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Research Article

On Properties of Pseudointegrals Based on Pseudoaddition Decomposable Measures

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We mainly discussed pseudointegrals based on a pseudoaddition decomposable measure. Particularly, we give the definition of the pseudointegral for a measurable function based on a strict pseudoaddition decomposable measure by generalizing the definition of the pseudointegral of a bounded measurable function. Furthermore, we got several important properties of the pseudointegral of a measurable function based on a strict pseudoaddition decomposable measure.

1. Introduction

The classical measure theory is one of the most important theories in mathematics [1, 2]. Although the additive measures are widely used, they do not allow modelling many phenomena involving interaction between criteria. For this reason, the fuzzy measure proposed by Sugeno is an extension of classical measure in which the additivity is replaced by a weaker condition, that is, monotonicity [3, 4]. Therefore, fuzzy measure and the corresponding integrals, for example, Choquet and Sugeno, are introduced [5–10].

So far, there have been many different fuzzy measures, such as the decomposable measure, the λ-additive measure, the belief measure, the possibility measure, and the plausibility measure. Among the fuzzy measures mentioned before, the decomposable measure was independently introduced by Dubois and Prade [11] and Weber [12]. Since the close relations with the classical measure theory, further developments of decomposable measures and related integrals have been extensive [13–18]. Decomposable measures include several well-known fuzzy measures such as the λ-additive measure and probability and possibility measures, and they provide a natural setting for relaxing probabilistic assumptions regarding the modeling of uncertainty [19, 20]. Decomposable measures and the corresponding integrals are very useful in decision theory and the theory of nonlinear differential and integral equations [21–24].

In many problems with uncertainty as in the theory of probabilistic metric spaces [20, 25, 26], multivalued logics [27, 28], and general measures [1, 4] often we work with many operations different from the usual addition and multiplication of reals. Some of them are triangular norms, triangular conorms, pseudoadditions, pseudomultiplications, and so forth [21, 29]. Based on the above-mentioned measures, pseudoanalysis as a generalization of the classical analysis is developed, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, +\infty]$ endowed with pseudoaddition $\oplus$ and with pseudomultiplication $\odot$ (see [13, 19, 30–33]). The families of the pseudooperations generated by a function g turn out to be solutions of well-known nonlinear functional equations [22–24].

In this paper, we will discuss pseudointegrals based on pseudoaddition decomposable measures. In Section 2, we recall the concepts of the pseudoaddition $\oplus$ and the pseudomultiplication $\odot$, which form a real semiring on the interval $[a, b] \subset [-\infty, +\infty]$ and the notion of the σ-$\oplus$-decomposable measure. Then we will give the definition of the pseudointegral of a measurable function based on a strict pseudoaddition decomposable measure by generalizing the definition of the pseudointegral of a bounded measurable function. In Section 3, we will discuss several important properties of the pseudointegral of a measurable function based on the strict pseudoaddition decomposable measure.

2. Preliminaries

Let $[a, b]$ be a closed subinterval of $\mathbb{R}$ (in some cases we will also take semiclosed subintervals). The total order on $[a, b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can also be another order. We will denote by Δ maximum element on $[a, b]$ (usually Δ is either a or b) with respect to this total order.

Definition 1 (see [34]). Let $\{x_n\}$ be a sequence from $[a, b]$.

(1) If $x_m \preceq x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is an increasing sequence.

(2) If $x_m \prec x_n$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is a strict increasing sequence.

(3) If $x_n \preceq x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is a decreasing sequence.

(4) If $x_n \prec x_m$ whenever $n > m$, then we say that the sequence $\{x_n\}$ is a strict decreasing sequence.

Let X be a nonempty set; we will denote by $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\mathcal{A}}$, and $\mathscr{B}_X$ algebra, σ-algebra, and Borel σ-algebra of subsets of a set X, respectively.

Denote by $\mathscr{F}(X)$ the set of all functionals from X to $[a, b]$. For each $\lambda \in [a, b]$ the constant functional in $\mathscr{F}(X)$ with value λ will also be denoted by λ. It will be clear from the context which usage is intended. A functional $f \in \mathscr{F}(X)$ is said to be finite if $f(x) \prec \Delta$ for all $x \in X$. The functional $f \in \mathscr{F}(X)$ is said to be bounded if there exists $\Omega \prec \Delta$, such that $f(x) \preceq \Omega$ for all $x \in X$. Denote by $\boldsymbol{\mathscr{B}}(X)$ the set of all bounded functionals.

Let f and h be two functions defined on X and with values in $[a, b]$ and let $\star$ be arbitrary binary operation on $[a, b]$. Then, we define for any $x \in X$

$$(f \star h)(x) = f(x) \star h(x), \tag{1}$$

and for any $\lambda \in [a, b]$, $(\lambda \star f)(x) = \lambda \star f(x)$. Let $\mathscr{A}$ be a subset of $\mathscr{F}(X)$. If $f \star h \in \mathscr{A}$ for all $f, h \in \mathscr{A}$, then $\mathscr{A}$ is $\star$-closed. The total order $\preceq$ on $[a, b]$ induces a partial order $\preceq$ on $\mathscr{F}(X)$ defined pointwise by stipulating that $f \preceq h$ if and only if $f(x) \preceq h(x)$ for all $x \in X$. Thus $(\mathscr{F}(X), \preceq)$ is a poset, and whenever we consider $\mathscr{F}(X)$ as a poset then it will always be with respect to this partial order. Let $\mathcal{S}[\lambda \prec f] = \{x \mid x \in X, \lambda \prec f(x), f \in \mathscr{F}(X)\}$.

Definition 2 (see [35]). A binary operation $\oplus : [a, b] \times [a, b] \to [a, b]$ is called a pseudoaddition, if it satisfies the following conditions, for all $x, y, z, w \in [a, b]$:

(1) $\mathbf{0} \oplus x = x$, where $\mathbf{0}$ is a zero element (usually $\mathbf{0}$ is either a or b) (boundary condition);

(2) $x \oplus z \preceq y \oplus w$ whenever $x \preceq y$ and $z \preceq w$ (monotonicity);

(3) $x \oplus y = y \oplus x$ (commutativity);

(4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (associativity).

A pseudoaddition $\oplus$ is said to be continuous if it is a continuous function in $[a, b]^2$; a pseudoaddition $\oplus$ is called strict if $\oplus$ is continuous and strictly monotone. The following are examples of pseudoadditions: $x \vee_{\oplus} y = y$ if and only if $x \preceq y$; $x \oplus y = g^{-1}(g(x) + g(y))$, where $g : [a, b] \to [0, 1]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \leq g(y)$. It is obvious that $\Delta \oplus x = \Delta$ for all $x \in [a, b]$.

Let $[a, b]_+ = \{x \mid x \in [a, b], \mathbf{0} \preceq x\}$. In this paper, we assume $[a, b] = [a, b]_+$.

Definition 3 (see [35]). A binary operation $\odot : [a, b] \times [a, b] \to [a, b]$ is called a pseudomultiplication, if it satisfies the following conditions, for all $x, y, z, w \in [a, b]$:

(1) $\mathbf{1} \odot x = x$, where $\mathbf{1} \in [a, b]$ is a unit element (boundary condition);

(2) $x \odot z \preceq y \odot w$ whenever $x \preceq y$ and $z \preceq w$ (monotonicity);

(3) $x \odot y = y \odot x$ (commutativity);

(4) $(x \odot y) \odot z = x \odot (y \odot z)$ (associativity).

A pseudomultiplication $\odot$ is said to be continuous if it is a continuous function in $[a, b]^2$. The following are examples of pseudomultiplications: $x \wedge_{\odot} y = x$ if and only if $x \preceq y$; $x \odot_g y = g^{-1}(g(x) \cdot g(y))$, where $g : [a, b] \to [0, 1]$ is a strictly monotone and continuous generator surjective function and $x \preceq y$ if and only if $g(x) \leq g(y)$. It is obvious that $g(\mathbf{0}) = 0$.

We assume also that $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudomultiplication with respect to $\oplus$; that is,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z). \tag{2}$$

The structure $([a, b], \oplus, \odot)$ is called a real semiring.

