Research Article

The EH Interpolation Spline and Its Approximation

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A new interpolation spline with two parameters, called EH interpolation spline, is presented in this paper, which is the extension of the standard cubic Hermite interpolation spline, and inherits the same properties of the standard cubic Hermite interpolation spline. Given the fixed interpolation conditions, the shape of the proposed splines can be adjusted by changing the values of the parameters. Also, the introduced spline could approximate to the interpolated function better than the standard cubic Hermite interpolation spline and the quartic Hermite interpolation splines with single parameter by a new algorithm.

1. Introduction

Spline interpolation is a useful and powerful tool for curves and surfaces modeling. Standard cubic Hermite spline is one of those interpolation functions. But, for the given interpolation condition, the cubic Hermite interpolation spline is fixed; that is to say, the shape of the interpolation curve or surface is fixed for the given interpolation data [1–6]. Since the interpolation function is unique for the given interpolation data, to modify the shape of the interpolation curve to approximate the given curve seems to be impossible and it is contradictory to the uniqueness of the interpolation function for the given interpolation data. For the given interpolation condition, how to improve the approximation accuracy of the interpolation spline is an important problem in the computer aided geometric design. In recent years, many authors have presented some new method to modify the shape of the interpolation curve to satisfy the industrial product design with several kinds of new interpolation splines with parameters [7–20].

These new splines all have similar properties of the standard cubic Hermite spline. For example, for the given interpolation data, if interpolation interval approaches zero, theoretically speaking, these splines can approximate the given curve and surface well. However, there exists a problem, in the process of the actual calculation; the amount of computation will increase dramatically if the length of the interpolating intervals decreases. On the other hand, the approximation accuracy of these new splines may not be better than the standard cubic Hermite spline.

In [7–17], many rational form interpolation splines with multiple parameters were presented. For the given interpolation data, the change of the parameters causes the change of the interpolation curve. Nevertheless, the computation of the splines with multiple parameters is very complicated. Several kinds of rational splines with a single parameter were presented in the papers [18, 19], which is simple to compute, but its approximation accuracy is not good for the given curves and surfaces. In general, polynomial-form splines are suitable to be used to design and compute. In [20], a polynomial-form spline, called quartic Hermite spline with single parameter, is presented as the extension of the standard Hermite spline. The quartic spline has a simple form, and its approximation rate to the given curves and surfaces is not high.

In this paper, a class of new quartic splines with two parameters is developed, which is the extension of the standard cubic Hermite interpolation spline and inherits the same properties of the standard cubic Hermite interpolation spline. For the given interpolation conditions, the shape of the proposed splines can be adjusted by changing the values of the parameters. Furthermore, the introduced splines could

approximate to the interpolated functions better than the standard cubic Hermite interpolation splines and the quartic Hermite interpolation splines with single parameter.

The remainder of the paper is organized as follows. Section 2 introduces the standard cubic Hermite spline and some of its properties. A kind of interpolation spline with two parameters is presented in Section 3. Section 4 discusses the approximation of the introduced spline curve with numerical examples. In the end, a summary and conclusions are given.

2. The Standard Cubic Hermite Spline and Its Basis Functions

Generally, for $t \in [0, 1]$, the following four basis functions,

$$
\begin{aligned}
\alpha_0(t) &= 1 - 3t^2 + 2t^3, \\
\alpha_1(t) &= 3t^2 - 2t^3, \\
\beta_0(t) &= t - 2t^2 + t^3, \\
\beta_1(t) &= -t^2 + t^3,
\end{aligned}
\tag{1}
$$

are called the standard cubic Hermite bases.

