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*Research Article*

# Asymptotics for the Solutions to Defective Renewal Equations

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This paper investigates the defective renewal equations under the nonconvolution equivalent distribution class. The asymptotics of the solution to the defective renewal equations have been given for the heavy-tailed and light-tailed cases, respectively.

## 1. Introduction

This paper will consider the defective renewal equation

$$Z(x) = z(x) + q\int_0^x Z(x-y)F(\mathrm{d}y), \quad x \geq 0, \tag{1}$$

where $F$ is a proper distribution on $[0,\infty)$, $z(x) \geq 0$ is a known and locally bounded function on $[0,\infty)$, and $0 < q < 1$. The only solution $Z(x)$ to (1) is given by

$$Z(x) = (1-q)^{-1}\int_0^x z(x-y)U_0(\mathrm{d}y), \quad x \geq 0, \tag{2}$$

where

$$U_0(x) = (1-q)\sum_{n=0}^{\infty} q^n F^{*n}(x), \quad x \geq 0 \tag{3}$$

(see e.g., Asmussen [1, Chapter V]) and here $F^{*n}$ is the $n$-fold convolution of $F$ with itself, $n \geq 2$, $F^{*1} = F$, and $F^{*0}$ is the distribution degenerate at zero.

Since in most cases it is not easy to calculate (2), more attention is paid to the asymptotics of the solution $Z(x)$. When $Z = F$, the asymptotics of $Z(x)$ have been investigated by many researchers, such as Embrechts et al. [2], Embrechts and Goldie [3], and Cline [4]. Asmussen [5] and Asmussen et al. [6] considered the case that $z$ is a subexponential density. Yin and Zhao [7] obtained the asymptotics of $Z(x)$ for the monotone function $z$. For the above case, K. Wang and Y. Wang [8] gave the local asymptotics of $Z(x)$. Cui et al. [9] considered a new case that

$$\lim_{x\to\infty}\frac{z(x)}{\overline{F}(x)} = c, \tag{4}$$

where $c$ is a positive constant. In Corollary 5.1 and Theorem 5.2 of Cui et al. [9], they obtained the asymptotics of $Z(x)$ under the condition that $F \in \mathcal{S}$ and $F \in \mathcal{S}(\gamma)$ for some $\gamma > 0$ with $\widehat{F}(\gamma) = \int_{-\infty}^{\infty} e^{\gamma u}F(\mathrm{d}u) < 1$, respectively. The classes $\mathcal{S}$ and $(\gamma)$, $\gamma > 0$, (the definitions of these distribution classes will be given below) are convolution equivalent distribution classes. But beyond the convolution equivalent distribution classes, there exist some other distributions. How to estimate the asymptotics of the solution $Z(x)$ for the nonconvolution equivalent distribution $F$ will be an interesting question. This paper will investigate this case. Under the conditions (4) and that $F$ may not belong to the convolution equivalent distribution class, this paper obtains the asymptotics of the solution $Z(x)$. In order to better illuminate our motivation and results, we will introduce some notions and notation.

Without special statement, in this paper a limit is taken as $x \to \infty$. For two nonnegative functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geq 1$, write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$, and write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$. For a proper distribution $V$ on $(-\infty,\infty)$, the tail of $V$ is $\overline{V} = 1 - V$. For a real number $\gamma$, denote by $\widehat{V}(\gamma) = \int_{-\infty}^{\infty} e^{\gamma u}V(\mathrm{d}u)$ the moment generating function of $V$.

Firstly, we will introduce some heavy-tailed and light-tailed distribution classes. Say that a random variable (r.v.) $\xi$ (or its corresponding distribution $V$) is heavy-tailed if for all $\lambda > 0$, $\widehat{V}(\gamma) = \infty$; otherwise, say that it is light-tailed. Let $V$ be a distribution on $(-\infty, \infty)$. Say that the distribution $V$ belongs to the class $\mathscr{L}(\gamma)$ for some $\gamma \geq 0$, if for any $t \in (-\infty, \infty)$,