Because of the associative property of the pseudoaddition $\oplus$, it can be extended by induction to n-ary operation by setting

$$\bigoplus_{i=1}^{n} x_i = \left(\bigoplus_{i=1}^{n-1} x_i\right) \oplus x_n. \tag{3}$$

Due to monotonicity, for each sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements of $[a, b]$, the following limit can be considered:

$$\bigoplus_{i=1}^{\infty} x_i = \lim_{n \to \infty} \bigoplus_{i=1}^{n} x_i. \tag{4}$$

Definition 4 (see [36]). Let A be a nonempty set and $\oplus$ a pseudoaddition. A binary operation $d_{\oplus} : A \times A \to [a, b]$ is called a pseudometric on A, if it satisfies the following conditions, for all $x, y, z \in A$:

(1) $d_{\oplus}(x, y) = \mathbf{0}$ if and only if $x = y$;

(2) $d_{\oplus}(x, y) = d_{\oplus}(y, x)$;

(3) there exists $\lambda \in [a, b]$ such that

$$d_{\oplus}(x, y) \preceq \lambda \odot (d_{\oplus}(x, z) \oplus d_{\oplus}(z, y)), \tag{5}$$

where $\odot$ is a distributive pseudomultiplication with respect to $\oplus$.

Let $\{x_n\}_{n\ge1}$ be a sequence from $[a,b]$. The sequence $\{x_n\}_{n\ge1}$ is said to be convergent, if for any $\mathbf{0} \prec \varepsilon$, there exists positive integer $N(\varepsilon)$, such that $d_{\oplus}(x_n, x) \prec \varepsilon$ for all $n \ge N(\varepsilon)$, denoted by $x = \lim_{n\to\infty} x_n$, and x is said to be the limit of the sequence $\{x_n\}_{n\ge1}$;

$$\underline{\lim}_{n\to\infty} x_n = \bigvee_{n=1}^{\infty}{}_{\oplus} \left(\bigwedge_{k\ge n}{}_{\odot} x_k \right) \tag{6}$$

is said to be the lower limit of the sequence $\{x_n\}_{n\ge1}$;

$$\overline{\lim}_{n\to\infty} x_n = \bigwedge_{n=1}^{\infty}{}_{\odot} \left(\bigvee_{k\ge n}{}_{\oplus} x_k \right) \tag{7}$$

is said to be the upper limit of the sequence $\{x_n\}_{n\ge1}$. It is obvious that $\underline{\lim}_{n\to\infty} x_n \preceq \overline{\lim}_{n\to\infty} x_n$. Let $\{f_n\}_{n\ge1}$ be a sequence from $\mathscr{F}(X)$. The sequence $\{f_n\}_{n\ge1}$ is said to be convergent, if for any $\mathbf{0} \prec \varepsilon$, and for each point $x_0 \in X$, there exists positive integer $N(\varepsilon, x_0)$, such that $d_{\oplus}(f_n(x_0), f(x_0)) \prec \varepsilon$ for all $n \ge N(\varepsilon, x_0)$, denoted by $f = \lim_{n\to\infty} f_n$, and f is said to be the limit functional of the functionals sequence $\{f_n\}_{n\ge1}$.

Let $\mathscr{A}$ be a subset of $\mathscr{F}(X)$. The poset $\mathscr{A}$ is said to be upper complete if $\lim_{n\to\infty} f_n \in \mathscr{A}$ for each increasing sequence $\{f_n\}_{n\ge1}$ from $\mathscr{A}$; the poset $\mathscr{A}$ is said to be lower complete if $\lim_{n\to\infty} f_n \in \mathscr{A}$ for each decreasing sequence $\{f_n\}_{n\ge1}$ from $\mathscr{A}$; the poset $\mathscr{A}$ is said to be complete if $\lim_{n\to\infty} f_n \in \mathscr{A}$ for each sequence $\{f_n\}_{n\ge1}$ from $\mathscr{A}$, where the limit of the sequence of functionals $\{f_n\}_{n\ge1}$ is given by $(\lim_{n\to\infty} f_n)(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$.

For any continuous pseudoaddition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, there exists at least one point $z \in [a,b]$ such that $y = x \oplus z$. If pseudoaddition $\oplus$ is strict, then there exists only one point $z \in [a,b]$ such that $y = x \oplus z$ for all $x, y \in [a,b]$ with $x \prec \Delta$. Thus we have the following concepts.

Definition 5 (see [34]). For any continuous pseudoaddition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the paracomplement set $y -_{\oplus} x$ is a nonempty set of all points z such that $y = x \oplus z$.

Example 6. Let the total order $\preceq$ on $[0,1]$ be the usual order of the real line and let the pseudoaddition $\oplus$ be the usual multiplication of the real numbers. It is obvious that zero element is 1. If $x = 0$, then $y = 0$ and $y -_{\oplus} x = [0,1]$. If $x \ne 0$, then for any $0 \le y < x$, we have $y -_{\oplus} x = \{y/x\} \subseteq [0,1]$.

Definition 7 (see [34]). For any continuous pseudoaddition $\oplus$, if $f, h \in \mathscr{F}(X)$, then define the paracomplement set $|f -_{\oplus} h|$ as the set of all those functionals φ such that

$$\varphi(x) = \begin{cases} f(x) -_{\oplus} h(x), & \text{if } h(x) \preceq f(x), \\ h(x) -_{\oplus} f(x), & \text{if } f(x) \prec h(x), \end{cases} \tag{8}$$

for all $x \in X$.

Definition 8 (see [34]). For any strict pseudoaddition $\oplus$ and $x, y \in [a,b]$ with $x \preceq y$, the complement $y -'_{\oplus} x$ is defined as

$$y -'_{\oplus} x = \begin{cases} z \in [a,b], & \text{such that } y = x \oplus z, \text{ if } x \prec \Delta, \\ \mathbf{0}, & \text{otherwise}. \end{cases} \tag{9}$$

Definition 9 (see [34]). For any strict pseudoaddition $\oplus$, if $f, h \in \mathscr{F}(X)$, then define the complement functional $|f -'_{\oplus} h|$ pointwise as

$$\left| f -'_{\oplus} h \right|(x) = \begin{cases} f(x) -'_{\oplus} h(x), & \text{if } h(x) \preceq f(x), \\ h(x) -'_{\oplus} f(x), & \text{if } f(x) \prec h(x), \end{cases} \tag{10}$$

for all $x \in X$.

Definition 10 (see [34]). For any pseudoaddition $\oplus$, a nonempty subset $\mathscr{K}$ of $\mathscr{F}(X)$ is said to be a functional space with respect to $\oplus$, denoted by $(\mathscr{K}, \oplus)$, if $(\lambda \odot f) \oplus (\mu \odot h) \in \mathscr{K}$ for all $f, h \in \mathscr{K}$ and $\lambda, \mu \in [a,b]$, where $\odot$ is a distributive pseudomultiplication with respect to $\oplus$.

It is clear that $(\mathscr{F}(X), \oplus)$ is the greatest functional space with respect to any pseudoaddition $\oplus$. Thus the functional space $(\mathscr{K}, \oplus)$ with $\mathscr{K} \subseteq \mathscr{F}(X)$ is also called a subspace of $(\mathscr{F}(X), \oplus)$. If $(\mathscr{K}, \oplus)$ is a functional space with respect to $\oplus$, then we just write $\mathscr{K}$ instead of $(\mathscr{K}, \oplus)$ whenever $\oplus$ can be determined from the context.

Definition 11 (see [34]). For each subset $\mathscr{A}$ of $\mathscr{F}(X)$ the upper closure of $\mathscr{A}$, denoted by $\widehat{\mathscr{A}}$, is the set of all elements of $\mathscr{F}(X)$ having the form $\lim_{n\to\infty} f_n$ for some increasing sequence $\{f_n\}_{n\ge1}$ from $\mathscr{A}$.

It follows from Definition 11 that $\mathscr{A} \subseteq \widehat{\mathscr{A}}$ and $\mathscr{A} = \widehat{\mathscr{A}}$ if and only if $\mathscr{A}$ is upper complete.

Definition 12 (see [34]). For any continuous pseudoaddition $\oplus$, a subspace $(\mathscr{K}, \oplus)$ will be called paracomplemented if $|f -_{\oplus} h| \subseteq \mathscr{K}$ for all $f, h \in \mathscr{K}$; for any strict pseudoaddition $\oplus$, a subspace $(\mathscr{K}, \oplus)$ will be called complemented if $|f -'_{\oplus} h| \in \mathscr{K}$ for all $f, h \in \mathscr{K}$.

