1), \lambda_{\min}(P_2) \} \|z(t)\|^2,
 \end{aligned}$$

$$\begin{aligned}
 &V(x(t_0), y(t_0)) \\
 &\leq \left[\lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \tau_1 \lambda_{\max}(Q_1) \right. \\
 &\quad + (\tau_2 - \tau_1) [\lambda_{\max}(Q_2) + \lambda_{\max}(Q_3)] \\
 &\quad + (\sigma_2 - \sigma_1) [\lambda_{\max}(Q_5) + \lambda_{\max}(Q_6)] \\
 &\quad + \sigma_1 \lambda_{\max}(Q_4) + \sigma_2 m^2 \lambda_{\max}(Q_7) \\
 &\quad + \frac{1}{2} \tau_1^3 \lambda_{\max}(W_1) + \frac{1}{2} (\tau_2 - \tau_1)^3 \lambda_{\max}(W_2) \\
 &\quad \left. + \frac{1}{2} \sigma_1^3 \lambda_{\max}(W_3) + \frac{1}{2} (\sigma_2 - \sigma_1)^3 \lambda_{\max}(W_4) \right] \|z(t_0)\|_{C_1}^2. \tag{36}
 \end{aligned}$$

Let

$$\begin{aligned}
 \lambda_1 &= \min \{ \lambda_{\min}(P_1), \lambda_{\min}(P_2) \}, \\
 \lambda_2 &= \lambda_{\max}(P_1) + \lambda_{\max}(P_2) + \tau_1 \lambda_{\max}(Q_1) \\
 &\quad + (\tau_2 - \tau_1) [\lambda_{\max}(Q_2) + \lambda_{\max}(Q_3)] \\
 &\quad + (\sigma_2 - \sigma_1) [\lambda_{\max}(Q_5) + \lambda_{\max}(Q_6)] \\
 &\quad + \sigma_1 \lambda_{\max}(Q_4) + \sigma_2 m^2 \lambda_{\max}(Q_7) \\
 &\quad + \frac{1}{2} \tau_1^3 \lambda_{\max}(W_1) + \frac{1}{2} (\tau_2 - \tau_1)^3 \lambda_{\max}(W_2) \\
 &\quad + \frac{1}{2} \sigma_1^3 \lambda_{\max}(W_3) + \frac{1}{2} (\sigma_2 - \sigma_1)^3 \lambda_{\max}(W_4). \tag{37}
 \end{aligned}$$

Then, by (35) and (36), we have

$$\lambda_1 \|z(t)\|^2 \leq V(x(t), y(t)) \leq \lambda_2 e^{-\lambda(t-t_0)} \|z(t_0)\|_{C_1}^2. \tag{38}$$

By (38), we get that

$$\|z(t)\|^2 \leq \frac{\lambda_2}{\lambda_1} e^{-\lambda(t-t_0)} \|z(t_0)\|_{C_1}^2. \tag{39}$$

Let $\alpha = (\lambda_2/\lambda_1)^{1/2}$, and, by Definition 5, the genetic regulatory networks in (10) are exponentially stable.

The proof is completed. \square

In the following, we consider the globally exponential stability of the genetic regulatory networks with time-varying delays and continuous distributed delays.

Theorem 9. For system (14) with Assumptions 1 and 3, the equilibrium point is globally exponentially stable (that is, there are two positive constants α' and λ such that $\|z(t)\| \leq \alpha e^{-\lambda t} \|z(t_0)\|_{C_1}$, for all $t \geq t_0$) if there exist positive definite matrices $H_2, H_4, P_1, P_2, S, Q_i, i = 1, \dots, 7, W_i, i = 1, \dots, 4, \Gamma_1 = \text{diag}(\gamma_{11}, \dots, \gamma_{1n})$, and $\Gamma_2 = \text{diag}(\gamma_{21}, \dots, \gamma_{2n})$, such that,

for any appropriate dimensions constant matrices H_1, H_3, X, Y , the following LMIs hold:

$$\begin{pmatrix} W_2 & X \\ X & W_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} W_4 & Y \\ Y & W_4 \end{pmatrix} \geq 0, \tag{40}$$

$$\begin{aligned}
 &\Omega'_1 + \Omega'_2 + L_3^T [e^{-\lambda\tau_2} Q_2 + e^{-\lambda\tau_1} Q_3] L_3 \\
 &\quad + L_8^T [e^{-\lambda\sigma_2} Q_5 + e^{-\lambda\sigma_1} Q_6] L_8 < 0, \tag{41}
 \end{aligned}$$