These bases satisfy

$$
\begin{aligned}
&\alpha_0(0) = \alpha_1(1) = 1, &&\alpha_0(1) = \alpha_1(0) = 0, \\
&\alpha_1'(0) = \alpha_1'(1) = 0, &&\alpha_0'(0) = \alpha_0'(1) = 0, \\
&\beta_0(0) = \beta_0(1) = 0, &&\beta_1(0) = \beta_1(1) = 0, \\
&\beta_0'(0) = \beta_1'(1) = 1, &&\beta_0'(1) = \beta_1'(0) = 0, \\
&\alpha_0(t) + \alpha_1(t) = 1, &&\beta_0(t) + \beta_1(1 - t) = 0.
\end{aligned}
\tag{2}
$$

For given knots $a = x_0 < x_1 < \cdots < x_n = b$ and data $\{(x_i, y_i, d_i), i = 0, 1, \ldots, n\}$, where y_i and d_i are the values of the function value and the first-order derivative value of the function being interpolated, let $h_i = x_{i+1} - x_i, t = (x - x_i)/h_i$ and then the standard cubic Hermite spline in the interval $[x_i, x_{i+1}]$ can be defined as follows:

$$
H_i(x) = \alpha_0(t) y_i + \alpha_1(t) y_{i+1} + \beta_0(t) h_i d_i + \beta_1(t) h_i d_{i+1},
$$
$$
i = 0, 1, \ldots, n - 1.
\tag{3}
$$

Obviously, we have $H_i(x_i) = y_i, H_i(x_{i+1}) = y_{i+1}, H_i'(x_i) = d_i, H_i'(x_{i+1}) = d_{i+1}$.

The standard cubic Hermite spline is C^1 continuous. However, if interpolation data is given, the shape and approximation of the spline cannot be changed.

3. The EH Interpolation Spline

In order to overcome the disadvantage of the standard cubic Hermite spline, we extend its basis functions firstly.

3.1. The Basis Functions of the EH Interpolation Spline

Definition 1. For arbitrary real number λ_i, μ_i and $0 \le t \le 1$, the following four functions with parameters λ_i, μ_i,

$$
\begin{aligned}
e\alpha_0(t) &= 1 + (\lambda_i - 3) t^2 + 2(1 - \lambda_i) t^3 + \lambda_i t^4, \\
e\alpha_1(t) &= (3 - \lambda_i) t^2 + 2(\lambda_i - 1) t^3 - \lambda_i t^4, \\
e\beta_0(t) &= t + (\mu_i - 2) t^2 + (1 - 2\mu_i) t^3 + \mu_i t^4, \\
e\beta_1(t) &= -(\mu_i + 1) t^2 + (1 + 2\mu_i) t^3 - \mu_i t^4,
\end{aligned}
\tag{4}
$$

are calledbasis functions of the EH interpolation spline, briefly EH bases.

The EH bases are the extension of the standard cubic Hermite bases. When $\lambda_i = \mu_i = 0$, the bases are the standard cubic Hermite bases. The bases have the similar properties of the standard cubic Hermite bases.

By computing directly, we have $e\alpha_0(0) = e\alpha_1(1) = 1, e\alpha_0(1) = e\alpha_1(0) = 0, e\alpha_0'(0) = e\alpha_1'(1) = 0, e\alpha_0'(0) = e\alpha_1'(1) = 0, e\beta_0(0) = e\beta_0(1) = 0, e\beta_1(0) = e\beta_1(1) = 0, e\beta_0'(0) = e\beta_1'(1) = 1, e\beta_1'(1) = e\beta_1'(0) = 0$, and $e\alpha_0(t) + e\alpha_1(t) = 1, e\beta_0(t) + e\beta_1(1 - t) = 0$.

When $\lambda_i = \mu_i$, the EH bases (4) are basis functions with single parameter in [20].

Figure 1 shows the four EH bases, where the solid lines are the standard Hermite bases, the parameters $\lambda_i = 2, \mu_i = -2$ are for the dot-dash lines, and $\lambda_i = -2, \mu_i = 2$ are for dashed line.

So, we may construct the Ferguson curve with two parameters based on the EH bases as follows:

$$
\text{EH}_i(t) = e\alpha_0(t) p_i + e\alpha_1(t) p_{i+1} + e\beta_0 p_i' + e\beta_1(t) p_{i+1}', \tag{5}
$$

where p_i, p_{i+1} and p_i', p_{i+1}' are two interpolation points and their tangent vectors, respectively.

