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Research Article

Wave-Breaking Phenomena and Existence of Peakons for a Generalized Compressible Elastic-Rod Equation

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Consideration in this paper is the Cauchy problem of a generalized hyperelastic-rod wave equation. We first derive a wave-breaking mechanism for strong solutions, which occurs in finite time for certain initial profiles. In addition, we determine the existence of some new peaked solitary wave solutions.

1. Introduction

Consideration herein is the following generalized hyperelastic-rod wave equation:

$$\begin{gathered} u_t - u_{txx} + \left(\frac{g(u)}{2}\right)_x = \gamma\left(2u_x u_{xx} + u u_{xxx}\right), \\ t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given C^∞-function and the real number γ with $\gamma \neq 0$ is a given parameter. In general, the constant γ is given in terms of the material constants and the prestress of the rod. In fact, the authors [1, 2] studied the special compressible materials which lead to values of γ ranging from -29.4760 to 3.4174.

When $\gamma = 1$, $g(u) = 3u^2$, (1) becomes the classical Camassa-Holm (CH) equation [3]

$$u_t - u_{xxt} + 3uu_x + ku_x = 2u_x u_{xx} + u u_{xxx} \tag{2}$$

with initial data $u(0, x) = u_0(x)$. While for $\gamma = 0$ and $g(u) = 3u^2$, (1) becomes the Benjamin-Bona-Mahony equation [4]. And when $g(u) = 3u^2$, $\gamma \in (-29.4760, 3.4174)$, (1) is so-called the hyperelastic-rod wave equation [2].

The classical CH equation (2) was originally proposed as a model for surface waves [3] and has been studied extensively in the last twenty years because of its many remarkable properties: infinity of conservation laws and complete integrability [3], existence of peaked solitons and multipeakons [3, 5, 6] (with $k = 0$), well-posedness and breaking waves, and meaning solutions that remain bounded, while its slope becomes unbounded in finite time [7, 8]. On the other hand, if $k = 0$, the CH equation (2) admits peaked solitary wave solutions (called peakons) which possess the form $u(t, x) = ce^{-|x-ct|}$ with speed $c \in \mathbb{R}$, $c \neq 0$, and their stability was studied in [9]. Recently, Gui et al. [10] proved that there exist some peaked functions which are global weak solutions to a modified Camassa-Holm equation. We should mention that the solutions to the CH equation (2) can be uniquely continued after wave-breaking as either a global conservative or global dissipative weak solution [11–14]. It is worth pointing out that there exists a global-in-time weak solution to the CH equation in the energy space [15].

For $\gamma \in \mathbb{R} \setminus \{0\}$, $g(u) = 2ku + (3/2)u^2$, (1) serves as a model equation for mechanical vibrations in a hyperelastic rod [1, 2]. Similar to CH equation, stability of solitary wave solutions has been studied in [16]. In addition, the solutions to the hyperelastic-rod wave equation can be uniquely continued after wave breaking as a global conservative weak solution

[17]. Moreover, there exists a global-in-time weak solution to the hyperelastic-rod wave equation in the energy space [18].

Motivated by the approaches in [9, 10], our goals in this paper are concerned with the wave-breaking phenomena and the existence of some new peakons of (1) with $g(u) = au^2 + 2ku$ $(a, k \in \mathbb{R}, a > 0)$, $\gamma \neq 0$. In this case, (1) may be read as

$$u_t - u_{txx} + \left(\frac{a}{2}u^2\right)_x + ku_x = \gamma\left(2u_x u_{xx} + uu_{xxx}\right), \quad t > 0, \quad x \in \mathbb{R}, \tag{3}$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}.$$

Introducing the momentum $m := u - u_{xx}$, we get from (3) that

$$m_t + \gamma u m_x + 2\gamma u_x m + (a - 3\gamma) uu_x + ku_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{4}$$
$$m(0,x) = m_0(x) := u_0 - u_{0xx}, \quad x \in \mathbb{R}.$$

Note that if $p(x) := (1/2)e^{-|x|}$, $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * (u - u_{xx}) = u$, where $*$ denotes convolution with respect to the spatial variable x. Therefore, (3) can also be rewritten as the following equivalent form

$$u_t + \gamma uu_x = -\partial_x p * \left(\frac{a-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2 + ku\right), \quad t > 0, \quad x \in \mathbb{R}, \tag{5}$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}.$$

We are now in a position to give the notions of strong and weak solutions.

Definition 1. If $u \in \mathscr{C}([0,T]; H^s) \cap \mathscr{C}^1([0,T]; H^{s-1})$ with $s > 3/2$ and some $T > 0$ satisfies (5), then u is called a strong solution on $[0,T]$. If u is a strong solution on $[0,T]$ for every $T > 0$, then it is called a global strong solution.

Definition 2. Given initial data $u_0 \in H^1(\mathbb{R})$, the function $u \in L^\infty_{\mathrm{loc}}([0,T); H^1_{\mathrm{loc}}(\mathbb{R}))$ is said to be a weak solution to the initial-value problem (5) if it satisfies the following identity:

$$\int_0^T \int_{\mathbb{R}} \left[u\, \varphi_t + \frac{\gamma}{2}\, u^2 \varphi_x + p * \left(\frac{a-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2 + ku\right)\partial_x \varphi\right] dx\, dt + \int_{\mathbb{R}} u_0(x)\, \varphi(0,x)\, dx = 0, \tag{6}$$

for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T) \times \mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

Our main results of the present paper are Theorem 9 (wave breaking) and Theorem 10 (existence of peakons).