where $\Omega'_1 = \text{diag}(\lambda P_1 + Q_1 - e^{-\lambda\tau_1} W_1 - H_1^T A - A^T H_1, e^{-\lambda\tau_1} (Q_1 - Q_2) - e^{-\lambda\tau_1} W_1 - e^{-\lambda\tau_2} W_2, e^{-\lambda\tau_1} (1 - \tau_4) Q_3 - e^{-\lambda\tau_2} [(1 - \tau_3) Q_2 + 2W_2 - X^T - X], -e^{-\lambda\tau_2} W_2, \tau_1^2 W_1 + (\tau_2 - \tau_1)^2 W_2 - H_2^T - H_2, \lambda P_2 + Q_4 - e^{-\lambda\sigma_1} W_3 - 2M_1^T \Gamma_1 M_0 - H_3^T C - C^T H_3, -e^{-\lambda\sigma_1} (Q_4 - Q_5 + W_3) - e^{-\lambda\sigma_2} W_4, e^{-\lambda\sigma_1} (1 - \sigma_4) Q_6 - e^{-\lambda\sigma_2} [(1 - \sigma_3) Q_5 + 2W_4 - Y^T - Y] - M_1^T \Gamma_2 M_0, -e^{-\lambda\sigma_2} (Q_4 + W_4), \sigma_1^2 W_3 + (\sigma_2 - \sigma_1)^2 W_4 - H_4^T - H_4, Q_7 - 2\Gamma_1 + \vartheta_1^2 S, -e^{-\lambda\sigma_2} Q_7 (1 - \sigma_4) - 2\Gamma_2, -e^{-\lambda\vartheta_1} S, -e^{-\lambda\vartheta_1} S)$, $\Omega'_2 = \text{sym}(L_1^T e^{-\lambda\tau_1} W_1 L_2 + L_1^T [P_1 - H_1^T - H_2^T A] L_5 + I_1^T H_1^T B L_{12} + L_2^T e^{-\lambda\tau_2} (W_2 - X) L_3 + L_2^T e^{-\lambda\tau_2} X L_4 + L_3^T e^{-\lambda\tau_2} (W_2 - X) L_4 + L_3^T H_3^T D L_6 + L_3^T H_4^T D L_{10} + I_5^T H_2^T B L_{12} + L_6^T e^{-\lambda\sigma_1} W_3 L_7 + L_6^T [P_2 - H_3^T - H_4^T C] L_{10} + L_6^T [\Gamma_1 M_0 + M_1^T \Gamma_1] L_{11} + L_7^T e^{-\lambda\sigma_2} (W_4 - Y) L_8 + L_7^T e^{-\lambda\sigma_2} Y L_9 + L_8^T e^{-\lambda\sigma_2} (W_4 - Y) L_9 + L_8^T [\Gamma_2 M_0 + M_1^T \Gamma_2] L_{12})$, and $L_i = [0_{n \times (i-1)n}, I_{n \times n}, 0_{n \times (15-i)n}]$, $i = 1, \dots, 12$.

Proof. Based on system (14), we construct the following Lyapunov-Krasovskii functional:

$$\begin{aligned}
 V(x(t), y(t)) &= V_1(x(t), y(t)) + V_2(x(t), y(t)) \\
 &\quad + V_3(x(t), y(t)) + V_4(x(t), y(t)), \tag{42}
 \end{aligned}$$

where $V_1(x(t), y(t))$, $V_2(x(t), y(t))$, and $V_3(x(t), y(t))$ are defined as in Theorem 8 and

$$\begin{aligned}
 &V_4(x(t), y(t)) \\
 &= \int_{-\vartheta_1}^0 \int_{t+\theta}^t \vartheta_1 e^{\lambda(s-t)} g^T(y(s)) S g(y(s)) ds d\theta. \tag{43}
 \end{aligned}$$

Taking the derivative of V_4 ,

$$\begin{aligned}
 &\dot{V}_4(x(t), y(t)) \\
 &\leq -\lambda \int_{-\vartheta_1}^0 \int_{t+\theta}^t \vartheta_1 e^{\lambda(s-t)} g^T(y(s)) S g(y(s)) ds d\theta \\
 &\quad + \vartheta_1^2 g^T(y(t)) S g(y(t)) \\
 &\quad - e^{-\lambda\vartheta_1} \int_{t-\vartheta_1}^t \vartheta_1 g^T(y(\theta)) S g(y(\theta)) d\theta. \tag{44}
 \end{aligned}$$