Definition 13 (see [34]). For any continuous pseudoaddition $\oplus$, a paracomplemented subspace $(\mathscr{K}, \oplus)$ is regular if it contains $\mathbf{1}$ and is closed under $\vee_{\oplus}$; for any strict pseudoaddition $\oplus$, a complemented subspace $(\mathscr{K}, \oplus)$ is normal if it contains $\mathbf{1}$ and is closed under $\vee_{\oplus}$.

Note that $(f \vee_{\oplus} h) \oplus (f \wedge_{\odot} h) = f \oplus h$ for all $f, h \in \mathscr{F}(X)$ and thus a paracomplemented subspace of $\mathscr{F}(X)$ is $\wedge_{\odot}$-closed if and only if it is $\vee_{\oplus}$-closed. It is obvious that regular and normal are closed under $\wedge_{\odot}$.

Definition 14 (see [37]). The pseudocharacteristic function of a set $E \subseteq X$ is defined with

$$I_E(x) = \begin{cases} \mathbf{0}, & x \notin E, \\ \mathbf{1}, & x \in E, \end{cases} \tag{11}$$

where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$.

Definition 15 (see [21]). A functional $\varphi \in \mathscr{F}(X)$ is said to be elementary if it has the following representation:

$$\varphi = \bigoplus_{i=1}^{n} \lambda_i \odot I_{E_i}, \tag{12}$$

for each $\lambda_i \in [a,b]$ and $E_i \in \mathscr{A}$ pairwise disjoint and with $X = \bigcup_{i=1}^{n} E_i$, and the set of such elementary functionals will be denoted by $\mathscr{E}(X)$. It is obvious that $I_E \in \mathscr{E}(X)$, for all $E \subseteq X$.

Definition 16 (see [21]). A set function $m : \mathscr{A} \to [a,b]$ (or semiclosed interval) is called a σ-$\oplus$-decomposable measure if it satisfies the following conditions:

(1) $m(\emptyset) = \mathbf{0}$;

(2) $m(E) \preceq m(F)$ for all $E, F \in \mathscr{A}$ with $E \subset F$;

(3) $m(E \cup F) = m(E) \oplus m(F)$ for all $E, F \in \mathscr{A}$ and $E \cap F = \emptyset$;

(4) $m(\bigcup_{i=1}^{\infty} E_i) = \oplus_{i=1}^{\infty} m(E_i)$ for any sequence $\{E_i\}_{i\geq 1}$ of pairwise disjoint sets from $\mathscr{A}$.

A pair $(X, \mathscr{A})$ consisting of a nonempty set X and a σ-algebra of subsets of X is called a measurable space. A functional $f : X \to [a,b]$ is said to be a measurable functional if $f^{-1}(\mathscr{B}_{[a,b]}) \subseteq \mathscr{A}$. Let $\mathscr{M}(\mathscr{A})$ be the set of all measurable mappings from $(X, \mathscr{A})$ to $([a,b], \mathscr{B}_{[a,b]})$; that is,

$$\mathscr{M}(\mathscr{A}) = \left\{ f \in \mathscr{F}(X) \mid f^{-1}\left(\mathscr{B}_{[a,b]}\right) \subseteq \mathscr{A} \right\}. \tag{13}$$

Then $\mathscr{E}(\mathscr{S})$ will denote the set of those elements $f \in \mathscr{E}(X)$ for which $f^{-1}(\lambda) = \{x \in X \mid f(x) = \lambda\} \in \mathscr{S}$ for each $\lambda \in f(X)$. In particular, this means that $\mathscr{E}(\mathscr{A}) = \mathscr{M}(\mathscr{A}) \cap \mathscr{E}(X)$. Denote by $\mathscr{B}(\mathscr{A})$ the set of all bounded measurable functionals.

Definition 17 (see [38]). Let $\oplus$ be a continuous pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functionals of a.e. pseudofinite on X. If there exists a measurable functional f of a.e. pseudofinite on X, such that

$$\lim_{n\to\infty} m\mathscr{S}\left[\sigma \preceq d_{\oplus}\left(f_n, f\right)\right] = \mathbf{0}, \tag{14}$$

for arbitrary $\mathbf{0} \prec \sigma \prec \Delta$, then the functionals sequence $\{f_n\}_{n\geq 1}$ is said to be convergent to f with respect to $\oplus$-measure, denoted by $f_n \Rightarrow f$. If the functionals sequence $\{f_n\}_{n\geq 1}$ does not converge to f with respect to $\oplus$-measure, denote by $f_n \not\Rightarrow f$.

Definition 18 (see [35]). Let $\oplus$ be a continuous pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure.

(i) If $m(X) \prec \Delta$, then the pseudointegral of an elementary measurable function $\varphi : X \to [a,b]$ is defined by

$$\int_X^{\oplus} \varphi \odot dm = \bigoplus_{i=1}^{n} \lambda_i \odot m\left(E_i\right), \tag{15}$$

for $\lambda_i \in [a,b]$ and $E_i \in \mathscr{A}$ pairwise disjoint and with $X = \bigcup_{i=1}^{n} E_i$.

(ii) If $m(X) \prec \Delta$ and $\{\varphi_n\}$ is the sequence of elementary measurable functions such that, for each $x \in X$,

$$d_{\oplus}\left(\varphi_n(x), f(x)\right) \longrightarrow \mathbf{0} \quad \text{uniformly as } n \longrightarrow \infty, \tag{16}$$

where a sequence of elementary functions $\{\varphi_n\}$ from the previous definition is constructed in [34], then the pseudointegral of a bounded measurable function $f : X \to [a,b]$ is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_X^{\oplus} \varphi_n \odot dm. \tag{17}$$

If there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$, then we say that X is σ-finite set of $\oplus$-measure and $\{E_n\}$ is a $\oplus$-measure finite and monotone cover of X. The sequence of bounded measurable functionals $[f]_n$ is given by

$$[f]_n(x) = \begin{cases} f(x), & \text{if } f(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec f(x), \end{cases} \tag{18}$$

$\mathbf{0} \prec \mu_1 \prec \mu_2 \prec \cdots \prec \mu_n \prec \cdots$, $\mu_n \oplus \mu_n = \mu_{2n}$ and $\lim_{n\to\infty}\mu_n = \Delta$. It is obvious that $\{[f]_n\}$ is an increasing functionals sequence.

Definition 19. Let $\oplus$ be a strict pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If X is σ-finite of $\oplus$-measure and $\{E_n\}$ is a $\oplus$-measure finite and monotone cover of X, then the pseudointegral of a measurable function $f : X \to [a,b]$ is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_{E_n}^{\oplus} [f]_n \odot dm. \tag{19}$$

3. Main Results

Lemma 20 (see [21]). *Let $\oplus$ be a continuous pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $m(X) \prec \Delta$, then for all $f, h \in \mathscr{B}(\mathscr{A})$, we have*

(1) $\int_X^{\oplus} (f \vee_{\oplus} h) \odot dm = \int_X^{\oplus} f \odot dm \vee_{\oplus} \int_X^{\oplus} h \odot dm$;

(2) $\int_X^{\oplus} (f \wedge_{\odot} h) \odot dm = \int_X^{\oplus} f \odot dm \wedge_{\odot} \int_X^{\oplus} h \odot dm$;

(3) *If $f \oplus h \in \mathscr{B}(\mathscr{A})$, then*

$$\int_X^{\oplus} (f \oplus h) \odot dm = \int_X^{\oplus} f \odot dm \oplus \int_X^{\oplus} h \odot dm; \tag{20}$$

(4) $f \preceq h \Rightarrow \int_X^{\oplus} f \odot dm \preceq \int_X^{\oplus} h \odot dm$;

(5) $\int_{X_1 \cup X_2}^{\oplus} f \odot dm = \int_{X_1}^{\oplus} f \odot dm \oplus \int_{X_2}^{\oplus} f \odot dm$, *where $X_1, X_2 \in \mathscr{A}$ with $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$;*

(6) $\int_E^{\oplus} f \odot dm = \mathbf{0}$ *whenever $E \in \mathscr{A}$ with $m(E) = \mathbf{0}$.*

Theorem 21. *Let $\oplus$ be a strict pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If X is σ-finite of $\oplus$-measure and $f \in \mathscr{M}(\mathscr{A})$. Let $\{E_n^{(i)}\}$ $(i = 1, 2)$ be two different $\oplus$-measure finite and monotone covers of X and let $\{k_n^{(j)}\}$ $(j = 1, 2)$ be two different positive integer sequences with $\lim_{n\to\infty} k_n^{(j)} = +\infty$. Then*

$$\lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm = \lim_{n\to\infty} \int_{E_n^{(2)}}^{\oplus} [f]_{k_n^{(2)}} \odot dm. \tag{21}$$