The remainder of the paper is organized as follows. In Section 2, the results of blow-up to strong solutions are presented in detail. It is shown that the solutions of (3) can only have singularities which correspond to wave breaking (Theorems 9). In Section 3, the existence of some new peaked solutions of (3) is verified. From this, we know that there exist some peaked solitary wave solutions for the case $k \neq 0$ (compared to the case in the Camassa-Holm equation; see Remark 11).

Notation 1. As above and henceforth, we denote the norm of the Lebesgue space $L^p(\mathbb{R})$ $(1 \leq p \leq \infty)$ by $\|\cdot\|_{L^p}$ and the norm of the Sobolev space $H^s(\mathbb{R})$ $(s \in \mathbb{R})$ by $\|\cdot\|_{H^s}$. We denote by $*$ the spatial convolution on $\mathbb{R}$.

2. Wave-Breaking Phenomena

By using the Kato's method [19], we may easily get the following results about the local well-posedness and blow-up criterion of strong solutions to (5), of which proofs are similar to the one as in the CH equation in [7] (up to a slight modification) and we omit it here.

Theorem 3. *Let $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Then there exists a time $T > 0$ such that the initial-value problem (5) has a unique strong solution $u \in \mathscr{C}([0,T]; H^s) \cap \mathscr{C}^1([0,T]; H^{s-1})$ and the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in H^s into $\mathscr{C}([0,T]; H^s) \cap \mathscr{C}^1([0,T]; H^{s-1})$.*

We are now in a position to state a blow-up criterion for (5).

Theorem 4. *Let $u_0 \in H^s(\mathbb{R})$ be as in Theorem 3 with $s > 3/2$. Let u be the corresponding solution to (5). Assume that $T^*_{u_0} > 0$ is the maximum time of existence. Then*

$$T^*_{u_0} < \infty \Longrightarrow \int_0^{T^*_{u_0}} \|u_x(\tau)\|_{L^\infty} d\tau = \infty. \tag{7}$$

Remark 5. The blow-up criterion (7) implies that the lifespan $T^*_{u_0}$ does not depend on the regularity index s of the initial data u_0. Indeed, let u_0 be in H^s for some $s > 3/2$ and consider some $s' \in (3/2, s)$. Denote by u_s (resp., $u_{s'}$) the corresponding maximal H^s (resp., $H^{s'}$) solution given by the above theorem. Denote by T^*_s (resp., $T^*_{s'}$) the lifespan of u_s (resp., $u_{s'}$). Since $H^s \hookrightarrow H^{s'}$, uniqueness ensures that $T^*_s \leq T^*_{s'}$ and that $u_s \equiv u_{s'}$ on $[0, T^*_s)$. Now, if $T^*_s < T^*_{s'}$, then we must have $u_{s'}$ in $\mathscr{C}([0,T^*_s]; H^{s'})$. Hence, $\partial_x u_{s'} \in L^1([0,T^*_s]; L^\infty)$, which contradicts the above blow-up criterion (7). Therefore, $T^*_s = T^*_{s'}$.

Remark 6. For a strong solution u in Theorem 3, the Hamiltonian functionals are conserved; that is

$$\frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx = 0, \qquad \frac{d}{dt}\int_{\mathbb{R}} \left(\frac{a}{3}u^3 + \gamma uu_x^2 + ku^2\right) dx = 0, \tag{8}$$

for all $t \in [0,T)$.

The following blow-up criterion shows that the wave-breaking depends only on the infimum of γu_x.

Theorem 7. *Let $u_0 \in H^s(\mathbb{R})$ be as in Theorem 3 with $s > 3/2$. Then the corresponding solution u to (5) blows up in finite time $T^*_{u_0} > 0$ if and only if*

$$\liminf_{t\uparrow T^*_{u_0}} \inf_{x\in\mathbb{R}} \{\gamma u_x(t,x)\} = -\infty. \tag{9}$$

Proof. Since, in view of Remark 5, the existence time $T^*_{u_0}$ is independent of the choice of s, we need only to consider the case $s = 3$, which relies on a simple density argument.

Multiplying (3) by m and integrating over $\mathbb{R}$ with respect to x and then integration by parts produce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m^2 dx &= -\frac{a}{2}\int_{\mathbb{R}} (u^2)_x m\,dx \\ &\quad + \gamma\int_{\mathbb{R}} m(2u_x u_{xx} + u u_{xxx})\,dx \\ &= -a\int_{\mathbb{R}} u u_x m\,dx + 2\gamma\int_{\mathbb{R}} u_x(u-m)m\,dx \\ &\quad + \gamma\int_{\mathbb{R}} u(u_x - m_x)m\,dx \\ &= (3\gamma - a)\int_{\mathbb{R}} u u_x m\,dx - \frac{3}{2}\gamma\int_{\mathbb{R}} u_x m^2\,dx.\end{aligned} \tag{10}$$

We next differentiate (3) with respect to x to get

$$\begin{aligned} m_{xt} &= (3\gamma - a)(u_x^2 + u^2 - um) \\ &\quad - 3\gamma u_x m_x - 2\gamma u m - \gamma u m_{xx} + k(m-u).\end{aligned} \tag{11}$$

