

Research Article

Mathematical Models Arising in the Fractal Forest Gap via Local Fractional Calculus

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The forest new gap models via local fractional calculus are investigated. The JABOWA and FORSKA models are extended to deal with the growth of individual trees defined on Cantor sets. The local fractional growth equations with local fractional derivative and difference are discussed. Our results are first attempted to show the key roles for the nondifferentiable growth of individual trees.

1. Introduction

Fractals had been used to describe special problems in biology and ecology [1–4] because of the measure of nature objects underlying the geometry, replacing the complex real-world objects by describing the Euclidean ideas. Fractal dimension was applied to describe the measure of the complexity in biology and ecology. In forestry, the fractal geometry had been applied to estimate stand density, predict forest succession, and describe the form of trees [5–7]. The scaling of dynamics in hierarchical structure was investigated in [8–12]. The ecological resilience example from boreal forest was presented in the context [8]. The quantitative theory of forest structure was discussed [9]. The allometric scaling laws in biology were proposed in the works [10, 11]. Based upon the cross-scale analysis, the geometry and dynamics of ecosystems were considered and the structure ecosystems across scales in time and space were discussed in [12, 13]. Fractal forestry was modeled by using the scaling of the testing parameters for ecological complexity.

Forest gap model (JABOWA) developed by Botkin et al. [14–16] was the first simulation model for gap-phase replacement. It was applied to describe a forest as a mosaic of closed canopies and simulate forest dynamics based upon

the establishment, growth, and death of individual trees [17–20]. The JABOWA model in the form of the FORET model (called JABOWA-FORET) was further developed in [21–24]. The JABOWA model of the simulation of stand structure in a forest gap model was improved in [24] and the FORSKA [25] was proposed by Botkin et al. In [14–16], the ecological functions are continuous. In [10–13], the ecological functions were expressed across scales in time and space. However, as it is shown in Figure 1 the ecological functions distinguishing hierarchical size scales in nature, such as the measures of tree size and measure of soil fertility, are defined on Cantor sets. The above approaches do not deal with them.

Local fractional calculus theory [26–38] was applied to handle the nondifferentiable functions defined on Cantor sets. The heat-conduction, transport, Maxwell, diffusion, wave, Fokker-Planck, and the mechanical structure equations were usefully shown (see for more details [28–36] and the cited references therein). In order to simulate forest dynamics on the basis of the establishment, growth, and death of individual trees defined on Cantor sets, the aim of this paper is to present the forest new gap models for simulating the gap-phase replacement by employing the local fractional calculus.

The paper has been organized as follows. In Section 2, we review the JABOWA and FORSKA models for the forest

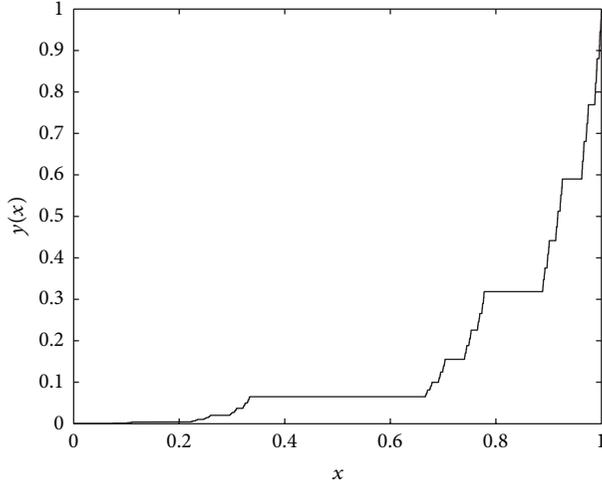


FIGURE 1: The ecological function $f(x)$ defined on Cantor set.

succession. In Section 3, we propose JABOWA and FORSKA models for the fractal forest succession. Finally, Section 4 is conclusions.

2. Growth Models for Forest Gap

In this section we will revise the JABOWA and FORSKA models.

2.1. The JABOWA Model. The growth equation with difference form is given by [15, 16, 24, 39]

$$\frac{\Delta D}{\Delta t} = G \cdot D \cdot \phi(D) \cdot \frac{1}{b(D)} \cdot f(e), \quad (1)$$

where the function D is diameter at breast height of the trees, the parameter H is tree height, G is a growth rate, the function $b(D)$ is a quantity encapsulating this allometric relationship, $f(e)$ is a quantity influencing the abiotic and biotic environment on tree growth, $\phi(D) = (1 - D \cdot H / (D_{\max} \cdot H_{\max}))$, and D_{\max} and H_{\max} are the maximum measures of the tree dimensions.