Proof. Let $s = \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm$. Since $\{\int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm\}$ is an increasing sequence, we have

$$\int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \preceq s, \tag{22}$$

for every positive integer n. Let $F \in \mathscr{A}$ with $m(F) \prec \Delta$ and k is an arbitrary positive integer. If $k_n^{(1)} > k$, then we have

$$\begin{aligned} &\int_F^{\oplus} [f]_k \odot dm \\ &= \int_{F\cap E_n^{(1)}}^{\oplus} [f]_k \odot dm \oplus \int_{F-E_n^{(1)}}^{\oplus} [f]_k \odot dm \\ &\preceq \int_{F\cap E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \oplus \left(\mu_k \odot m\left(F - E_n^{(1)}\right)\right) \\ &\preceq \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \oplus \left(\mu_k \odot m\left(F - E_n^{(1)}\right)\right) \\ &\preceq s \oplus \left(\mu_k \odot m\left(F - E_n^{(1)}\right)\right). \end{aligned} \tag{23}$$

Since $\{F - E_n^{(1)}\}$ is a decreasing sequence and

$$\bigcap_{n=1}^{\infty}\left(F - E_n^{(1)}\right) = F - \bigcup_{n=1}^{\infty} E_n^{(1)} = F - X = \emptyset, \tag{24}$$

by Theorem 3.3 in [38], we have

$$\lim_{n\to\infty} m\left(F - E_n^{(1)}\right) = m\left(\lim_{n\to\infty}\left(F - E_n^{(1)}\right)\right) = \mathbf{0}, \tag{25}$$

which implies that

$$\begin{aligned} \int_F^{\oplus} [f]_k \odot dm &\preceq s \oplus \left(\mu_k \odot \lim_{n\to\infty} m\left(F - E_n^{(1)}\right)\right) \\ &= s = \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm. \end{aligned} \tag{26}$$

In particular, let $F = E_l^{(2)}$ and $k = k_l^{(2)}$. Then we have

$$\int_{E_l^{(2)}}^{\oplus} [f]_{k_l^{(2)}} \odot dm \preceq \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm, \tag{27}$$

for every positive integer l. Hence, we get that

$$\lim_{l\to\infty} \int_{E_l^{(2)}}^{\oplus} [f]_{k_l^{(2)}} \odot dm \preceq \lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm. \tag{28}$$

On the contrary, using a similar argument, we can obtain

$$\lim_{n\to\infty} \int_{E_n^{(1)}}^{\oplus} [f]_{k_n^{(1)}} \odot dm \preceq \lim_{l\to\infty} \int_{E_l^{(2)}}^{\oplus} [f]_{k_l^{(2)}} \odot dm. \tag{29}$$

□

In Theorem 21, put $k_n^{(1)} = n$ and $k_l^{(2)} = l$. Then we can easily see that the pseudointegral in Definition 19 has a unique value. In particular, we can get some elementary properties of the pseudointegral in the following theorem.

Theorem 22. *Let $\oplus$ be a strict pseudoaddition and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$, then for all $f, h \in \mathscr{M}(\mathscr{A})$, we have*

(1) $\int_X^{\oplus} (f \vee_{\oplus} h) \odot dm = \int_X^{\oplus} f \odot dm \vee_{\oplus} \int_X^{\oplus} h \odot dm$;

(2) $\int_X^{\oplus} (f \wedge_{\odot} h) \odot dm = \int_X^{\oplus} f \odot dm \wedge_{\odot} \int_X^{\oplus} h \odot dm$;

(3) $\int_X^{\oplus} (f \oplus h) \odot dm = \int_X^{\oplus} f \odot dm \oplus \int_X^{\oplus} h \odot dm$;

(4) $f \preceq h \Rightarrow \int_X^{\oplus} f \odot dm \preceq \int_X^{\oplus} h \odot dm$;

(5) $\int_{X_1 \cup X_2}^{\oplus} f \odot dm = \int_{X_1}^{\oplus} f \odot dm \oplus \int_{X_2}^{\oplus} f \odot dm$, *where* $X_1, X_2 \in \mathscr{A}$ *with* $X_1 \cup X_2 = X$ *and* $X_1 \cap X_2 = \emptyset$;

(6) $\int_E^{\oplus} f \odot dm = \mathbf{0}$ *whenever* $E \in \mathscr{A}$ *with* $m(E) = \mathbf{0}$.

Proof. For (1) and (2), we only prove (1) holds. By a similar proof, we can prove (2) holds. Since

$$\begin{aligned} [f]_n(x) &= \begin{cases} f(x), & \text{if } f(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec f(x), \end{cases} \\ [h]_n(x) &= \begin{cases} h(x), & \text{if } h(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec h(x), \end{cases} \end{aligned} \tag{30}$$

$n = 1, 2, \ldots$, we get that

$$\left([f]_n \vee_{\oplus} [h]_n\right)(x) = \begin{cases} (f \vee_{\oplus} h)(x), & \text{if } (f \vee_{\oplus} h)(x) \preceq \mu_n, \\ \mu_n, & \text{if } \mu_n \prec (f \vee_{\oplus} h)(x), \end{cases} \tag{31}$$

which implies that

$$[f \vee_{\oplus} h]_n = \left([f]_n \vee_{\oplus} [h]_n\right). \tag{32}$$

Thus, by (1) of Lemma 20, we have

$$\begin{aligned} &\int_X^{\oplus} (f \vee_{\oplus} h) \odot dm \\ &= \lim_{n\to\infty} \int_{E_n}^{\oplus} [f \vee_{\oplus} h]_n \odot dm \\ &= \lim_{n\to\infty} \int_{E_n}^{\oplus} \left([f]_n \vee_{\oplus} [h]_n\right) \odot dm \\ &= \lim_{n\to\infty} \int_{E_n}^{\oplus} [f]_n \odot dm \vee_{\oplus} \lim_{n\to\infty} \int_{E_n}^{\oplus} [h]_n \odot dm \\ &= \int_X^{\oplus} f \odot dm \vee_{\oplus} \int_X^{\oplus} h \odot dm. \end{aligned} \tag{33}$$

(3) Since

$$[f\oplus h]_n(x)=\begin{cases}(f\oplus h)(x), & \text{if } (f\oplus h)(x)\preceq\mu_n,\\ \mu_n, & \text{if } \mu_n\prec(f\oplus h)(x),\end{cases}$$

$$\begin{aligned}&([f]_n\oplus[h]_n)(x)\\ &=\begin{cases}(f\oplus h)(x), & \text{if } (f\vee_{\oplus}h)(x)\preceq\mu_n,\\ \mu_n\oplus(f\wedge_{\odot}h)(x), & \text{if } (f\wedge_{\odot}h)(x)\preceq\mu_n\prec(f\vee_{\oplus}h)(x),\\ \mu_n\oplus\mu_n=\mu_{2n}, & \text{if } \mu_n\prec(f\wedge_{\odot}h)(x),\end{cases}\end{aligned}\tag{34}$$

$n=1,2,\ldots$, we get that

$$[f\oplus h]_n\preceq[f]_n\oplus[h]_n\preceq[f\oplus h]_{2n}.\tag{35}$$

Thus, we have

$$\begin{aligned}\int_{E_n}^{\oplus}[f\oplus h]_n\odot dm&\preceq\int_{E_n}^{\oplus}([f]_n\oplus[h]_n)\odot dm\\ &\preceq\int_{E_n}^{\oplus}[f\oplus h]_{2n}\odot dm\\ &\preceq\int_{E_{2n}}^{\oplus}[f\oplus h]_{2n}\odot dm.\end{aligned}\tag{36}$$

By (3) of Lemma 20, we have

$$\int_{E_n}^{\oplus}([f]_n\oplus[h]_n)\odot dm=\int_{E_n}^{\oplus}[f]_n\odot dm\oplus\int_{E_n}^{\oplus}[h]_n\odot dm,\tag{37}$$

which implies that

$$\begin{aligned}\int_{E_n}^{\oplus}[f\oplus h]_n\odot dm&\preceq\int_{E_n}^{\oplus}[f]_n\odot dm\oplus\int_{E_n}^{\oplus}[h]_n\odot dm\\ &\preceq\int_{E_{2n}}^{\oplus}[f\oplus h]_{2n}\odot dm.\end{aligned}\tag{38}$$