Multiplying by m_x then integrating over $\mathbb{R}$ with respect to x lead to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m_x^2\,dx &= -2(3\gamma - a)\int_{\mathbb{R}} u_x(u-m)m\,dx \\ &\quad - 2(3\gamma - a)\int_{\mathbb{R}} u_x u m\,dx \\ &\quad - (3\gamma - a)\int_{\mathbb{R}} u m m_x dx - 3\gamma\int_{\mathbb{R}} u_x m_x^2 dx \\ &\quad - 2\gamma\int_{\mathbb{R}} u m m_x\,dx + \frac{\gamma}{2}\int_{\mathbb{R}} u_x m_x^2\,dx \\ &= -4(3\gamma - a)\int_{\mathbb{R}} u_x u m\,dx \\ &\quad + (5a - 17\gamma)\int_{\mathbb{R}} u m m_x\,dx - \frac{5}{2}\gamma\int_{\mathbb{R}} u_x m_x^2\,dx.\end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} (m^2 + m_x^2)\,dx &= -6(3\gamma - a)\int_{\mathbb{R}} u_x u m\,dx + 2(5a - 17\gamma)\int_{\mathbb{R}} u m m_x\,dx \\ &\quad - \int_{\mathbb{R}} (\gamma u_x)(3m^2 + 5m_x^2)\,dx.\end{aligned} \tag{13}$$

If γu_x is bounded from below on $[0, T^*_{m_0}) \times \mathbb{R}$, that is, there exists a positive constant $C_1 > 0$ such that $\gamma u_x \geq -C_1$ on $[0, T^*_{m_0}) \times \mathbb{R}$, then the above estimate implies

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} (m^2 + m_x^2)\,dx &\leq 5C_1\int_{\mathbb{R}} (m^2 + m_x^2)\,dx \\ &\quad + |2(3\gamma - a)|\,\|u(t,\cdot)\|_{L^\infty}\int_{\mathbb{R}} (u_x^2 + m^2)\,dx \\ &\quad + |5a - 17\gamma|\,\|u(t,\cdot)\|_{L^\infty}\int_{\mathbb{R}} (m^2 + m_x^2)\,dx \\ &\leq C_2(a, \gamma, \|u_0\|_{H^1})\int_{\mathbb{R}} (m^2 + m_x^2)\,dx,\end{aligned} \tag{14}$$

where we used (8) so that $\|u(t,\cdot)\|^2_{L^\infty} \leq (1/2)E(u(t,\cdot)) = (1/2)E(u_0) = (1/2)\|u_0\|^2_{H^1}$. Applying Gronwall's inequality then yields

$$\|m(t)\|^2_{H^1} = \int_{\mathbb{R}} (m^2 + m_x^2)\,dx \leq e^{C_2 t}\|m_0\|^2_{H^1} \tag{15}$$

for $t \in [0, T^*_{m_0})$, which ensures that the solution $m(t,x)$ does not blow up in finite time.

On the other hand, if

$$\liminf_{t\uparrow T^*_{u_0}} \left[\inf_{x\in\mathbb{R}} (\gamma u_x(t,x))\right] = -\infty, \tag{16}$$

by Theorem 3 for the existence of local strong solutions and the Sobolev embedding theorem, we infer that the solution will blow up in finite time. The proof of Theorem 7 is then complete. □

For $\gamma \neq 0$, we define

$$\mathfrak{s}(t) \stackrel{\text{def}}{=} \operatorname{sign}\{\gamma\}\inf_{x\in\mathbb{R}} [u_x(t,x)\operatorname{sign}\{\gamma\}], \quad t \geq 0, \tag{17}$$

where $\operatorname{sign}(\gamma)$ is the sign function of $\gamma \in \mathbb{R}$, and set $\mathfrak{s}_0 := \mathfrak{s}(0)$. Then, thanks to Theorem 3, for every $t \in [0, T)$ there exists at least one point $\xi(t) \in \mathbb{R}$ with $\mathfrak{s}(t) = u_x(t, \xi(t))$. Just as the proof given in [8], one can show the following property of $\mathfrak{s}(t)$.

Lemma 8. *Let $u(t)$ be the solution to (5) on $[0, T)$ with initial data $u_0 \in H^2(\mathbb{R})$, as given by Theorem 3. Then the function $\mathfrak{s}(t)$ is almost everywhere differentiable on $[0, T)$, with*

$$\frac{d\mathfrak{s}}{dt} = u_{tx}(t, \xi(t)) \quad \text{a.e. on } (0, T). \tag{18}$$

With Lemma 8 in hand, we may get the following result about the wave-breaking phenomena of solutions to the generalized hyperelastic-rod wave equation (3).

Theorem 9. *Let $a > 0$, $\gamma \neq 0$, and $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Assume that*

$$\inf_{x\in\mathbb{R}} \{u_0'(x)\} < -K \quad if\ \gamma > 0, \qquad or \quad \sup_{x\in\mathbb{R}} \{u_0'(x)\} > K \quad if\ \gamma < 0, \tag{19}$$

where

$$K := \begin{cases} \sqrt{\dfrac{|a-\gamma|}{2|\gamma|}}\|u_0\|_{H^1}, & if\ \gamma \in (-\infty, 0) \cup (a, +\infty), \\ \sqrt{\dfrac{a-(1+\alpha_0)\gamma}{2\gamma}}\|u_0\|_{H^1}, & if\ \gamma \in \left(0, \dfrac{a}{1+\alpha_0}\right], \\ 0, & if\ \gamma \in \left(\dfrac{a}{1+\alpha_0}, a\right) \end{cases} \tag{20}$$

with $\alpha_0 = \sqrt{(4a-3\gamma)/4\gamma} - (1/2)$ for $\gamma \in (0, a)$. Then the corresponding solution to (5) blows up in finite time. Moreover, the maximal time of existence is estimated above by

$$0 < T \leq \frac{-2\inf_{x\in\mathbb{R}}\{\operatorname{sign}(\gamma)\, u_0'(x)\}}{|\gamma|\left[(\inf_{x\in\mathbb{R}}\{\operatorname{sign}(\gamma)\, u_0'(x)\})^2 - K^2\right]} < +\infty. \tag{21}$$

Proof. Let $T > 0$ be the maximal existence time of the solution $u(t,x)$ to (5) with initial data u_0 as stated in Theorem 3. Applying Theorem 3, Remark 5, and a simple density argument, we need only to show that Theorem 9 holds with some $s \geq 5$. Here we assume $s = 5$ to prove the theorem.