The parameter $f(e)$ is simulated as follows [24]:

$$f(e) = g_1(\text{AL}) \cdot g_2(\text{SBAR}) \cdot g_3(\text{DD}), \quad (2)$$

where $g_1(\text{AL})$ is a quantity of available light, $g_2(\text{SBAR})$ is a quantity of stand basal area, and $g_3(\text{DD})$ is a quantity of the annual degree-day sum. It was referred to as Liebig's law of the minimum [24].

The allometric relationship with a parabolic form is written as follows [24, 40]:

$$b(D) = b_1 + b_2 D + b_3 D^2, \quad (3)$$

where b_1 , b_2 , and b_3 are parameters.

Leaf area index reads as follows [24]:

$$\text{LAI} = \mu D^2, \quad (4)$$

where $\mu = c/k$ with a species-specific parameter c and the scale leaf weight per tree to the projected leaf area k .

In order to implement a new height-diameter relationship [40], the differential form of growth equation in the JABOWA model was suggested as follows [41]:

$$\frac{dD}{dt} = G \cdot D \cdot \varphi(D), \quad (5)$$

where the function has the following form:

$$\varphi(D) = \frac{1 - H/H_{\max}}{2H_{\max} - b(-sD/b + 2)e^{-Ds/b}} \quad (6)$$

with the parameter $b = H_{\max} - 137$ and the initial slope value of the height diameter relationship s .

2.2. The FORSKA Model. The FORSKA model was developed for unmanaged natural forests and tree height had H - D relationship with the FORSKA model given by [24, 25, 39, 41]

$$H = 1.3 + (H_{\max} - 1.3) \cdot (1 - e^{-sD/(H_{\max}-1.3)}), \quad (7)$$

where the parameter s is the initial slope value of the height diameter relationship at $D = 0$, H_{\max} is the maximum measure of the tree dimension, D is diameter at breast height of the trees, and H is tree height.

As it is known, the trees in the real forest do not follow the H - D relationship. The growth equation with differential form can be written as follows [24, 25]:

$$\frac{dH}{dD} = s \cdot f(H), \quad (8)$$

where

$$f(H) = \frac{H - 1.3}{H_{\max} - 1.3}. \quad (9)$$

3. The JABOWA and FORSKA Models for the Fractal Forest Succession

In this section, based upon the local fractional calculus theory, we show the JABOWA and FORSKA models for the fractal forest succession. At first, we start with the local fractional derivative.

3.1. Local Fractional Derivative. We now give the local fractional calculus and the recent results.

If

$$|f(x) - f(x_0)| < \varepsilon^\alpha \quad (10)$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$, then we denote [26-28]

$$f(x) \in C_\alpha(a, b). \quad (11)$$

If $f(x) \in C_\alpha(a, b)$, then we have [26]

$$\dim_H(F \cap (a, b)) = \dim_H(C_\alpha(a, b)) = \alpha, \quad (12)$$

where $C_\alpha(a, b) = \{f : f(x) \text{ is local fractional continuous, } x \in F \cap (a, b)\}$.

Let $f(x) \in C_\alpha(a, b)$. The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined as [26–34]

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad (13)$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1 + \alpha)\Delta(f(x) - f(x_0))$.

For $0 < \alpha \leq 1$, the increment of $f(x)$ can be written as follows [26, 27]:

$$\Delta^\alpha f(x) = f^{(\alpha)}(x) (\Delta x)^\alpha + \lambda (\Delta x)^\alpha, \quad (14)$$

where Δx is an increment of x and $\lambda \rightarrow 0$ as $\Delta x \rightarrow 0$.

For $0 < \alpha \leq 1$, the α -local fractional differential of $f(x)$ reads as [26, 27]

$$d^\alpha f = f^{(\alpha)}(x) (dx)^\alpha. \quad (15)$$

From (14), we have approximate formula in the form

$$\Delta^\alpha f(x) \cong \Gamma(1 + \alpha) \Delta(f(x) - f(x_0)). \quad (16)$$

Let $f(x) \in C_\alpha(a, b)$. The local fractional integral of $f(x)$ of order α is given by [26–31]

$$\begin{aligned} {}_a J_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{j=N-1} f(t_j) (\Delta t_j)^\alpha, \end{aligned} \quad (17)$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \dots\}$, and $[t_j, t_{j+1}]$, $j = 0, \dots, N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

The α -dimensional Hausdorff measure H_α is calculated by [26]

$$H_\alpha(F \cap (0, x)) = {}_0 I_x^{(\alpha)} 1 = \frac{x^\alpha}{\Gamma(1 + \alpha)}. \quad (18)$$

3.2. The Local Fractional JABOWA Models (LFJABOWA). Here, we structure the LFJABOWA models via local fractional derivative and difference.