For the given two interpolation points and tangent vectors, with the different parameters λ_i, μ_i, we may obtain different shape of the Ferguson curve with two parameters accordingly.

Figure 2 shows the Ferguson curves with different parameters, where the solid line is the standard Ferguson spline curve with $\lambda_i = \mu_i = 0$, the parameters $\lambda_i = 2, \mu_i = -2$ are for the dot-dash line, and $\lambda_i = -2, \mu_i = 2$ are for dashed line.

3.2. The EH Interpolation Spline

Definition 2. Given a data set $\{(x_i, y_i, d_i), i = 0, 1, \ldots, n\}$, where y_i and d_i are the values of the function value and the first-order derivative value of the function being interpolated and $a = x_0 < x_1 < \cdots < x_n = b$ is the knot spacing, let
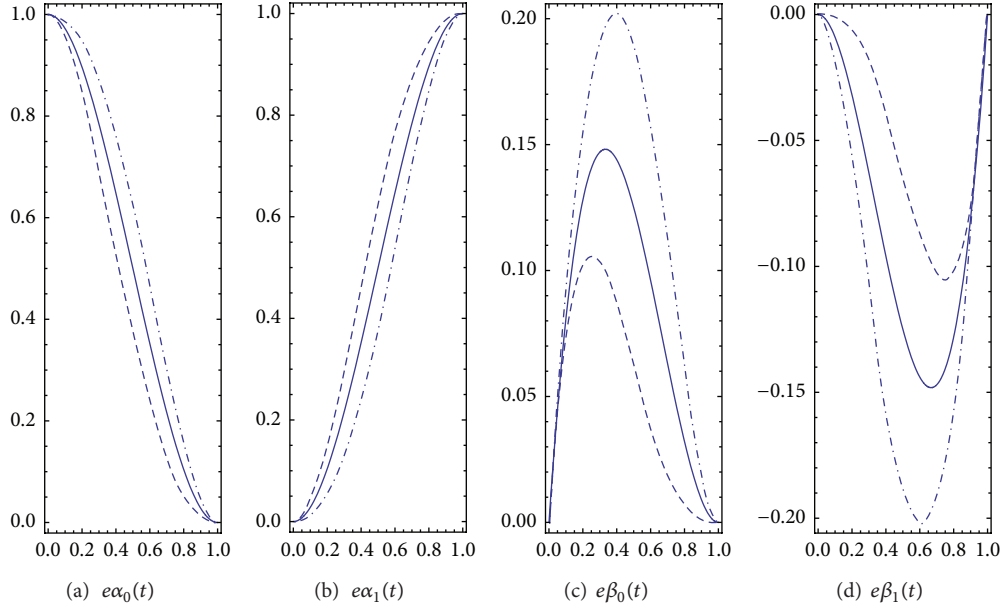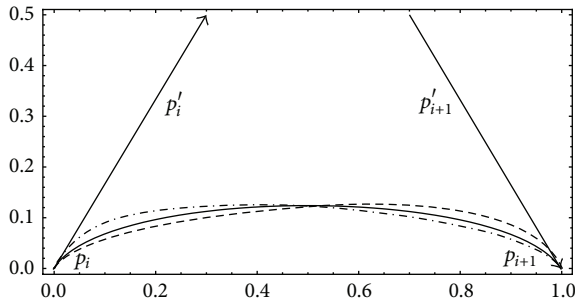
FIGURE 1: The graph of the four EH bases.

FIGURE 2: The Ferguson curve with different parameters.