Firstly, thanks to Lemma 8 and the definition of $\mathfrak{s}(t)$, there are $x_0 \in \mathbb{R}$ and $\xi(t) \in \mathbb{R}$ such that $u_0'(x_0) = \inf_{x\in\mathbb{R}} u_0'(x)$, (18) holds, and

$$u_{xx}(t, \xi(t)) = 0, \quad \forall t \in [0, T). \tag{22}$$

Differentiating (5) with respect to x and using the identity $\partial_x^2 p * f = p * f - f$, we obtain

$$u_{xt} = -\frac{\gamma}{2}u_x^2 - \gamma u u_{xx} + \frac{a-\gamma}{2}u^2 - p * \left(\frac{a-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right) - k u_{xx}, \tag{23}$$

which along with (22) implies

$$\frac{d}{dt}\mathfrak{s}(t) = -\frac{\gamma}{2}\mathfrak{s}(t)^2 + \frac{a-\gamma}{2}u^2(t,\xi(t)) - p * \left(\frac{a-\gamma}{2}u^2 + \frac{\gamma}{2}u_x^2\right)(t,\xi(t)) \quad \text{a.e. on } (0,T). \tag{24}$$

For the case $\gamma \geq a$, thanks to the Young's inequality and the Sobolev inequality, we get from Remark 6 that, for all $t \in [0, T)$,

$$\left\| p * \frac{a-\gamma}{2}u^2 \right\|_{L^\infty} \leq \frac{\gamma-a}{2}\|u(t)\|_{L^\infty}^2 \leq \frac{\gamma-a}{4}\|u(t)\|_{H^1}^2 = \frac{\gamma-a}{4}\|u_0\|_{H^1}^2. \tag{25}$$

Note that $p * ((\gamma/2)u_x^2) \geq 0$ and $((a-\gamma)/2)u^2(t,\xi(t)) \leq 0$, we infer that

$$\frac{d}{dt}\mathfrak{s}(t) \leq -\frac{\gamma}{2}\mathfrak{s}^2(t) + \frac{\gamma-a}{4}\|u_0\|_{H^1}^2 \quad \text{a.e. on } (0,T). \tag{26}$$

Set $K := \sqrt{(\gamma-a)/2\gamma}\|u_0\|_{H^1}$; we have

$$\frac{d}{dt}\mathfrak{s}(t) \leq -\frac{\gamma}{2}\mathfrak{s}^2(t) + \frac{\gamma}{2}K^2 \quad \text{a.e. on } (0,T). \tag{27}$$

We now claim that if $\mathfrak{s}(0) < -K$, then $\mathfrak{s}(t) < -K$ for all $t \in (0, T)$.

In fact, assuming that the contrary would, in view of $\mathfrak{s}(t)(t)$ being continuous, ensure the existence of some $t_0 \in (0,T)$, such that $\mathfrak{s}(t) < -K$ in $(0,t_0)$ and $\mathfrak{s}(t_0) = -K$. Combining this with (27) would give $(d/dt)\mathfrak{s}(t) < 0$ a.e. on $[0,t_0)$. Since $\mathfrak{s}(t)$ is absolutely continuous on $[0,t_0]$, an integration of this inequality would give the following inequality and we get the contradiction $\mathfrak{s}(t_0) < \mathfrak{s}(0) < -K$. This proves the previous claim.

Therefore it follows from (27) that $(d/dt)\mathfrak{s}(t) < 0$ a.e. on $(0,T)$ which implies that $\mathfrak{s}(t)$ is strictly decreasing on $[0,T)$.

Set $\delta := (\gamma/2)(1 - (K/u_0'(x_0))^2) \in (0, \gamma/2)$. Then we have $K^2/(1-(2/\gamma)\delta) = (u_0'(x_0))^2 < \mathfrak{s}^2(t)$, that is, $K^2 < (1-(2/\gamma)\delta)\mathfrak{s}^2(t)$. Therefore it follows that

$$\frac{d}{dt}\mathfrak{s}(t) \leq -\frac{\gamma}{2}\mathfrak{s}^2(t)\left[1 - \left(1 - \frac{2}{\gamma}\delta\right)\right] = -\delta\mathfrak{s}^2(t), \quad \text{a.e. on } (0,T), \tag{28}$$

which leads to

$$\mathfrak{s}(t) \leq \frac{u_0'(x_0)}{1 + \delta t u_0'(x_0)} \longrightarrow -\infty, \quad \text{as } t \longrightarrow -\frac{1}{\delta u_0'(x_0)}. \tag{29}$$

This implies that the maximal time of existence T satisfies

$$T \leq -\frac{1}{\delta u_0'(x_0)} = \frac{2\inf_{x\in\mathbb{R}} u_0'(x)}{\gamma\left[K^2 - (\inf_{x\in\mathbb{R}} u_0'(x))^2\right]} < +\infty. \tag{30}$$

In consequence, we have

$$\liminf_{t\uparrow T}\left(\inf_{x\in\mathbb{R}} \gamma u_x(t,x)\right) = -\infty, \tag{31}$$

which along with Theorem 7 applied completes the proof of Theorem 9 for the case $\gamma \geq a$.