$$\overline{V}(x - t) \sim e^{\gamma t}\overline{V}(x), \tag{5}$$

where, when $\gamma > 0$ and $V$ is a lattice distribution, $x$ and $t$ are both taken as a multiple of the lattice step. Say that the distribution $V$ belongs to the class $\mathscr{S}(\gamma)$ for some $\gamma \geq 0$, if $V \in \mathscr{L}(\gamma)$, $\widehat{V}(\gamma) < \infty$, and

$$\overline{V^{*2}}(x) \sim 2\widehat{V}(\gamma)\overline{V}(x). \tag{6}$$

The class $\mathscr{S}(\gamma)$, $\gamma \geq 0$, is called the convolution equivalent distribution class and was introduced by [10] and Chover et al. [11, 12] for distributions on $[0, \infty)$ and by Pakes [13] for distributions on $(-\infty, \infty)$. Especially, we call $\mathscr{S}(0)$ and $\mathscr{L}(0)$ the subexponential distribution class and the long-tailed distribution class, denoted by $\mathscr{S}$ and $\mathscr{L}$, respectively.

This paper will mainly investigate the case that the distribution $F$ may not be convolution equivalent. We will introduce another distribution class. Say that the distribution $V$ belongs to the class $\mathscr{OS}$, if $\overline{V}(x) > 0$ for sufficiently large $x$ and

$$C_V^* = \limsup_{x \to \infty} \frac{\overline{V^{2*}}(x)}{\overline{V}(x)} < \infty. \tag{7}$$

Clearly, if $V \in \mathscr{S}(\gamma)$ for some $\gamma \geq 0$ then $C_V^* = 2\widehat{V}(\gamma)$. Therefore, for each $\gamma \geq 0$, $\mathscr{S}(\gamma) \subset \mathscr{OS}$. If $V \in \mathscr{L}(\gamma)$ for some $\gamma \geq 0$ then $C_V^* \geq 2\widehat{V}(\gamma)$, which can be obtained by Lemma 2.4 of Embrechts and Goldie [3] and Theorems 1.1 and 1.2 of Yu et al. [14]. The class $\mathscr{OS}$ is first introduced by Klüppelberg [15] and detailedly studied in Klüppelberg and Villasenor [16], Shimura and Watanabe [17], Watanabe and Yamamura [18], Lin and Wang [19], Yang and Wang [20], and Wang et al. [21], among others. This paper will consider the case that $F \in \mathscr{L}(\gamma) \cap \mathscr{OS}$, $\gamma \geq 0$. As noted by Wang et al. [21], for each $\gamma > 0$, $\mathscr{S}(\gamma) \subset \mathscr{L}(\gamma) \cap \mathscr{OS}$ and the class $(\mathscr{L}(\gamma) \cap \mathscr{OS}) \setminus \mathscr{S}(\gamma)$ is nonempty.

We first present the main result for the heavy-tailed case.

**Theorem 1.** *For the renewal equation* (1)*, assume that* (4) *holds. If $F \in \mathscr{L} \cap \mathscr{OS}$ and $C_F^* < 1 + q^{-1}$, then*

$$\frac{c}{1-q}\overline{F}(x) \lesssim Z(x) \lesssim \left(\frac{1}{1-q} + \frac{C_{U_0}^* - 2}{1 - q\left(C_F^* - 1\right)}\right) c\overline{F}(x). \tag{8}$$

*Remark 2.* If $F \in \mathscr{S}$ then $U_0 \in \mathscr{S}$ by Theorem 1 of Cline [4]. Hence, $C_{U_0}^* = 2$. The result of Theorem 1 implies that

$$Z(x) \sim \frac{c}{1-q}\overline{F}(x), \tag{9}$$

which is Corollary 5.1 (ii) of Cui et al. [9].

In the following, we give the result for the light-tailed case.