By Lemma 6, we get that

$$\begin{aligned}
 & \dot{V}_4(x(t), y(t)) \\
 & \leq -\lambda \int_{-\vartheta_1}^0 \int_{t+\theta}^t \vartheta_1 e^{\lambda(s-t)} g^T(y(s)) Sg(y(s)) ds d\theta \\
 & \quad + \vartheta_1^2 g^T(y(t)) Sg(y(t)) - e^{-\lambda\vartheta_1} \frac{\vartheta_1}{\vartheta_1 - \vartheta(t)} \\
 & \quad \times \left(\int_{t-\vartheta_1}^{t-\vartheta(t)} g(y(\theta)) d\theta \right)^T S \left(\int_{t-\vartheta_1}^{t-\vartheta(t)} g(y(\theta)) d\theta \right) \\
 & \quad - e^{-\lambda\vartheta_1} \frac{\vartheta_1}{\vartheta(t)} \\
 & \quad \times \left(\int_{t-\vartheta(t)}^t g(y(\theta)) d\theta \right)^T S \left(\int_{t-\vartheta(t)}^t g(y(\theta)) d\theta \right) \\
 & \leq -\lambda \int_{-\vartheta_1}^0 \int_{t+\theta}^t \vartheta_1 e^{\lambda(s-t)} g^T(y(s)) Sg(y(s)) ds d\theta \\
 & \quad + \vartheta_1^2 g^T(y(t)) Sg(y(t)) - e^{-\lambda\vartheta_1} \left(1 + \frac{\vartheta(t)}{\vartheta_1} \right) \\
 & \quad \times \left(\int_{t-\vartheta_1}^{t-\vartheta(t)} g(y(\theta)) d\theta \right)^T S \left(\int_{t-\vartheta_1}^{t-\vartheta(t)} g(y(\theta)) d\theta \right) \\
 & \quad - e^{-\lambda\vartheta_1} \left(1 + \frac{\vartheta_1 - \vartheta(t)}{\vartheta_1} \right) \\
 & \quad \times \left(\int_{t-\vartheta(t)}^t g(y(\theta)) d\theta \right)^T S \left(\int_{t-\vartheta(t)}^t g(y(\theta)) d\theta \right). \tag{45}
 \end{aligned}$$

Combining (23)–(31), (42), and (45), we get

$$\begin{aligned}
 & \dot{V}(x(t), y(t)) + \lambda V(x(t), y(t)) \\
 & \leq \xi_1^T(t) \left\{ \Omega + L_3^T \left[e^{-\lambda\tau_2} (\dot{\tau}(t) - \tau_3) Q_2 \right. \right. \\
 & \quad \left. \left. + e^{-\lambda\tau_1} (\tau_4 - \dot{\tau}(t)) Q_3 \right] L_3 \right. \\
 & \quad \left. + L_8^T \left[e^{-\lambda\sigma_2} (\dot{\sigma}(t) - \sigma_3) Q_5 \right. \right. \\
 & \quad \left. \left. + e^{-\lambda\sigma_1} (\sigma_4 - \dot{\sigma}(t)) Q_6 \right] L_8 \right\} \xi_1(t), \tag{46}
 \end{aligned}$$

where

$$\begin{aligned}
 \xi_1^T(t) = & \left[x^T(t), x^T(t - \tau_1), x^T(t - \tau(t)), \right. \\
 & x^T(t - \tau_2), \dot{x}^T(t), y^T(t), y^T(t - \sigma_1), \\
 & \left. y^T(t - \sigma(t)), y^T(t - \sigma_2), \dot{y}^T(t), g^T(y(t)), \right.
 \end{aligned}$$

$$\begin{aligned}
 & g^T(y(t - \sigma(t))), \left(\int_{t-\vartheta_1}^{t-\vartheta(t)} g(y(\theta)) d\theta \right)^T, \\
 & \left. \left(\int_{t-\vartheta(t)}^t g(y(\theta)) d\theta \right)^T \right]. \tag{47}
 \end{aligned}$$

By (40) and (41), we get that $\dot{V}(x(t), y(t)) + \lambda V(x(t), y(t)) < 0$. Similar to the proof of Theorem 8 we have

$$\|z(t)\|^2 \leq \frac{\lambda'_2}{\lambda_1} e^{-\lambda(t-t_0)} \|z(t_0)\|_{C_1}^2. \tag{48}$$

where $\lambda'_2 = \lambda_2 + (1/2)\vartheta_1^3 m^2 \lambda_{\max}(S)$ and λ_1, λ_2 are defined as Theorem 8. Let $\alpha' = (\lambda'_2/\lambda_1)^{1/2}$, and, by Definition 5, the genetic regulatory network (14) is exponentially stable.