Hence, we get that

$$\begin{aligned}&\lim_{n\to\infty}\int_{E_n}^{\oplus}[f\oplus h]_n\odot dm\\ &\quad\preceq\lim_{n\to\infty}\left(\int_{E_n}^{\oplus}[f]_n\odot dm\oplus\int_{E_n}^{\oplus}[h]_n\odot dm\right)\\ &\quad=\lim_{n\to\infty}\int_{E_n}^{\oplus}[f]_n\odot dm\oplus\lim_{n\to\infty}\int_{E_n}^{\oplus}[h]_n\odot dm\\ &\quad\preceq\lim_{n\to\infty}\int_{E_{2n}}^{\oplus}[f\oplus h]_{2n}\odot dm,\end{aligned}\tag{39}$$

which implies that

$$\begin{aligned}\int_X^{\oplus}(f\oplus h)\odot dm&\preceq\int_X^{\oplus}f\odot dm\oplus\int_X^{\oplus}h\odot dm\\ &\preceq\int_X^{\oplus}(f\oplus h)\odot dm;\end{aligned}\tag{40}$$

that is,

$$\int_X^{\oplus}(f\oplus h)\odot dm=\int_X^{\oplus}f\odot dm\oplus\int_X^{\oplus}h\odot dm.\tag{41}$$

(4) If $f\preceq h$, then $[f]_n\preceq[h]_n$, $n=1,2,\ldots$. Thus, by (4) of Lemma 20, we have

$$\int_{E_n}^{\oplus}[f]_n\odot dm\preceq\int_{E_n}^{\oplus}[h]_n\odot dm.\tag{42}$$

Hence, we get that

$$\lim_{n\to\infty}\int_{E_n}^{\oplus}[f]_n\odot dm\preceq\lim_{n\to\infty}\int_{E_n}^{\oplus}[h]_n\odot dm,\tag{43}$$

that is,

$$\int_X^{\oplus}f\odot dm\preceq\int_X^{\oplus}h\odot dm;\tag{44}$$

(5) Since $X=\bigcup_{n=1}^{\infty}E_n$ with $m(E_n)\prec\Delta$, we have $X_1=\bigcup_{n=1}^{\infty}(E_n\cap X_1)$ with $m(E_n\cap X_1)\prec\Delta$ and $X_2=\bigcup_{n=1}^{\infty}(E_n\cap X_2)$ with $m(E_n\cap X_2)\prec\Delta$. By (5) of Lemma 20, we have

$$\int_{E_n}^{\oplus}[f]_n\odot dm=\int_{E_n\cap X_1}^{\oplus}[f]_n\odot dm\oplus\int_{E_n\cap X_2}^{\oplus}[f]_n\odot dm,\tag{45}$$

which implies that

$$\begin{aligned}&\int_X^{\oplus}f\odot dm\\ &\quad=\lim_{n\to\infty}\int_{E_n}^{\oplus}[f]_n\odot dm\\ &\quad=\lim_{n\to\infty}\int_{E_n\cap X_1}^{\oplus}[f]_n\odot dm\oplus\int_{E_n\cap X_2}^{\oplus}[f]_n\odot dm\\ &\quad=\lim_{n\to\infty}\int_{E_n\cap X_1}^{\oplus}[f]_n\odot dm\oplus\lim_{n\to\infty}\int_{E_n\cap X_2}^{\oplus}[f]_n\odot dm\\ &\quad=\int_{X_1}^{\oplus}f\odot dm\oplus\int_{X_2}^{\oplus}f\odot dm.\end{aligned}\tag{46}$$

(6) Since $X=\bigcup_{n=1}^{\infty}E_n$, we have $E=\bigcup_{n=1}^{\infty}(E_n\cap E)$. By the monotonicity of σ-$\oplus$-decomposable measure m, we get that if $m(E)=\mathbf{0}$, then $m(E_n\cap E)=\mathbf{0}$. By (6) of Theorem 22, we have

$$\int_{E\cap E_n}^{\oplus}[f]_n\odot dm=\mathbf{0},\tag{47}$$

which implies that

$$\int_E^{\oplus}f\odot dm=\lim_{n\to\infty}\int_{E\cap E_n}^{\oplus}[f]_n\odot dm=\mathbf{0}.\tag{48}$$

□

Theorem 23. *Let $\oplus$ be a strict pseudoaddition and $m:\mathscr{A}\to[a,b]$ a σ-$\oplus$-decomposable measure.*

(1) *If* $f \in \mathcal{M}(\mathscr{A})$ *and* $E \in \mathscr{A}$ *is a* σ*-finite set of* $\oplus$*-measure, then* $\int_E^{\oplus} f \odot dm = \mathbf{0}$ *if and only if* $f = \mathbf{0}$ *a.e. on* E.

(2) *If* $f \in \mathcal{M}(\mathscr{A})$, *then for any* $E \in \mathscr{A}$, $\lim_{mE \to \mathbf{0}} \int_E^{\oplus} f \odot dm = \mathbf{0}$.

Proof. (1) Suppose $\int_E^{\oplus} f \odot dm = \mathbf{0}$. For arbitrary $\mathbf{0} \prec \delta$, let $E_\delta = \{x \in E \mid \delta \preceq f(x)\} \in \mathscr{A}$. Then we get that

$$\delta \odot m(E_\delta) \preceq \int_{E_\delta}^{\oplus} f \odot dm \preceq \int_{E_\delta}^{\oplus} f \odot dm \oplus \int_{E-E_\delta}^{\oplus} f \odot dm = \int_E^{\oplus} f \odot dm = \mathbf{0}. \tag{49}$$

Thus, we have $m(E_\delta) = \mathbf{0}$. Since $\mathbf{0} \prec \delta$ is arbitrary, we have $m(\mathcal{S}[\mathbf{0} \prec f] \cap E) = \mathbf{0}$.

Suppose $f = \mathbf{0}$ a.e. on E, that is, $m(E \cap \mathcal{S}[\mathbf{0} \prec f]) = \mathbf{0}$. By (6) of Theorem 22, we have

$$\int_E^{\oplus} f \odot dm = \int_{E\cap\mathcal{S}[\mathbf{0}\prec f]}^{\oplus} f \odot dm \oplus \int_{E\cap\mathcal{S}[f=\mathbf{0}]}^{\oplus} f \odot dm = \mathbf{0}. \tag{50}$$

(2) If there exists $\Omega \prec \Delta$, such that $f(x) \preceq \Omega$ for all $x \in E$, then

$$\int_E^{\oplus} f \odot dm \preceq \Omega \odot m(E), \quad \text{i.e., } \lim_{mE \to \mathbf{0}} \int_E^{\oplus} f \odot dm = \mathbf{0}. \tag{51}$$

For any $f \in \mathcal{M}(\mathscr{A})$, we have

$$\int_E^{\oplus} f \odot dm = \lim_{n\to\infty} \int_E^{\oplus} [f]_n \odot dm, \tag{52}$$

which implies that

$$\lim_{mE\to 0} \int_E^{\oplus} f \odot dm = \lim_{mE\to 0}\lim_{n\to\infty} \int_E^{\oplus} [f]_n \odot dm = \lim_{n\to\infty}\lim_{mE\to 0} \int_E^{\oplus} [f]_n \odot dm = \mathbf{0}. \tag{53}$$

□

Lemma 24 (see [38]). *Let* $\oplus$ *be a strict pseudoaddition. The function* $d_\oplus : [a,b]^2 \to [a,b]$ *given by*

$$d_\oplus(x,y) = \left|x -'_{\oplus} y\right| = \begin{cases} y -'_{\oplus} x, & \text{if } x \preceq y, \\ x -'_{\oplus} y, & \text{if } y \prec x, \end{cases} \tag{54}$$

is a pseudometric on $[a,b]$ *with* $\lambda = \mathbf{1}$.