**Theorem 3.** *For the renewal equation* (1)*, assume that* (4) *holds. If $F \in \mathscr{L}(\gamma) \cap \mathscr{OS}$ for some $\gamma > 0$ satisfying $\widehat{F}(\gamma) < 1$ and $C_F^* < 1 + \widehat{F}(\gamma)$, then*

$$c_1\overline{F}(x) \lesssim Z(x) \lesssim c_2\overline{F}(x), \tag{10}$$

*where*

$$\begin{aligned} c_1 &= \frac{c}{1 - q\widehat{F}(\gamma)} + \frac{qI}{\left(1 - q\widehat{F}(\gamma)\right)^2}, \\ c_2 &= \frac{c}{1 - q\widehat{F}(\gamma)} + \frac{c\left(1 - q\widehat{F}(\gamma)\right)}{1 - q\left(C_F^* - \widehat{F}(\gamma)\right)}\left(\frac{C_{U_0}^*}{1-q} - \frac{2}{1 - q\widehat{F}(\gamma)}\right) \\ &\quad + \frac{qI}{\left(1 - q\left(C_F^* - \widehat{F}(\gamma)\right)\right)\left(1 - q\widehat{F}(\gamma)\right)}, \\ I &= \gamma\int_0^\infty e^{\gamma y} z(y)dy. \end{aligned} \tag{11}$$

*Remark 4.* If $F \in \mathscr{S}(\gamma)$ for some $\gamma > 0$, by Theorem 1 of Cline [4], then $C_F^* = 2\widehat{F}(\gamma)$ and $U_0 \in \mathscr{S}(\gamma)$. Therefore, $C_{U_0}^* = 2\widehat{U_0}(\gamma) = 2(1-q)/(1 - q\widehat{F}(\gamma))$. Then we can obtain from Theorem 3 that

$$Z(x) \sim c_1\overline{F}(x), \tag{12}$$

which is Theorem 5.2 (ii) of Cui et al. [9].

## 2. Proofs of Theorems

Before giving the proof of Theorems 1 and 3, we first give some lemmas. The first lemma comes from Lemma 2.2 of Yu and Wang [22], which will need the following notation. For a distribution $V$ on $(-\infty, \infty)$ and any $\gamma \geq 0$, define

$$\begin{aligned} \mathscr{H}_V(\gamma) = \{h \text{ on } [0, \infty) : h(x) \uparrow \infty,\ x^{-1}h(x) \longrightarrow 0, \\ \overline{V}(x - y) \sim e^{\gamma y}\overline{V}(x) \\ \text{uniformly for } |y| \leq h(x)\}. \end{aligned} \tag{13}$$

**Lemma 5.** *Suppose that $V$ is a distribution on $(-\infty, \infty)$ and belongs to the class $\mathscr{L}(\gamma) \cap \mathscr{OS}$ for some $\gamma \geq 0$. Then for any $h \in \mathscr{H}_V(\gamma)$,*

$$\limsup \int_{h(x)}^{x-h(x)} \frac{\overline{V}(x-u)}{\overline{V}(x)} V(du) = C_V^* - 2\widehat{V}(\gamma). \tag{14}$$

**Lemma 6.** *For the random sum* (3)*, assume that $F \in \mathscr{L}(\gamma) \cap \mathscr{OS}$ for some $\gamma \geq 0$. When $\gamma = 0$, let $C_F^* < 1 + q^{-1}$; when $\gamma > 0$, let $\widehat{F}(\gamma) < 1$ and $C_F^* < 1 + \widehat{F}(\gamma)$. Then $U_0 \in \mathscr{L}(\gamma) \cap \mathscr{OS}$,*

$$\liminf \frac{\overline{U_0}(x)}{\overline{F}(x)} = \frac{q(1-q)}{\left(1 - q\widehat{F}(\gamma)\right)^2}, \tag{15}$$

$$\limsup \frac{\overline{U_0}(x)}{\overline{F}(x)} \leq \frac{q(1-q)}{\left(1 - q\left(C_F^* - \widehat{F}(\gamma)\right)\right)\left(1 - q\widehat{F}(\gamma)\right)}. \tag{16}$$

*Proof.* We first prove (15). Let $\tau$ be a r.v. with a distribution $P(\tau = n) = (1-q)q^n$, $n = 0, 1, \ldots$. When $\gamma = 0$ and since $F$ is heavy-tailed and $\tau$ is light-tailed, by Theorem 2 of Denisov et al. [23], it holds that

$$\liminf \frac{\overline{U_0}(x)}{\overline{F}(x)} = \frac{q}{(1-q)}. \tag{17}$$