The proof is completed. \square

Remark 10. In the proof of Theorems 8 and 9, we use convex combination and interactive convex combination definition to estimate the upper bound of derivative function of the Lyapunov-Krasovskii functional and obtain some new conservative weaker sufficient conditions.

Remark 11. When the lower bound of derivatives of time-varying delays is immeasurable, let $\int_{t-\tau_2}^{t-\tau(t)} e^{\lambda(s-t)} x^T(s) Q_3 x(s) ds = 0$ and $\int_{t-\sigma_2}^{t-\sigma(t)} e^{\lambda(s-t)} y^T(s) Q_6 y(s) ds = 0$ in Theorem 8 (20) or Theorem 9 (41); our results are true still.

4. Numerical Examples

In this section, two examples are given to illustrate the effectiveness of our theoretical results.

Example 1. Consider a genetic regulatory network model reported by Elowitz and Leibler [24], which studied the dynamics of repressilator which is cyclic negative-feedback loop comprising three repressor genes (*lacl*, *tetR*, and *cl*) and their promoters (*cl*, *lacl*, and *tetR*):

$$\frac{dx_i}{dt} = -x_i + \frac{\alpha}{1 + y_j^n} + \alpha_0, \tag{49}$$

$$\frac{dy_i}{dt} = \beta(x_i - y_i).$$

Taking time-varying delays into account and shifting the equilibrium point to the origin, one gets the following model:

$$\begin{aligned}
 \dot{x}(t) & = -Ax(t) + Bg(y(t - \sigma(t))), \\
 \dot{y}(t) & = -Cy(t) + Dx(t - \tau(t)), \tag{50}
 \end{aligned}$$

where $A = \text{diag}(2, 2, 2)$, $C = \text{diag}(3, 3, 3)$, $D = \text{diag}(1, 1, 1)$, and the coupling matrix

$$B = 1.5 \times \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \tag{51}$$

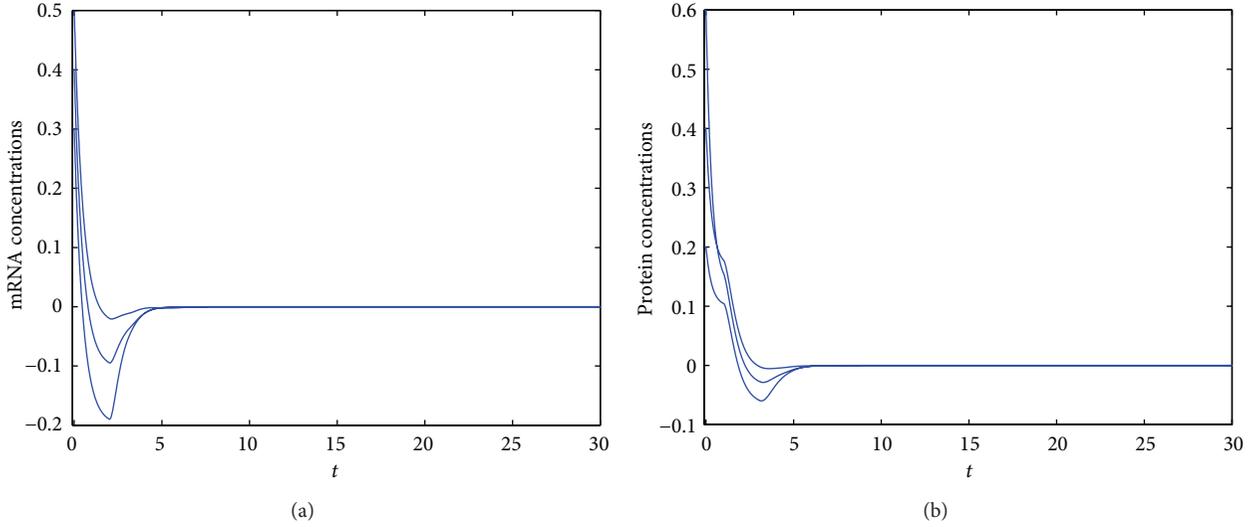


FIGURE 1: (a) mRNA concentrations $x(t)$. (b) Protein concentrations $y(t)$.