From (5), when

$$G \cdot \varphi(D) = \lambda_0 D^{\beta-1}, \quad (19)$$

the Enquist growth model in JABOWA model reads as [11, 42]

$$\frac{dD}{dt} = \lambda_0 D^\beta, \quad (20)$$

where D is the diameter at breast height, λ_0 is the scaling coefficient, and β is the fractal dimension.

Making use of the fractional complex transform [29] and (20), the growth equation in the JABOWA model with local fractional derivative (LFJABOWA) is suggested by

$$\frac{d^\alpha D}{dt^\alpha} = \lambda_0 D^\beta, \quad (21)$$

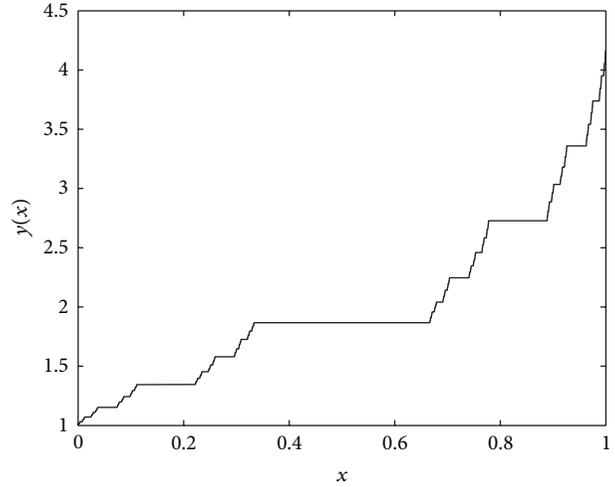


FIGURE 2: The graph of (22) with parameters $\beta = 1$, $\lambda_0 = 1$, and $t_0 = 0$.

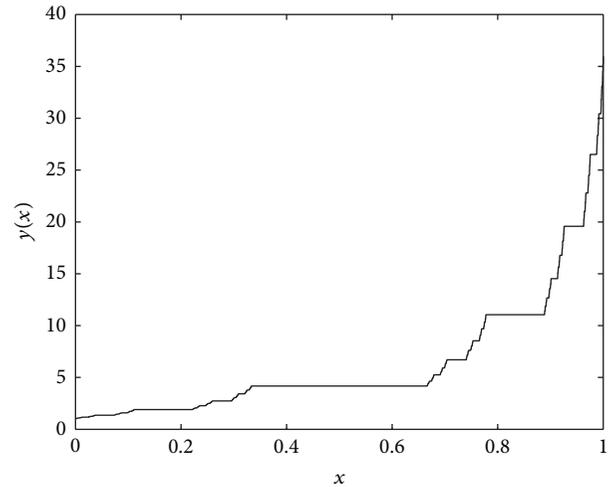


FIGURE 3: The graph of (22) with parameters $\beta = 1$, $\lambda_0 = 2$, and $t_0 = 0$.

where D is the diameter at breast height, λ_0 is the scaling coefficient, and α and β are the fractal dimensions. In order to illustrate the difference from the works presented in [42], we consider the following case: when $\beta = 1$, (21) can be integrated to give

$$D(t) = E_\alpha(\lambda_0 t^\alpha) - E_\alpha(\lambda_0 t_0^\alpha). \quad (22)$$

For the parameters $\beta = 1$, $t = 0$, the solutions of (21) with different values $\lambda_0 = 1$, $\lambda_0 = 2$, $\lambda_0 = 3$, and $\lambda_0 = 4$ are, respectively, shown in Figures 2, 3, 4, and 5.

Using the fractional complex transform, (5) becomes into

$$\frac{d^\alpha D}{dt^\alpha} = G \cdot D \cdot \varphi(D). \quad (23)$$

Comparing (22) and (23), we have

$$\frac{d^\alpha D}{dt^\alpha} = \psi(t, D), \quad (24)$$

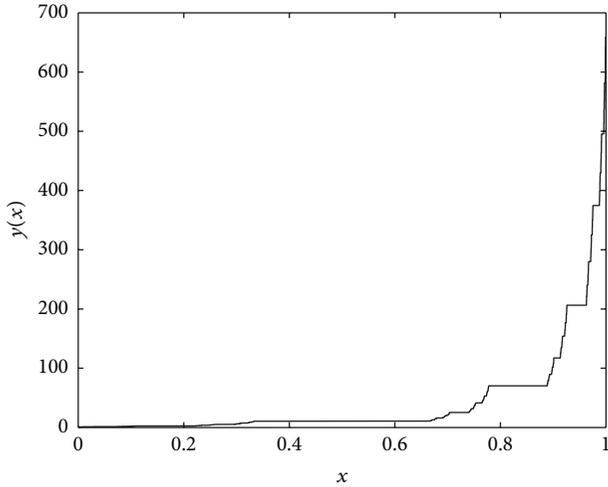


FIGURE 4: The graph of (22) with parameters $\beta = 1$, $\lambda_0 = 3$, and $t_0 = 0$.