When $\gamma > 0$ and since there exists $\epsilon_1 > 0$ such that

$$E\left(\max\left\{\widehat{F}(\gamma) + \epsilon_1, 1\right\}\right)^{\tau} < \infty, \tag{18}$$

by Theorem 1.2 of Yu et al. [14], it holds that

$$\begin{aligned}\liminf \frac{\overline{U_0}(x)}{\overline{F}(x)} &\le E\tau\left(\widehat{F}(\gamma)\right)^{\tau-1} \\ &= \frac{q(1-q)}{\left(1-q\widehat{F}(\gamma)\right)^2}.\end{aligned} \tag{19}$$

On the other hand, since $F \in \mathscr{L}(\gamma)$, by Fatou's lemma and Lemma 5.4 of Pakes [13], we have

$$\begin{aligned}\liminf \frac{\overline{U_0}(x)}{\overline{F}(x)} &\ge E\tau\left(\widehat{F}(\gamma)\right)^{\tau-1} \\ &= \frac{q(1-q)}{\left(1-q\widehat{F}(\gamma)\right)^2}.\end{aligned} \tag{20}$$

This completes the proof of (15).

Now we prove (16). Since $C_F^* < 1 + q^{-1}$ for $\gamma = 0$ and $C_F^* < 1 + \widehat{F}(\gamma)$ for $\gamma > 0$, there exists $\epsilon_2 > 0$ such that for $\gamma \ge 0$,

$$E\left(C_F^* - \widehat{F}(\gamma) + \epsilon_2\right)^{\tau} < \infty. \tag{21}$$

Hence, by Corollary 1 of Yu and Wang [22], we get $U_0 \in \mathscr{L}(\gamma) \cap \mathscr{OS}$ and (16) holds. □

*Proof of Theorem 1.* Since $F \in \mathscr{L} \cap \mathscr{OS}$, we get $U_0 \in \mathscr{L} \cap \mathscr{OS}$ by Lemma 6. For any fixed positive constant $M$, when $x$ is sufficiently large, we get

$$\begin{aligned}Z(x) &= (1-q)^{-1}\left(\int_0^M + \int_M^{x-M} + \int_{x-M}^x\right) z(x-y)\, U_0(\mathrm{d}y) \\ &=: J_1(x,M) + J_2(x,M) + J_3(x,M).\end{aligned} \tag{22}$$

By (4) and $F \in \mathscr{L}$, we get

$$\begin{aligned}J_1(x,M) &\sim c(1-q)^{-1}\int_0^M \overline{F}(x-y)\, U_0(\mathrm{d}y) \\ &\sim c(1-q)^{-1}\overline{F}(x)\, U_0(M).\end{aligned} \tag{23}$$

Hence,

$$\lim_{M\to\infty}\lim_{x\to\infty}\frac{J_1(x,M)}{\overline{F}(x)} = \frac{c}{1-q}. \tag{24}$$

For $J_3(x,M)$, since $\sup_{t\in[0,M]} z(t) < \infty$, by $U_0 \in \mathscr{L}$ and Lemma 6, it holds that

$$\begin{aligned}J_3(x,M) &\le (1-q)^{-1}\sup_{t\in[0,M]} z(t)\left(\overline{U_0}(x-M) - \overline{U_0}(x)\right) \\ &= o\left(\overline{U_0}(x)\right) = o\left(\overline{F}(x)\right).\end{aligned} \tag{25}$$

For $J_2(x,M)$, we first estimate the asymptotics of

$$\int_M^{x-M} \overline{U_0}(x-y)\, U_0(\mathrm{d}y) \tag{26}$$

as firstly letting $x \to \infty$ and then letting $M \to \infty$. For any $x \ge 0$, it holds that

$$\begin{aligned}\overline{U^{*2}}(x) &= 2\int_0^M \overline{U_0}(x-y)\, U_0(\mathrm{d}y) \\ &\quad + \int_M^{x-M} \overline{U_0}(x-y)\, U_0(\mathrm{d}y) \\ &\quad + \overline{U_0}(x-M)\,\overline{U_0}(M).\end{aligned} \tag{27}$$