The gene regulation function is taken as $g(x) = x^2/(1 + x^2)$, $M_0 = \text{diag}(0, 0, 0)$, and $M_1 = \text{diag}(0.65, 0.65, 0.65)$. The time delays $\sigma(t)$ and $\tau(t)$ are assumed to be

$$\sigma(t) = 1 + 2\sin^2 t, \quad \tau(t) = 1 + \cos^2 t. \quad (52)$$

We can get the parameters as follows:

$$\begin{aligned} \sigma_1 &= 1, & \sigma_2 &= 3, & \sigma_3 &= -2, & \sigma_4 &= 2, \\ \tau_1 &= 1, & \tau_2 &= 2, & \tau_3 &= -1, & \tau_4 &= 1, \end{aligned} \quad (53)$$

and choose $\lambda = 0.5$. In accordance with the condition in Theorem 8, system (50) is exponentially stable. By using the MATLAB LMI toolbox, we can get the feasible solutions. Due to the space limitation, we only list matrices P_1 and P_2 here as follows:

$$\begin{aligned} P_1 &= \begin{pmatrix} 17.1041 & 7.9997 & 7.9997 \\ 7.9997 & 17.1041 & 7.9997 \\ 7.9997 & 7.9997 & 17.1041 \end{pmatrix}, \\ P_2 &= \begin{pmatrix} 13.8943 & 0.5573 & 0.5573 \\ 0.5573 & 13.8943 & 0.5573 \\ 0.5573 & 0.5573 & 13.8943 \end{pmatrix}. \end{aligned} \quad (54)$$

The initial condition is $x(0) = (0.3, 0.5, 0.4)^T$ and $y(0) = (0.2, 0.4, 0.6)^T$. The simulation results of the trajectories are shown in Figure 1.

Example 2. In this example, we consider the genetic regulatory network (14) with time-varying delays and continuous distributed delays, in which the parameters are listed as follows:

$$\begin{aligned} A &= \text{diag}(1, 2, 3), & C &= \text{diag}(5, 4, 5), \\ D &= \text{diag}(0.3, 0.2, 0.4), & B &= \begin{pmatrix} 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \\ 0.8 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (55)$$

and $g(x) = x^2/(1 + x^2)$, $M_0 = \text{diag}(0, 0, 0)$, and $M_1 = \text{diag}(0.65, 0.65, 0.65)$. The time delays $\sigma(t)$, $\tau(t)$, and $\vartheta(t)$ are assumed to be

$$\begin{aligned} \sigma(t) &= 0.5 + 0.3\sin^2 t, & \tau(t) &= 0.4 + 0.1\cos^2 t, \\ \vartheta(t) &= 2\sin^2(t). \end{aligned} \quad (56)$$

We can get the parameters as follows:

$$\begin{aligned} \sigma_1 &= 0.5, & \sigma_2 &= 0.8, & \sigma_3 &= -0.3, & \sigma_4 &= 0.3, \\ \tau_1 &= 0.4, & \tau_2 &= 0.5, & \tau_3 &= -0.1, \end{aligned} \quad (57)$$

$$\tau_4 = 0.1, \quad \vartheta_1 = 2 \quad (58)$$

and choose $\lambda = 0.2$. By using the MATLAB LMI toolbox, we can get the feasible solutions. Due to the space limitation, we only list matrices P_1 and P_2 here as follows:

$$\begin{aligned} P_1 &= \begin{pmatrix} 7.1735 & -0.4713 & -0.2152 \\ -0.4713 & 7.2143 & -0.3071 \\ -0.2152 & -0.3071 & 6.9607 \end{pmatrix}, \\ P_2 &= \begin{pmatrix} 7.9363 & 0 & 0 \\ 0 & 8.0199 & 0 \\ 0 & 0 & 7.9001 \end{pmatrix}. \end{aligned} \quad (59)$$

5. Concluding Remarks

This paper has investigated the exponential stability of genetic regulatory networks with time-varying delays and continuous distributed delays. By using the novel Lyapunov-Krasovskii functions and employing the Jensen inequality and the interactive convex combination method, some sufficient criteria are given to ensure the exponential stability with less conservative. All the obtained conditions are dependent on the delays and on linear matrix inequalities. Two examples are provided to illustrate the effectiveness of our results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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