$h_i = x_{i+1} - x_i, t = (x - x_i)/h_i$; then the EH interpolation spline in the interval $[x_i, x_{i+1}]$ can be defined as follows:

$$
\begin{aligned}
& \text{EH}\,(x)|_{[x_i, x_{i+1}]} \\
& \quad = y_i e\alpha_0\,(t) + y_{i+1} e\alpha_1\,(t) + d_i h_i e\beta_0\,(t) + d_{i+1} h_i e\beta_1\,(t)\,, \\
& \hspace{5cm} i = 0, 1, \ldots, n-1,
\end{aligned}
$$

(6)

where $e\alpha_0(t), e\alpha_1(t), e\beta_0(t), e\beta_1(t)$ are the EH bases.

Obviously, for the data set $\{(x_i, y_i, d_i),\ i = 0, 1, \ldots, n\}$, $\text{EH}(x)$ satisfies

$$
\begin{aligned}
\text{RH}\,(x_i) = y_i\,, & \qquad \text{RH}'\,(x_i) = d_i\,, \\
& i = 0, 1, \ldots, n.
\end{aligned}
$$

(7)

If $\lambda_i = \mu_i = 0$, it is just the standard cubic Hermite spline. It is of interest that, for suitable selected parameters λ_i, μ_i, the piecewise interpolation function $\text{EH}(x)$ can be

C^2-continuous in the interval $[x_0, x_n]$. In fact, denote $\Delta_i = (y_{i+1} - y_i)/h_i$, and let

$$
\text{EH}''\,(x_i+) = \text{EH}''\,(x_i-)\,, \quad i = 1, 2, \ldots, n-1; \quad (8)
$$

then the equations connecting the parameters λ_i and μ_i,

$$
\begin{aligned}
& h_i \left[\Delta_{i-1}\left(\lambda_{i-1} + 3\right) - \left(d_i\left(2 - \mu_{i-1}\right) + d_{i-1}\left(\mu_{i-1} + 1\right)\right)\right] \\
& \quad = h_{i-1}\left[\Delta_i\left(\lambda_i - 3\right) + \left(d_{i+1}\left(1 + \mu_i\right) + d_i\left(2 - \mu_i\right)\right)\right], \\
& \hspace{5cm} i = 1, 2, \ldots, n-1,
\end{aligned}
$$

(9)

may be obtained. If the successive parameters $(\lambda_{i-1}, \mu_{i-1})$ and (λ_i, μ_i) satisfy (9) at $i = 1, 2, \ldots, n-1$, then $\text{EH}(x) \in C^2(x_0, x_n)$. Furthermore, if the knots are equally spaced and $\lambda_i = \mu_i = 0$, then (9) becomes the well-known tridiagonal system for a cubic spline

$$
d_{i-1} + 4d_i + d_{i+1} = 3\left(\Delta_{i-1} + \Delta_i\right), \quad i = 1, 2, \ldots, n-1. \quad (10)
$$

Hence, if given the parameter values λ_0, μ_0 in the interval $[x_0, x_1]$, by (9), we may obtain the λ_1 and μ_1 and so on. Thus, we can construct a C^2-continuous interpolation curve.

4. The Approximation of the EH Interpolation Spline

According to the interpolation reminder of cubic Hermite spline, when interpolation interval approaches zero, the cubic Hermite spline curve can approximate well to the function being interpolated. However, for the EH interpolation spline we constructed, it can approximate well to the function being

TABLE 1: The parameters λ_i and μ_i for EH interpolation spline and the max error.

x_i	y_i	d_i	λ_i	μ_i	$RH\varepsilon_i$	$H\varepsilon_i$
0.0000	1.0000	1.0000	0.0421	0.0412	0.2569×10^{-4}	0.9062×10^{-3}
0.5000	1.2071	−0.1107	0.0146	0.0129	0.2111×10^{-4}	0.2569×10^{-3}
1.0000	1.0000	−0.5708	0.2451	0.2783	0.3769×10^{-4}	0.2569×10^{-3}
1.5000	0.7955	−0.1451	0.0188	0.0192	0.2974×10^{-4}	0.1069×10^{-2}
2.0000	1.0000	1.0000	0.0108	0.0108	0.1735×10^{-4}	0.8647×10^{-3}
2.5000	1.7929	2.1107				

interpolated without interpolation interval approaching zero, and it can approximate to the interpolated functions better than the standard cubic Hermite interpolation spline.