Next, to deal with the case $0 < \gamma < a$, we need first to verify the following inequality for any positive real number α:

$$p * \left(\left(\alpha^2 + \alpha\right) u^2 + u_x^2\right)(x) \geq \alpha u^2(x) \quad \text{for } \forall u \in H^3(\mathbb{R}). \tag{32}$$

Indeed, observe that the inequality

$$\begin{aligned} e^{-x} \int_{-\infty}^{x} e^{y} \left(\alpha^2 u^2 + u_x^2\right)(y)\, dy &\geq 2\alpha e^{-x} \int_{-\infty}^{x} e^{y} u(y) u_x(y)\, dy \\ &= \alpha e^{-x} \int_{-\infty}^{x} e^{y} \frac{d}{dy}\left(u^2(y)\right) dy \\ &= \alpha u^2(x) - \alpha e^{-x} \int_{-\infty}^{x} e^{y} u^2(y)\, dy \end{aligned} \tag{33}$$

yields

$$e^{-x} \int_{-\infty}^{x} e^{y} \left(\left(\alpha^2 + \alpha\right) u^2 + u_x^2\right)(y)\, dy \geq \alpha u^2(x). \tag{34}$$

Whereas the inequality

$$\begin{aligned} e^{x} \int_{x}^{\infty} e^{-y} \left(\alpha^2 u^2 + u_x^2\right)(y)\, dy &\geq -2\alpha e^{x} \int_{x}^{\infty} e^{-y} u(y) u_x(y)\, dy \\ &= -\alpha e^{x} \int_{x}^{\infty} e^{-y} \frac{d}{dy}\left(u^2(y)\right) dy \\ &= \alpha u^2(x) - \alpha e^{x} \int_{x}^{\infty} e^{-y} u^2(y)\, dy \end{aligned} \tag{35}$$

leads to

$$e^{x} \int_{x}^{\infty} e^{-y} \left(\left(\alpha^2 + \alpha\right) u^2 + u_x^2\right)(y)\, dy \geq \alpha u^2(x), \tag{36}$$

which along with (34) gives rise to (32).

Notice that $\alpha_0 = \sqrt{(4a - 3\gamma)/4\gamma} - (1/2)$ is the only positive root of $\alpha^2 + \alpha = (a - \gamma)/\gamma$; we get from (32) that

$$p * \left(\frac{a - \gamma}{2} u^2 + \frac{\gamma}{2} u_x^2\right)(t, \xi(t)) \geq \frac{\gamma \alpha_0}{2} u_x^2(t, \xi(t)), \tag{37}$$

which together with (24) implies that

$$\begin{aligned} \frac{d}{dt}\mathfrak{s}(t) \leq& -\frac{\gamma}{2}\mathfrak{s}(t)^2 \\ &+ \frac{a - (1 + \alpha_0)\gamma}{2} u^2(t, \xi(t)) \quad \text{a.e. on } (0, T). \end{aligned} \tag{38}$$

Thanks to the Sobolev inequality and Remark 6, we get that, for any $t \in (0, T)$, $\|u(t)\|_{L^\infty}^2 \leq (1/2)\|u(t)\|_{H^1}^2 = (1/2)\|u_0\|_{H^1}^2$, which along with (38) implies that (27) holds for

$$K := \begin{cases} \sqrt{\dfrac{a - (1 + \alpha_0)\gamma}{2\gamma}} \|u_0\|_{H^1}, & \text{if } \gamma \in \left(0, \dfrac{a}{1 + \alpha_0}\right], \\ 0, & \text{if } \gamma \in \left(\dfrac{a}{1 + \alpha_0}, a\right). \end{cases} \tag{39}$$

Repeating the above argument in the proof of (31) for the case $\gamma \geq a$ leads to (31), and then the proof of Theorem 9 is complete for the case $0 < \gamma < a$.

Let us now consider the case $\gamma < 0$. It follows from (24) that

$$\frac{d}{dt}\mathfrak{s}(t) \geq \frac{|\gamma|}{2}\mathfrak{s}^2(t) - \frac{a - \gamma}{4}\|u_0\|_{H^1}^2 \quad \text{a.e. on } (0, T), \tag{40}$$

where we used Young's inequality so that, for all $t \in [0, T)$,

$$\left\|p * u^2\right\|_{L^\infty} \leq \|u(t)\|_{L^\infty}^2 \leq \frac{1}{2}\|u(t)\|_{H^1}^2. \tag{41}$$

Therefore, set $K := \sqrt{(a - \gamma)/ - 2\gamma}\|u_0\|_{H^1}$; we may get

$$\frac{d}{dt}\mathfrak{s}(t) \geq \frac{|\gamma|}{2}\mathfrak{s}^2(t) - \frac{|\gamma|}{2}K^2 \quad \text{a.e. on } (0, T). \tag{42}$$

Repeating the above argument in the proof of (31) for the case $\gamma \geq a$ again ends the proof of Theorem 9. □

3. Peaked Solitary Wave Solution

In this section we consider the existence of peaked solitary wave solutions of (3) in the case of $g(u) = au^2 + 2ku$ ($a, k \in \mathbb{R}$ and $k \neq 0$), which can be understood as global weak solutions.

Theorem 10. *For any $\gamma \in \mathbb{R} \setminus \{a/3, 0\}$, $a\ \gamma > 0$, and $k \neq 0$, the peaked function of the form*

$$\overline{u}_c(t, x) = \frac{c}{\gamma} e^{-\sqrt{a/3\gamma}|x - ct|}, \quad \text{with } c = \frac{3k\gamma}{3\gamma - a} \tag{43}$$

is a global weak solution to (3) *in the sense of Definition* (2).

Remark 11. It is known that for the Camassa-Holm equation (that is, (5) with $\gamma = 1$ and $a = 3$), for any $c \neq 0$, the peaked function of the form $u_c(t, x) = ce^{-|x - ct|}$ is a global weak solution to the Camassa-Holm equation with the case $k = 0$ in the sense of Definition (2), and if $k \neq 0$, the Camassa-Holm equation has no peaked solution. Here Theorem 10 implies that, if $k \neq 0$, (5) still has some peaked solutions, as long as $\gamma \neq a/3$ and 0.