Theorem 25. *Let* $\oplus$ *be a strict pseudoaddition and* X *a* σ*-finite set of* $\oplus$*-measure. If* $m : \mathscr{A} \to [a,b]$ *is a* σ*-*$\oplus$*-decomposable measure, then for any* $f, h \in \mathcal{M}(\mathscr{A})$,

$$\left|\int_X^{\oplus} f \odot dm -'_{\oplus} \int_X^{\oplus} h \odot dm\right| \preceq \int_X^{\oplus} \left|f -'_{\oplus} h\right| \odot dm. \tag{55}$$

Proof. Let $E = \{x \mid h(x) \preceq f(x), x \in X\}$ and $F = \{x \mid f(x) \prec h(x), x \in X\}$. Then E and F are two $\oplus$-measure σ-finite sets of X. By (4) of Theorem 22, we have

$$\int_E^{\oplus} h \odot dm \preceq \int_E^{\oplus} f \odot dm, \qquad \int_F^{\oplus} f \odot dm \preceq \int_F^{\oplus} h \odot dm. \tag{56}$$

Thus, by (3) of Theorem 22, we have

$$\begin{aligned}\int_E^{\oplus} f \odot dm &= \int_E^{\oplus} \left(\left|f -'_{\oplus} h\right| \oplus h\right) \odot dm \\ &= \int_E^{\oplus} \left|f -'_{\oplus} h\right| \odot dm \oplus \int_E^{\oplus} h \odot dm, \\ \int_F^{\oplus} h \odot dm &= \int_F^{\oplus} \left(\left|f -'_{\oplus} h\right| \oplus f\right) \odot dm \\ &= \int_F^{\oplus} \left|f -'_{\oplus} h\right| \odot dm \oplus \int_F^{\oplus} f \odot dm,\end{aligned} \tag{57}$$

which implies that

$$\begin{aligned}&\int_F^{\oplus} \left|f -'_{\oplus} h\right| \odot dm \oplus \int_X^{\oplus} f \odot dm \\ &\quad = \int_E^{\oplus} \left|f -'_{\oplus} h\right| \odot dm \oplus \int_X^{\oplus} h \odot dm.\end{aligned} \tag{58}$$

If $\int_X^{\oplus} h \odot dm \preceq \int_X^{\oplus} f \odot dm$, then we have

$$\begin{aligned}\int_X^{\oplus} f \odot dm &= \left|\int_X^{\oplus} f \odot dm -'_{\oplus} \int_X^{\oplus} h \odot dm\right| \oplus \int_X^{\oplus} h \odot dm \\ &\preceq \int_E^{\oplus} \left|f -'_{\oplus} h\right| \odot dm \oplus \int_X^{\oplus} h \odot dm \\ &\preceq \int_X^{\oplus} \left|f -'_{\oplus} h\right| \odot dm \oplus \int_X^{\oplus} h \odot dm,\end{aligned} \tag{59}$$

which implies that

$$\left|\int_X^{\oplus} f \odot dm -'_{\oplus} \int_X^{\oplus} h \odot dm\right| \preceq \int_X^{\oplus} \left|f -'_{\oplus} h\right| \odot dm. \tag{60}$$

Similarly, if $\int_X^{\oplus} f \odot dm \prec \int_X^{\oplus} h \odot dm$, we can also get this conclusion. □

Theorem 26. *Let* $\oplus$ *be a strict pseudoaddition, and let* X *be a* σ*-finite set of* $\oplus$*-measure and* $m : \mathscr{A} \to [a,b]$ *a* σ*-*$\oplus$*-decomposable measure. If*

(1) $\{f_n\} \subset \mathcal{M}(\mathscr{A})$;

(2) $f_n \preceq F$ *a.e. on* X, $n = 1, 2, \ldots$, *and* $F \in \mathcal{M}(\mathscr{A})$;

(3) $f_n \Rightarrow f$,

then $f \in \mathcal{M}(\mathscr{A})$ *and*

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{61}$$

Proof. Since $f_n \rightrightarrows f$ on X, by Theorem 3.8 in [38], there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ that a.e. converges to f on X. By Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$.

(I) Suppose $m(X) \prec \Delta$. By (2) of Theorem 23, for arbitrary $\mathbf{0} \prec \varepsilon = \varepsilon_1 \oplus \varepsilon_1$, there exists $\mathbf{0} \prec \delta$ such that if $E \subset X$ with $m(E) \prec \delta$, we have

$$\int_E^{\oplus} F \odot dm \prec \varepsilon_1. \tag{62}$$

Since $f_n \Rightarrow f$, there exists a natural number $N > 0$, such that $m(\mathcal{S}[\sigma \preceq |f -'_{\oplus} f_n|]) \prec \delta$ for all $n \geq N$, where $\varepsilon_1 = \sigma \odot m(X)$. Thus, we get that

$$\int_{\mathcal{S}[\sigma \preceq |f -'_{\oplus} f_n|]}^{\oplus} F \odot dm \prec \varepsilon_1. \tag{63}$$

Hence, by Theorem 25, we have

$$\begin{aligned} &\left| \int_X^{\oplus} f \odot dm -'_{\oplus} \int_X^{\oplus} f_n \odot dm \right| \\ &\quad \preceq \int_X^{\oplus} \left| f -'_{\oplus} f_n \right| \odot dm \\ &\quad = \int_{\mathcal{S}[\sigma \preceq |f -'_{\oplus} f_n|]}^{\oplus} \left| f -'_{\oplus} f_n \right| \odot dm \\ &\qquad \oplus \int_{\mathcal{S}[|f -'_{\oplus} f_n| \prec \sigma]}^{\oplus} \left| f -'_{\oplus} f_n \right| \odot dm \\ &\quad \preceq \int_{\mathcal{S}[\sigma \preceq |f -'_{\oplus} f_n|]}^{\oplus} F \odot dm \\ &\qquad \oplus \left(\sigma \odot m \left(\mathcal{S} \left[\left| f -'_{\oplus} f_n \right| \prec \sigma \right] \right) \right) \\ &\quad \prec \varepsilon_1 \oplus (\sigma \odot m(X)) = \varepsilon_1 \oplus \varepsilon_1 = \varepsilon. \end{aligned} \tag{64}$$

By Lemma 24, we obtain that

$$\lim_{n \to \infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{65}$$

(II) Suppose $m(X) = \Delta$. For arbitrary $\mathbf{0} \prec \varepsilon = \varepsilon_1 \oplus \varepsilon_1$, there exists $E_k \subseteq X$ with $m(E_k) \prec \Delta$, such that

$$\int_X^{\oplus} F \odot dm \prec \int_{E_k}^{\oplus} [F]_k \odot dm \oplus \varepsilon_1. \tag{66}$$

Thus, we have

$$\begin{aligned} &\int_{E_k}^{\oplus} [F]_k \odot dm \oplus \int_{X - E_k}^{\oplus} F \odot dm \\ &\quad \preceq \int_{E_k}^{\oplus} F \odot dm \oplus \int_{X - E_k}^{\oplus} F \odot dm \\ &\quad = \int_X^{\oplus} F \odot dm \prec \int_{E_k}^{\oplus} [F]_k \odot dm \oplus \varepsilon_1; \end{aligned} \tag{67}$$

that is, $\int_{X-E_k}^{\oplus} F \odot dm \prec \varepsilon_1$. Since the measurable functionals sequence $\{|f -'_{\oplus} f_n|\}$ satisfies

(i) $|f -'_{\oplus} f_n| \preceq F$ a.e. on E_k;

(ii) $|f -'_{\oplus} f_n| \Rightarrow \mathbf{0}$ on E_k,

by (I), we get that there exists a natural number $N > 0$, such that

$$\int_{E_k}^{\oplus} \left| f -'_{\oplus} f_n \right| \odot dm \prec \varepsilon_1, \tag{68}$$

for all $n > N$. Hence, by Theorem 25, we have

$$\begin{aligned} &\left| \int_X^{\oplus} f \odot dm -'_{\oplus} \int_X^{\oplus} f_n \odot dm \right| \\ &\quad \preceq \int_X^{\oplus} \left| f -'_{\oplus} f_n \right| \odot dm \\ &\quad = \int_{X - E_k}^{\oplus} \left| f -'_{\oplus} f_n \right| \odot dm \oplus \int_{E_k}^{\oplus} \left| f -'_{\oplus} f_n \right| \odot dm \\ &\quad \preceq \int_{X - E_k}^{\oplus} F \odot dm \oplus \varepsilon_1 \prec \varepsilon_1 \oplus \varepsilon_1 = \varepsilon. \end{aligned} \tag{69}$$

Consequently, we obtain that

$$\lim_{n \to \infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{70}$$

□

Corollary 27. *If the condition (3) of Theorem 26 is replaced by $f_n \to f$ a.e. on X, then the conclusion of Theorem 26 holds.*

Proof. Since $f_n \to f$ a.e. on X, by Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$.