Firstly, we give the definition of the "good approximation."

Definition 3. Let $H_i(x)$ be the standard cubic Hermite spline, $EH_i(x)$ be the EH interpolation spline, and $y(x)$ be the function being interpolated. Denoting $EH\varepsilon_i = \max_{x_i < x < x_{i+1}} |EH_i(x) - y(x)|$, $H\varepsilon_i = \max_{x_i < x < x_{i+1}} |H_i(x) - y(x)|$, then if $EH\varepsilon_i < H\varepsilon_i$, we can call $RH_i(x)$ has "good approximation" to the interpolated function $y(x)$ better than $H_i(x)$.

According to the Definition 3, if $EH\varepsilon_i < H\varepsilon_i$, we can get the range of the parameters value, λ_i and μ_i. In the range of the parameters value, selecting the arbitrary values of the parameters λ_i and μ_i, we have a "good approximation" curve.

Example 4. Given the function $y(x) = x + \cos((\pi/2)x)$ and knots $x_i = (i/2)$ ($i = 0, 1, \ldots, 5$), namely, $h_i = (i/2)$ ($i = 0, \ldots, 4$). According to the inequality $EH\varepsilon_i < H\varepsilon_i$, we may get the range of the parameters λ_i and μ_i. For the fixed interpolation condition, the max error and the parameters λ_i

and μ_i are given for every interval $[x_i, x_{i+1}]$ in Table 1. The error curves of the $EH(x)$ and $H(x)$ to $y(x)$ are shown in Figure 3.

By using the tensor product method, we can construct the EH interpolation spline surfaces, which has the similar EH interpolation spline curve.

Definition 5. Let $\Omega: [a, b] \times [c, d]$ be the plane region and $f(x, y)$ a bivariate function defined in the region Ω and let $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$ be the knot sequences. Denote $h_i = x_{i+1} - x_i, h_j = y_{j+1} - y_j$, $u = (x - x_i)/h_i, v = (y - y_j)/h_j$; then the EH interpolation spline surface on the region $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$ can be defined as follows:

$$EH(x, y)\Big|_{[x_i, x_{i+1}]}^{[y_i, y_{i+1}]}$$

$$= (e\alpha_0(u), e\alpha_1(u), e\beta_0(u), e\beta_1(u)) M \begin{pmatrix} e\alpha_0(v) \\ e\alpha_1(v) \\ e\beta_0(v) \\ e\beta_1(v) \end{pmatrix}, \quad (11)$$

where

$$M = \begin{pmatrix} f(x_i, y_i) & f(x_i, y_{i+1}) & h_j f_v'(x_i, y_i) & h_j f_v'(x_i, y_{i+1}) \\ f(x_{i+1}, y_i) & f(x_{i+1}, y_{i+1}) & h_j f_v'(x_{i+1}, y_i) & h_j f_v'(x_{i+1}, y_{i+1}) \\ h_i f_u'(x_i, y_i) & h_i f_u'(x_i, y_{i+1}) & h_i h_j f_{uv}''(x_i, y_i) & h_i h_j f_{uv}''(x_i, y_{i+1}) \\ h_i f_u'(x_{i+1}, y_i) & h_i f_u'(x_{i+1}, y_{i+1}) & h_i h_j f_{uv}''(x_{i+1}, y_i) & h_i h_j f_{uv}''(x_{i+1}, y_{i+1}) \end{pmatrix}. \quad (12)$$

Given the end-points, the first order partial derivative and the second-order blending partial derivative of the function interpolated, with proper parameters, the EH interpolation spline surfaces could approximate to the bivariate functions being interpolated better than the standard cubic Hermite spline surfaces.