Proof of Theorem 10. The proof of the theorem is motivated by the method in [10]. First, we can reduce the result to the one with the case $\gamma = 1$. In fact, multiplying γ with (1) and setting $\overline{u} = \gamma u$, (1) become the following form:

$$\overline{u}_t - \overline{u}_{txx} + \left(\frac{a}{2\gamma}\overline{u}^2 + k\overline{u}\right)_x = 2\overline{u}_x\overline{u}_{xx} + \overline{u}\,\overline{u}_{xxx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{44}$$

$$\overline{u}(0,x) = \overline{u}_0(x), \quad x \in \mathbb{R}. \tag{45}$$

We claim that, for all $t \in \mathbb{R}^+$,

$$\partial_x\overline{u}_c(t,x) = -\sqrt{\frac{a}{3\gamma}}\operatorname{sign}(x-ct)\,\overline{u}_c(t,x) \in L^\infty(\mathbb{R}), \tag{46}$$

in the sense of distribution $\mathcal{S}'(\mathbb{R})$.

Indeed, for any test function $\varphi(\cdot) \in C_c^\infty(\mathbb{R})$, we may get by using integration by parts that

$$\begin{aligned}
&\int_{\mathbb{R}} \operatorname{sign}(y)\, e^{-\sqrt{a/3\gamma}|y|}\varphi(y)\,dy \\
&= -\int_{-\infty}^{0} e^{\sqrt{a/3\gamma}y}\varphi(y)\,dy + \int_0^{+\infty} e^{-\sqrt{a/3\gamma}y}\varphi(y)\,dy \\
&= -\varphi(0) + \sqrt{\frac{3\gamma}{a}}\int_{-\infty}^{0} e^{\sqrt{a/3\gamma}y}\varphi'(y)\,dy + \varphi(0) \\
&\quad + \sqrt{\frac{3\gamma}{a}}\int_0^{+\infty} e^{-\sqrt{a/3\gamma}y}\varphi'(y)\,dy \\
&= \sqrt{\frac{3\gamma}{a}}\int_{\mathbb{R}} e^{-\sqrt{a/3\gamma}|y|}\varphi'(y)\,dy,
\end{aligned} \tag{47}$$

which along with the fact that $\operatorname{sign}(\cdot - ct)\overline{u}_c(t,\cdot) \in L^\infty(\mathbb{R})$ ends the proof of the claim.

Let us now set $\overline{u}_{0,c}(x) := \overline{u}_c(0,x)$ for $x \in \mathbb{R}$. Then we have

$$\lim_{t\to 0^+}\left\|\overline{u}_c(t,\cdot) - \overline{u}_{0,c}(\cdot)\right\|_{W^{1,\infty}} = 0. \tag{48}$$

Similar to the proof of (46), we get that, for all $t \geq 0$,

$$\partial_t\overline{u}_c(t,x) = c\operatorname{sign}(x-ct)\,\overline{u}_c(t,x) \in L^\infty(\mathbb{R}). \tag{49}$$

Therefore, from (46), (48), (49), and $c = 3k\gamma/(3\gamma - a)$ and $\gamma = 1$, together with integration by parts applied, we obtain that, for every test function $\phi(t,x) \in C_c^\infty([0,+\infty)\times\mathbb{R})$,

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}}\left(\overline{u}_d\partial_t\phi + \frac{1}{2}\overline{u}_c^2\partial_x\phi\right)dx\,dt \\
&\quad + \int_{\mathbb{R}}\overline{u}_c(0,x)\,\phi(0)\,dx\,dt \\
&= -\int_0^{+\infty}\int_{\mathbb{R}}\phi\left(\partial_t\overline{u}_c + \overline{u}_c\partial_x\overline{u}_c\right)dx\,dt \\
&= -\int_0^{+\infty}\int_{\mathbb{R}}\phi\left(c\sqrt{\frac{a}{3}}\operatorname{sign}(x-ct)\,ce^{-\sqrt{a/3}|x-ct|}\right. \\
&\qquad - \sqrt{\frac{a}{3}}\operatorname{sign}(x-ct) \\
&\qquad \left.\times c^2e^{-2\sqrt{a/3}|x-ct|}\right)dx\,dt \\
&= -\int_0^{+\infty}\int_{\mathbb{R}}\phi c^2\sqrt{\frac{a}{3}}\operatorname{sign}(x-ct) \\
&\qquad \times\left(e^{-\sqrt{a/3}|x-ct|} - e^{-2\sqrt{a/3}|x-ct|}\right)dx\,dt,
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x\overline{u}_c)^2 + k\overline{u}_c\right)\phi_x dx\,dt \\
&= -\int_0^{+\infty}\int_{\mathbb{R}}\phi\partial_x p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x\overline{u}_c)^2 + k\overline{u}_c\right)dx\,dt.
\end{aligned} \tag{51}$$

Notice from (46) that $\partial_x p = -(1/2)\operatorname{sign}(x)e^{-|x|}$ for $x \in \mathbb{R}$; we have

$$\begin{aligned}
&\partial_x p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x\overline{u}_c)^2 + k\overline{u}_c\right) \\
&= -\frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sign}(x-y)\,e^{-|x-y|} \\
&\qquad \times\left(\frac{4a-3}{6}c^2e^{-2\sqrt{a/3}|y-ct|} + kce^{-\sqrt{a/3}|y-ct|}\right)dy.
\end{aligned} \tag{52}$$