where

$$\psi(t, D) = G \cdot D \cdot \varphi(D) \tag{25}$$

or

$$\psi(t, D) = \lambda_0 D^\beta. \tag{26}$$

From (25) and (26), we could get

$$\varphi(D) = D^{\beta-1}, \tag{27}$$

$$\lambda_0 = G. \tag{28}$$

Hence, the local fractional JABOWA model (LFJABOWA) reads as follows:

$$\frac{d^\alpha D}{dt^\alpha} = \psi(t, D), \tag{29}$$

where $\psi(t, D)$ is a nondifferentiable function and D is the diameter at breast height.

In view of (16), from (29) we give the difference form of local fractional JABOWA model (LFJABOWA) in the following form:

$$\frac{\Delta^\alpha D}{(\Delta t)^\alpha} = \psi(t, D), \tag{30}$$

where D is the diameter at breast height and $\Delta^\alpha D(t) \cong \Gamma(1 + \alpha)\Delta(D(t) - D(t_0))$.

When the fractal dimension α is equal to 1, we get the generalized form of (1); namely,

$$\frac{\Delta D}{\Delta t} = \psi(t, D), \tag{31}$$

where

$$\psi(t, D) = G \cdot D \cdot \phi(D) \cdot \frac{1}{b(D)} \cdot f(e). \tag{32}$$

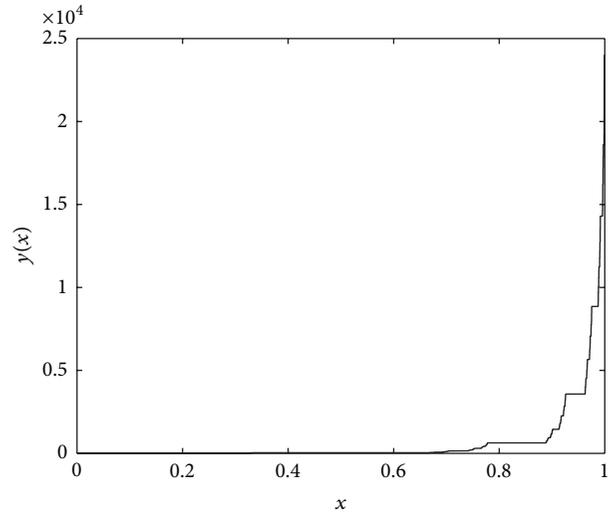


FIGURE 5: The graph of (22) with parameters $\beta = 1$, $\lambda_0 = 4$, and $t_0 = 0$.

3.3. *The Local Fractional FORSKA Models (LFFORSKA).* Here, we present the LFFORSKA models via local fractional derivative and difference.

The tree height H is the nondifferentiable function; namely,

$$|H(D) - H(D_0)| < \varsigma^\vartheta, \tag{33}$$

where for $\varsigma, \vartheta > 0$, $\varsigma, \vartheta \in R$, and $|D - D_0| < \vartheta$.

Following the fractional complex transform [29], from (8) we have the local fractional growth equation in the following form:

$$\frac{d^\alpha H}{dD^\alpha} = s \cdot f(H), \tag{34}$$

where $f(H)$ is a local fractional continuous function, H is the tree height, and s is a parameter.

Therefore, the generalized form of (34) is suggested as follows:

$$\frac{d^\alpha H}{dD^\alpha} = \chi(D, H), \tag{35}$$

where D is the diameter at breast height and H is the tree height. In view of (16), (35) can be rewritten as follows:

$$\frac{\Delta^\alpha H}{(\Delta D)^\alpha} = \chi(D, H), \tag{36}$$

where

$$|D - D_0| < \vartheta, \tag{37}$$

$$|H(D) - H(D_0)| < \varsigma^\vartheta.$$

The expression (36) is the difference form of local fractional FORSKA model (LFFORSKA).

4. Conclusions

In this work we investigated the local fractional models for the fractal forest succession. Based on the local fractional operators, we suggested the differential and difference forms of the local fractional JABOWA and FORSKA models. The nondifferentiable growths of individual trees were discussed. It is a good start for solving the rhetorical models for the fractal forest succession in the mathematical analysis.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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