Example 6. Given the bivariate function being interpolated $f(x, y) = \sin(\pi/2)x \cos(\pi/2)y$, let $a = 0 < 1 < 2 = b$ and $c = -1 < 0 < 1 = d$ be the knot sequences. Denote $h_i = x_{i+1} - x_i$, $h_j = y_{j+1} - y_j, u = (x - x_i)/h_i$, and $v = (y - y_j)/h_j$. By selecting $\lambda_0 = \lambda_1 = 0.3208$, $\mu_0 = \mu_1 = 0.6995$, we can work out that

the max error of the $EH(x, y) - f(x, y)$ equals 0.5069×10^{-3}, but the max error of the $H(x, y) - f(x, y)$ equals 0.1061×10^{-1}.

Figure 4 shows the error surface of the $EH(x, y) - f(x, y)$. Figure 5 shows the error surface of the $H(x, y) - f(x, y)$.

5. Conclusion

This paper introduced a kind of EH interpolation spline, which is the extension of the standard cubic Hermite interpolation spline. The shape of the proposed splines can be adjusted by changing the values of the parameters for the
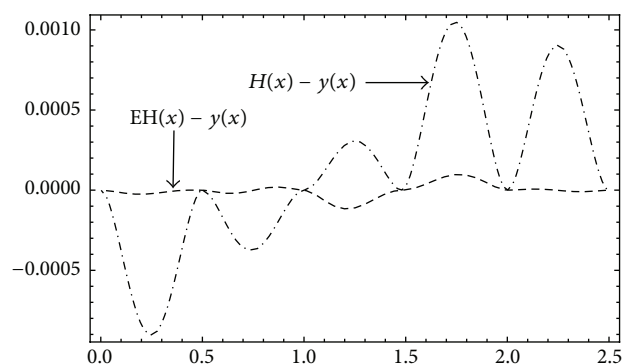
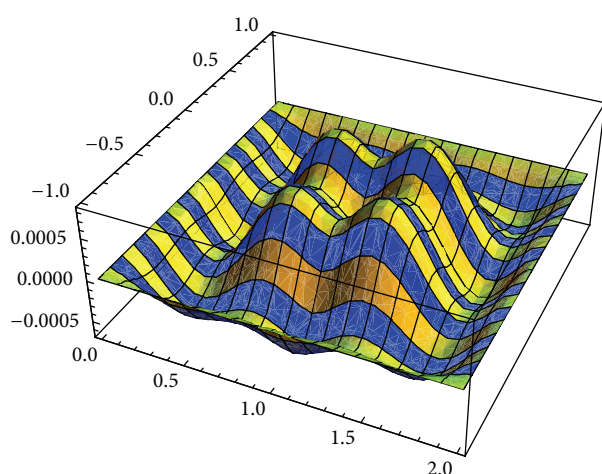Figure 3: The error curves of the $EH(x) - y(x)$ and $H(x) - y(x)$.

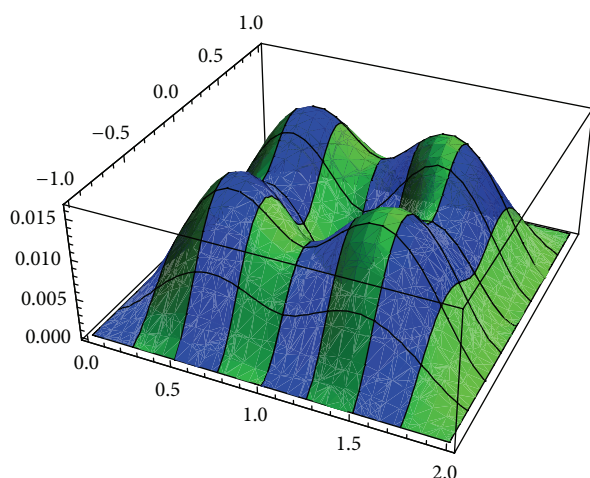Figure 4: The error surface of the $EH(x, y) - f(x, y)$.

Figure 5: The error surface of the $H(x, y) - f(x, y)$.

fixed interpolation conditions. Also, the introduced spline could approximate to the interpolated function better than the standard cubic Hermite interpolation spline.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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