When $x > ct$, we split the right hand side of (52) into the following three parts:

$$\begin{aligned}
&\partial_x p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x\overline{u}_c)^2 + k\overline{u}_c\right) \\
&= -\frac{1}{2}\left(\int_{-\infty}^{ct} + \int_{ct}^{x} + \int_x^{+\infty}\right)\operatorname{sign}(x-y)\,e^{-|x-y|} \\
&\qquad \times\left(\frac{4a-3}{6}c^2e^{-2\sqrt{a/3}|y-ct|} + kce^{-\sqrt{a/3}|y-ct|}\right)dy \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{53}$$

We directly compute I_1 as follows:

$$\begin{aligned} I_1 &= -\frac{1}{2}\int_{-\infty}^{ct} e^{-(x-y)}\left(\frac{4a-3}{6}c^2 e^{2\sqrt{a/3}(y-ct)} \right.\\ &\qquad \left. + kce^{\sqrt{a/3}(y-ct)}\right)dy \\ &= \left(-\frac{4a-3}{12\left(1+2\sqrt{a/3}\right)}d^2 - \frac{kc}{2\left(1+\sqrt{a/3}\right)}\right)e^{-(x-ct)}. \end{aligned} \tag{54}$$

Similarly, we have

$$\begin{aligned} I_2 &= \left(-\frac{4a-3}{12\left(1-2\sqrt{a/3}\right)}c^2\right)e^{-2\sqrt{a/3}(x-ct)} \\ &\quad + \left(-\frac{kc}{2\left(1-\sqrt{a/3}\right)}\right)e^{-\sqrt{a/3}(x-ct)} \\ &\quad + \left(\frac{4a-3}{12\left(1-2\sqrt{a/3}\right)}c^2 + \frac{kc}{2\left(1-\sqrt{a/3}\right)}\right)e^{-(x-ct)} \end{aligned} \tag{55}$$

and $I_3 = ((4a-3)/(12(1+2\sqrt{a/3})))c^2e^{-2\sqrt{a/3}(x-ct)} + (kc/(2(1+\sqrt{a/3})))e^{-\sqrt{a/3}(x-ct)}$. From this, we deduce that, for $x > ct$,

$$\begin{aligned} &\partial_x p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x \overline{u}_c)^2 + k\overline{u}_c\right) \\ &\quad = \sqrt{\frac{a}{3}}c^2\left(e^{-2\sqrt{a/3}(x-ct)} - e^{-\sqrt{a/3}(x-ct)}\right). \end{aligned} \tag{56}$$

And for the case $x \le ct$, we spilt the right hand side of (51) into the following three parts:

$$\begin{aligned} &\partial_x p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x \overline{u}_c)^2 + k\overline{u}_c\right) \\ &\quad = -\frac{1}{2}\left(\int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{+\infty}\right)\operatorname{sign}(x-y)\,e^{-|x-y|} \\ &\qquad \times \left(\frac{4a-3}{6}c^2e^{-2\sqrt{a/3}|y-ct|} + kce^{-\sqrt{a/3}|y-ct|}\right)dy \\ &\quad =: II_1 + II_2 + II_3. \end{aligned} \tag{57}$$

We compute

$$\begin{aligned} II_1 &= -\frac{1}{2}\int_{-\infty}^{x} e^{-(x-y)}\left(\frac{4a-3}{6}c^2e^{2\sqrt{a/3}(y-ct)} \right. \\ &\qquad \left. + kce^{\sqrt{a/3}(y-ct)}\right)dy \\ &= -\frac{4a-3}{12\left(1+2\sqrt{a/3}\right)}c^2e^{2\sqrt{a/3}(x-ct)} \\ &\quad - \frac{kc}{2\left(1+\sqrt{a/3}\right)}e^{\sqrt{a/3}(x-ct)}. \end{aligned} \tag{58}$$

Similarly, we have

$$\begin{aligned} II_2 &= \frac{4a-3}{12\left(2\sqrt{a/3}-1\right)}c^2\left(e^{x-ct} - e^{2\sqrt{a/3}(x-ct)}\right) \\ &\quad + \frac{kc}{2\left(\sqrt{a/3}-1\right)}\left(e^{x-ct} - e^{\sqrt{a/3}(x-ct)}\right) \end{aligned} \tag{59}$$

and $II_3 = (((4a-3)/(12(1+2\sqrt{a/3})))c^2 + kc/(2(1+\sqrt{a/3})))e^{x-ct}$, which along with (57) and (58) implies that, for $x \le ct$,

$$\begin{aligned} &\partial_x p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x \overline{u}_c)^2 + k\overline{u}_c\right) \\ &\quad = \sqrt{\frac{a}{3}}c^2\left(e^{\sqrt{a/3}(x-ct)} - e^{2\sqrt{a/3}(x-ct)}\right). \end{aligned} \tag{60}$$

From (56) and (60), we obtain

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}} p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}(\partial_x \overline{u}_c)^2 + k\overline{u}_c\right)\phi_x dx\,dt \\ &= \begin{cases} -\sqrt{\frac{a}{3}}c^2 \times \int_0^{+\infty}\int_{\mathbb{R}} \phi\left(e^{-2\sqrt{a/3}(x-ct)} - e^{-\sqrt{a/3}(x-ct)}\right)dx\,dt, & x > ct, \\ -\sqrt{\frac{a}{3}}c^2 \times \int_0^{+\infty}\int_{\mathbb{R}} \phi\left(e^{\sqrt{a/3}(x-ct)} - e^{2\sqrt{a/3}(x-ct)}\right)dx\,dt, & x \le ct. \end{cases} \end{aligned} \tag{61}$$