(I) Suppose $m(X) \prec \Delta$. By Theorem 3.9 in [38], if $f_n \to f$ a.e. on X, then $f_n \Rightarrow f$. By Theorem 26 (I), we have

$$\lim_{n \to \infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{71}$$

(II) Suppose $m(X) = \Delta$. Since X is σ-finite set of $\oplus$-measure, there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$. For any E_k, $k = 1, 2, \ldots$, the sequence of measurable functionals $\{|f -'_{\oplus} f_n|\}$ satisfies

(i) $|f -'_{\oplus} f_n| \preceq F$ a.e. on E_k, $k = 1, 2, \ldots$;

(ii) $|f -'_{\oplus} f_n| \to \mathbf{0}$ a.e. on E_k, $k = 1, 2, \ldots$.

By Theorem 3.9 in [38], we have

(ii)$'$ $|f -'_{\oplus} f_n| \Rightarrow \mathbf{0}$ on E_k, $k = 1, 2, \ldots$.

By (I) and proof of Theorem 26 (II), we have

$$\lim_{n \to \infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm. \tag{72}$$

□

Lemma 28. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{x_n\}$ is a monotone sequence, then the sequence $\{x_n\}$ is convergence.*

Proof. If $\{x_n\}$ is an increasing sequence, then

$$\begin{aligned} \underline{\lim}_{n\to\infty} x_n &= \bigvee_{\oplus\, n=1}^{\infty} \left(\bigwedge_{\odot\, k\geq n} x_k \right) = \bigvee_{\oplus\, n=1}^{\infty} x_n, \\ \overline{\lim}_{n\to\infty} x_n &= \bigwedge_{\odot\, n=1}^{\infty} \left(\bigvee_{\oplus\, k\geq n} x_k \right) = \bigvee_{\oplus\, n=1}^{\infty} x_n. \end{aligned} \tag{73}$$

If $\{x_n\}$ is a decreasing sequence, then

$$\begin{aligned} \underline{\lim}_{n\to\infty} x_n &= \bigvee_{\oplus\, n=1}^{\infty} \left(\bigwedge_{\odot\, k\geq n} x_k \right) = \bigwedge_{\odot\, n=1}^{\infty} x_n, \\ \overline{\lim}_{n\to\infty} x_n &= \bigwedge_{\odot\, n=1}^{\infty} \left(\bigvee_{\oplus\, k\geq n} x_k \right) = \bigwedge_{\odot\, n=1}^{\infty} x_n. \end{aligned} \tag{74}$$

Thus, we have

$$\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n. \tag{75}$$

By Theorem 3.2 in [38], we get that the sequence $\{x_n\}$ is convergent. □

Theorem 29. *Let $\oplus$ be a strict pseudoaddition and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is an increasing sequence of measurable functionals on X, then*

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} \lim_{n\to\infty} f_n \odot dm. \tag{76}$$

Proof. Let $\{f_n\}$ be an increasing sequence of measurable functionals on X. By Lemma 28, we get that the sequence of measurable functionals $\{f_n\}$ is convergent. Let $f = \lim_{n\to\infty} f_n$. By Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$ with $f_n \preceq f$ on X. By (4) of Theorem 22, we get that

$$\int_X^{\oplus} f_n \odot dm \preceq \int_X^{\oplus} f \odot dm, \tag{77}$$

which implies that

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm \preceq \int_X^{\oplus} f \odot dm. \tag{78}$$

On the contrary, since X is σ-finite set of $\oplus$-measure, there exists an increasing sequence of sets $\{E_n\} \subset \mathscr{A}$ with $m(E_n) \prec \Delta$, $n = 1, 2, \ldots$, such that $X = \bigcup_{n=1}^{\infty} E_n$. For any given integer $k > 0$, $\{[f_n]_k\}_{n\geq k}$ is an increasing sequence of measurable functionals and $[f_n]_k \preceq f$ on X, for all $n \geq k$. Now we show that

$$\lim_{n\to\infty} [f_n]_k = [f]_k. \tag{79}$$

For arbitrary $x_0 \in X$,

(i) if $f_n(x_0) \preceq \mu_k$, that is, $[f_n]_k(x_0) = f_n(x_0)$ for all $n \geq k$, then $f(x_0) \preceq \mu_k$, that is, $[f]_k(x_0) = f(x_0)$. Thus, we have

$$\lim_{n\to\infty} [f_n]_k (x_0) = [f]_k (x_0); \tag{80}$$

(ii) if there exists $n_0 \geq k$, such that $\mu_k \prec f_{n_0}(x_0)$, then $\mu_k \prec f_n(x_0)$; that is, $[f_n]_k(x_0) = \mu_k$ for all $n \geq n_0$; it follows that $\mu_k \preceq f(x_0)$; that is, $[f]_k(x_0) = \mu_k$. Thus, we have

$$\lim_{n\to\infty} [f_n]_k (x_0) = [f]_k (x_0) = \mu_k. \tag{81}$$

Hence, by Corollary 27, we get that

$$\int_{E_k}^{\oplus} [f]_k \odot dm = \lim_{n\to\infty} \int_{E_k}^{\oplus} [f_n]_k \odot dm \preceq \lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm, \tag{82}$$

which implies that

$$\int_X^{\oplus} f \odot dm = \lim_{k\to\infty} \int_{E_k}^{\oplus} [f]_k \odot dm \preceq \lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm. \tag{83}$$

Consequently, we obtain that

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} f \odot dm = \int_X^{\oplus} \lim_{n\to\infty} f_n \odot dm. \tag{84}$$

□

Theorem 30. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is a decreasing sequence of finite measurable functionals and pseudointegral of f_1 is finite on X, then*

$$\lim_{n\to\infty} \int_X^{\oplus} f_n \odot dm = \int_X^{\oplus} \lim_{n\to\infty} f_n \odot dm. \tag{85}$$

Proof. Let $\{f_n\}$ be a decreasing sequence of measurable functionals on X. By Lemma 28, we get that the sequence of measurable functionals $\{f_n\}$ is convergent. Let $f = \lim_{n\to\infty} f_n$. By Theorem 3.5 in [38], we have $f \in \mathscr{M}(\mathscr{A})$. Since $\{f_1 -'_{\oplus} f_n\}$ is an increasing sequence of measurable functionals, by Theorem 29, we have

$$\lim_{n\to\infty} \int_X^{\oplus} \left(f_1 -'_{\oplus} f_n\right) \odot dm = \int_X^{\oplus} \lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) \odot dm. \tag{86}$$

Since $f_1 = (f_1 -'_{\oplus} f_n) \oplus f_n$ and $\oplus$ is continuous, we have

$$f_1 = \lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) \oplus \lim_{n\to\infty} f_n = \lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) \oplus f. \tag{87}$$

Since $f_1 = (f_1 -'_{\oplus} f) \oplus f \prec \Delta$ and $\oplus$ is strict, we get that

$$\lim_{n\to\infty} \left(f_1 -'_{\oplus} f_n\right) = f_1 -'_{\oplus} f, \tag{88}$$

which implies that

$$\lim_{n\to\infty}\int_X^{\oplus}\left(f_1 -'_{\oplus} f_n\right)\odot dm = \int_X^{\oplus}\left(f_1 -'_{\oplus} f\right)\odot dm. \tag{89}$$

By (3) of Theorem 22, we have

$$\begin{gathered}\int_X^{\oplus} f_1\odot dm = \int_X^{\oplus}\left(f_1 -'_{\oplus} f_n\right)\odot dm \oplus \int_X^{\oplus} f_n\odot dm,\\ \int_X^{\oplus} f_1\odot dm = \int_X^{\oplus}\left(f_1 -'_{\oplus} f\right)\odot dm \oplus \int_X^{\oplus} f\odot dm.\end{gathered} \tag{90}$$

Thus, we get that

$$\begin{aligned}&\int_X^{\oplus} f_1\odot dm\\ &\quad = \lim_{n\to\infty}\int_X^{\oplus}\left(f_1 -'_{\oplus} f_n\right)\odot dm \oplus \lim_{n\to\infty}\int_X^{\oplus} f_n\odot dm\\ &\quad = \int_X^{\oplus}\left(f_1 -'_{\oplus} f\right)\odot dm \oplus \lim_{n\to\infty}\int_X^{\oplus} f_n\odot dm.\end{aligned} \tag{91}$$

Since $\int_X^{\oplus} f_1\odot dm \prec \Delta$ and $\oplus$ is strict, we obtain that

$$\lim_{n\to\infty}\int_X^{\oplus} f_n\odot dm = \int_X^{\oplus} f\odot dm = \int_X^{\oplus}\lim_{n\to\infty} f_n\odot dm. \tag{92}$$