On the other hand, by using the form of $\overline{u}_c$, we get from (50) that

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}}\left(\overline{u}_c\partial_t\phi + \frac{1}{2}\overline{u}_c^2\partial_x\phi\right)dx\,dt \\ &\quad + \int_{\mathbb{R}} \overline{u}_d(0,x)\,\phi(0)\,dx\,dt \\ &= \begin{cases} -\sqrt{\frac{a}{3}}c^2 \times \int_0^{+\infty}\int_{\mathbb{R}} \phi\left(e^{-\sqrt{a/3}(x-ct)} - e^{-2\sqrt{a/3}(x-ct)}\right)dx\,dt, & x > ct, \\ \sqrt{\frac{a}{3}}c^2 \times \int_0^{+\infty}\int_{\mathbb{R}} \phi\left(e^{\sqrt{a/3}(x-ct)} - e^{2\sqrt{a/3}(x-ct)}\right)dx\,dt, & x \le ct. \end{cases} \end{aligned} \tag{62}$$

Therefore, for every test function $\phi(t,x) \in C_c^{\infty}([0,+\infty)\times\mathbb{R})$, we conclude that

$$\int_0^{+\infty}\int_{\mathbb{R}}\left[\overline{u}_c\partial_t\phi + \frac{1}{2}\overline{u}_c^2\partial_x\phi \right.$$
$$\left. + p * \left(\frac{a-1}{2}\overline{u}_c^2 + \frac{1}{2}\left(\partial_x\overline{u}_c\right)^2 + k\overline{u}_c\right)\partial_x\phi\right] dx\, dt$$
$$+ \int_{\mathbb{R}} \overline{u}_c(0,x)\,\phi(0,x)\,dx = 0, \tag{63}$$

which completes the proof of Theorem 10. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] H.-H. Dai, "Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod," *Acta Mechanica*, vol. 127, no. 1–4, pp. 193–207, 1998.

[2] H.-H. Dai and Y. Huo, "Solitary shock waves and other travelling waves in a general compressible hyperelastic rod," *Proceedings of the Royal Society of London A*, vol. 456, no. 1994, pp. 331–363, 2000.

[3] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.

[4] T. B. Benjamin, J. L. Bona, and J. J. Mahony, "Model equations for long waves in nonlinear dispersive systems," *Philosophical Transactions of the Royal Society of London A*, vol. 272, no. 1220, pp. 47–78, 1972.

[5] M. S. Alber, R. Camassa, D. D. Holm, and J. E. Marsden, "The geometry of peaked solitons and billiard solutions of a class of integrable PDEs," *Letters in Mathematical Physics*, vol. 32, no. 2, pp. 137–151, 1994.

[6] C. Cao, D. D. Holm, and E. S. Titi, "Traveling wave solutions for a class of one-dimensional nonlinear shallow water wave models," *Journal of Dynamics and Differential Equations*, vol. 16, no. 1, pp. 167–178, 2004.

[7] A. Constantin and J. Escher, "Global existence and blow-up for a shallow water equation," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze*, vol. 26, no. 2, pp. 303–328, 1998.

[8] A. Constantin and J. Escher, "Wave breaking for nonlinear nonlocal shallow water equations," *Acta Mathematica*, vol. 181, no. 2, pp. 229–243, 1998.

[9] A. Constantin and W. A. Strauss, "Stability of peakons," *Communications on Pure and Applied Mathematics*, vol. 53, no. 5, pp. 603–610, 2000.

[10] G. Gui, Y. Liu, P. J. Olver, and C. Qu, "Wave-breaking and peakons for a modified Camassa-Holm equation," *Communications in Mathematical Physics*, vol. 319, no. 3, pp. 731–759, 2013.

[11] A. Bressan and A. Constantin, "Global conservative solutions of the Camassa-Holm equation," *Archive for Rational Mechanics and Analysis*, vol. 183, no. 2, pp. 215–239, 2007.

[12] A. Bressan and A. Constantin, "Global dissipative solutions of the Camassa-Holm equation," *Analysis and Applications*, vol. 5, no. 1, pp. 1–27, 2007.

[13] H. Holden and X. Raynaud, "Global conservative solutions of the Camassa-Holm equation—a Lagrangian point of view," *Communications in Partial Differential Equations*, vol. 32, no. 10–12, pp. 1511–1549, 2007.

[14] H. Holden and X. Raynaud, "Dissipative solutions for the Camassa-Holm equation," *Discrete and Continuous Dynamical Systems A*, vol. 24, no. 4, pp. 1047–1112, 2009.

[15] Z. Xin and P. Zhang, "On the weak solutions to a shallow water equation," *Communications on Pure and Applied Mathematics*, vol. 53, no. 11, pp. 1411–1433, 2000.

[16] A. Constantin and W. A. Strauss, "Stability of a class of solitary waves in compressible elastic rods," *Physics Letters A*, vol. 270, no. 3-4, pp. 140–148, 2000.

[17] H. Holden and X. Raynaud, "Global conservative solutions of the generalized hyperelastic-rod wave equation," *Journal of Differential Equations*, vol. 233, no. 2, pp. 448–484, 2007.

[18] G. M. Coclite, H. Holden, and K. H. Karlsen, "Global weak solutions to a generalized hyperelastic-rod wave equation," *SIAM Journal on Mathematical Analysis*, vol. 37, no. 4, pp. 1044–1069, 2005.

[19] T. Kato, "Quasi-linear equations of evolution, with applications to partial differential equations," in *Spectral Theory and Differential Equations*, W. N. Everitt, Ed., vol. 448 of *Lecture Notes in Mathematics*, pp. 25–70, Springer, Berlin, Germany, 1975.