□

Theorem 31. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is a sequence of measurable functionals on X, then*

$$\int_X^{\oplus}\left(\bigoplus_{n=1}^{\infty} f_n\right)\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{93}$$

Proof. Let $h_n = \oplus_{i=1}^{n} f_i$, $n = 1,2,\ldots$. Then $\{h_n\}$ is an increasing sequence of measurable functionals on X. By Theorem 29, we have

$$\lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm = \int_X^{\oplus}\lim_{n\to\infty} h_n\odot dm. \tag{94}$$

By (3) of Theorem 22, we have

$$\int_X^{\oplus} h_n\odot dm = \int_X^{\oplus}\bigoplus_{i=1}^{n} f_i\odot dm = \bigoplus_{i=1}^{n}\int_X^{\oplus} f_i\odot dm; \tag{95}$$

that is,

$$\lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{96}$$

Since $\lim_{n\to\infty} h_n = \oplus_{n=1}^{\infty} f_n$, we have

$$\int_X^{\oplus}\left(\bigoplus_{n=1}^{\infty} f_n\right)\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{97}$$

□

Theorem 32. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If f is a measurable functional on X,*

$$\int_X^{\oplus} f\odot dm = \bigoplus_{n=1}^{\infty}\int_{E_n}^{\oplus} f\odot dm, \tag{98}$$

for any sequence $\{E_n\}$ of pairwise disjoint sets from $\mathscr{A}$ with $X = \bigcup_{n=1}^{\infty} E_n$.

Proof. A functionals sequence $[f]_n$ is given by

$$f_n(x) = \begin{cases} f(x), & \text{if } x\in E_n,\\ \mathbf{0}, & \text{if } x\in X - E_n,\end{cases} \qquad n = 1,2,\ldots, \tag{99}$$

then $f = \oplus_{n=1}^{\infty} f_n$ and

$$\int_X^{\oplus} f_n\odot dm = \int_{E_n}^{\oplus} f_n\odot dm \oplus \int_{X-E_n}^{\oplus} f_n\odot dm = \int_{E_n}^{\oplus} f\odot dm. \tag{100}$$

By Theorem 31, we have

$$\int_X^{\oplus}\bigoplus_{n=1}^{\infty} f_n\odot dm = \bigoplus_{n=1}^{\infty}\int_X^{\oplus} f_n\odot dm. \tag{101}$$

Hence, we obtain that

$$\int_X^{\oplus} f\odot dm = \bigoplus_{n=1}^{\infty}\int_{E_n}^{\oplus} f\odot dm. \tag{102}$$

□

Theorem 33. *Let $\oplus$ be a strict pseudoaddition, and let X be a σ-finite set of $\oplus$-measure and $m : \mathscr{A} \to [a,b]$ a σ-$\oplus$-decomposable measure. If $\{f_n\}$ is a sequence of measurable functionals on X, then*

$$\int_X^{\oplus}\varliminf_{n\to\infty} f_n\odot dm \preceq \varliminf_{n\to\infty}\int_X^{\oplus} f_n\odot dm. \tag{103}$$

Proof. Let $h_n = \wedge_{\odot k=n}^{\infty} f_k$, $n = 1,2,\ldots$. Then $\{h_n\}$ is an increasing sequence of measurable functionals on X. By proof of Theorem 29, we have

$$\lim_{n\to\infty} h_n = \bigvee_{\oplus\, n=1}^{\infty} h_n = \bigvee_{\oplus\, n=1}^{\infty}\bigwedge_{\odot\, k=n}^{\infty} f_k = \varliminf_{n\to\infty} f_n. \tag{104}$$

By Theorem 29, we have

$$\int_X^{\oplus}\lim_{n\to\infty} h_n\odot dm = \lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm, \tag{105}$$

which implies that

$$\int_X^{\oplus}\varliminf_{n\to\infty} f_n\odot dm = \lim_{n\to\infty}\int_X^{\oplus} h_n\odot dm. \tag{106}$$

By (4) of Theorem 22 and $h_n \preceq f_k$ for all $k \geq n$, we have

$$\int_X^{\oplus} h_n\odot dm \preceq \int_X^{\oplus} f_k\odot dm, \tag{107}$$

for all $k \geq n$, which implies that

$$\int_X^{\oplus} h_n \odot dm \preceq \bigwedge_{k=n}^{\infty}{}_{\odot} \int_X^{\oplus} f_k \odot dm. \quad (108)$$

By (4) of Theorem 22 and the monotonicity of $\{h_n\}$, we have $\{\int_X^{\oplus} h_n \odot dm\}$ is an increasing sequence. Thus, by proof of Theorem 29, we have

$$\lim_{n\to\infty} \int_X^{\oplus} h_n \odot dm = \bigvee_{n=1}^{\infty}{}_{\oplus} \int_X^{\oplus} h_n \odot dm. \quad (109)$$

Hence, we obtain that

$$\int_X^{\oplus} \underline{\lim}_{n\to\infty} f_n \odot dm \preceq \bigvee_{n=1}^{\infty}{}_{\oplus} \bigwedge_{k=n}^{\infty}{}_{\odot} \int_X^{\oplus} f_k \odot dm = \underline{\lim}_{n\to\infty} \int_X^{\oplus} f_n \odot dm. \quad (110)$$

□

Example 34. Let the total order $\preceq$ on $[0, +\infty)$ be the usual order of the real line and the pseudoaddition $\oplus$ is defined by

$$x \oplus y = \begin{cases} \dfrac{x+y}{2}, & \text{if } x, y \in (0, \infty), \\ \max\{x, y\}, & \text{if } x = 0 \text{ or } y = 0, \end{cases} \quad (111)$$

and the pseudomultiplication $\odot$ is the usual multiplication of the real numbers. It is obvious that zero element is 0 and unit element is 1. Let the decomposable measure m be Lebesgue measure on $[0, 1]$. We know that the pseudointegral is

$$\int_{[0,1]}^{\oplus} f \odot dm = \frac{1}{2} \int_0^1 f(x)\, dx, \quad (112)$$

for each $f \in \mathscr{M}(\mathscr{A}([0, 1]))$, where the right hand side is the Lebesgue integral. Let

$$f_n(x) = \begin{cases} n, & \dfrac{1}{2n} \leq x \leq \dfrac{1}{n}, \\ 0, & \dfrac{1}{n} < x \leq 1 \text{ or } 0 \leq x < \dfrac{1}{2n}. \end{cases} \quad (113)$$

Then, we get that

$$\int_{[0,1]}^{\oplus} f_n \odot dm = \int_{[0,1/2n]}^{\oplus} 0 \odot dm \oplus \int_{[1/2n,1/n]}^{\oplus} n \odot dm \oplus \int_{[1/n,1]}^{\oplus} 0 \odot dm = \frac{1}{4}; \quad (114)$$

that is, $\underline{\lim}_{n\to\infty} \int_{[0,1]}^{\oplus} f_n \odot dm = 1/4$ and $\underline{\lim}_{n\to\infty} f_n = 0$, which implies that $\int_{[0,1]}^{\oplus} \underline{\lim}_{n\to\infty} f_n \odot dm = 0$. Hence, we obtain that

$$\int_{[0,1]}^{\oplus} \underline{\lim}_{n\to\infty} f_n \odot dm \leq \underline{\lim}_{n\to\infty} \int_{[0,1]}^{\oplus} f_n \odot dm. \quad (115)$$

4. Conclusions

In this paper, we mainly discussed pseudointegral based on pseudoaddition decomposable measure. Particularly, we have given the definition of the pseudointegral of a measurable function based on a strict pseudoaddition decomposable measure by generalizing the definition of the pseudointegral of a bounded measurable function. Furthermore, we have derived several important properties of the pseudointegral of a measurable function based on strict pseudoaddition decomposable measure. Finally, we have obtained that some theorems on the integral and the limit can be exchanged.

Recently, pseudoanalysis has obtained rapid development in the mechanical, chemical, biological, medical, and computer fields and has solved some uncertainty problems of knowledge. Pseudoanalysis theory has important applications in the field of computer image processing [39, 40]; for example, it can analyze and grasp the variation range of the image gray value, solve the relationship between the grey value and image color change, and take appropriate grey value to achieve better image processing effect. With the development of computer technology, pseudoanalysis will also get more and more widely used in computer science. We also hope that our results in this paper may lead to significant, new, and innovative results in other related fields.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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