Since $U_0 \in \mathscr{L}$, we have

$$\begin{aligned}&\lim_{M\to\infty}\lim_{x\to\infty}\int_0^M \frac{\overline{U_0}(x-y)}{\overline{U_0}(x)} U_0(\mathrm{d}y) = 1, \\ &\lim_{M\to\infty}\lim_{x\to\infty}\frac{\overline{U_0}(x-M)\,\overline{U_0}(M)}{\overline{U_0}(x)} = 0.\end{aligned} \tag{28}$$

Hence, $U_0 \in \mathscr{OS}$ means that

$$\limsup_{M\to\infty}\limsup_{x\to\infty}\int_M^{x-M}\frac{\overline{U_0}(x-y)}{\overline{U_0}(x)} U_0(\mathrm{d}y) = C_{U_0}^* - 2, \tag{29}$$

which, combining with (4) and Lemma 6, yields that

$$\begin{aligned}&\limsup_{M\to\infty}\limsup_{x\to\infty}\frac{J_2(x,M)}{\overline{F}(x)} \\ &\quad\le \frac{c}{1-q(C_F^*-1)} \\ &\quad\quad\times \limsup_{M\to\infty}\limsup_{x\to\infty}\int_M^{x-M}\frac{\overline{U_0}(x-y)}{\overline{U_0}(x)} U_0(\mathrm{d}y) \\ &\quad= \frac{c}{1-q(C_F^*-1)}\left(C_{U_0}^* - 2\right).\end{aligned} \tag{30}$$

Hence, (8) can be obtained by (22)–(25) and (30). □

*Proof of Theorem 3.* It follows from Lemma 6 and $F \in \mathscr{L}(\gamma) \cap \mathscr{OS}$ that $U_0 \in \mathscr{L}(\gamma) \cap \mathscr{OS}$. Taking $h \in \mathscr{H}_{U_0}(\gamma)$, when $x$ is sufficiently large, we get

$$\begin{aligned}Z(x) &= (1-q)^{-1}\left(\int_0^{h(x)} + \int_{h(x)}^{x-h(x)} + \int_{x-h(x)}^x\right) \\ &\quad\times z(x-y)\, U_0(\mathrm{d}y) \\ &=: I_1(x) + I_2(x) + I_3(x).\end{aligned} \tag{31}$$

By (4) and $F \in \mathscr{L}(\gamma)$, we have

$$\begin{aligned} I_1(x) &\sim c(1-q)^{-1}\int_0^{h(x)} \overline{F}(x-y)\,U_0(\mathrm{d}y) \\ &\sim \frac{c}{1-q\widehat{F}(\gamma)}\overline{F}(x). \end{aligned} \tag{32}$$

By (4) and Lemmas 5 and 6, it holds that

$$\begin{aligned} I_2(x) &\sim c(1-q)^{-1}\int_{h(x)}^{x-h(x)} \overline{F}(x-y)\,U_0(\mathrm{d}y) \\ &\lesssim \frac{c\left(1-q\widehat{F}(\gamma)\right)^2}{q(1-q)^2}\int_{h(x)}^{x-h(x)} \overline{U_0}(x-y)\,U_0(\mathrm{d}y) \\ &\lesssim \frac{c\left(1-q\widehat{F}(\gamma)\right)^2}{q(1-q)^2}\left(C_{U_0}^{*}-2\widehat{U_0}(\gamma)\right)\overline{U_0}(x) \\ &\lesssim \frac{c\left(1-q\widehat{F}(\gamma)\right)}{1-q\left(C_F^{*}-\widehat{F}(\gamma)\right)}\left(\frac{C_{U_0}^{*}}{1-q}-\frac{2}{1-q\widehat{F}(\gamma)}\right)\overline{F}(x). \end{aligned} \tag{33}$$

For $I_3(x)$, using Lemma 6 and the way of dealing with $Z_3(x)$ in Theorem 5.2 of Cui et al. [9], we can get

$$\begin{aligned} &\limsup \frac{I_3(x)}{\overline{F}(x)} \le \frac{qI}{\left(1-q\left(C_F^{*}-\widehat{F}(\gamma)\right)\right)\left(1-q\widehat{F}(\gamma)\right)}, \\ &\liminf \frac{I_3(x)}{\overline{F}(x)} \ge \frac{qI}{\left(1-q\widehat{F}(\gamma)\right)^2}. \end{aligned} \tag{34}$$

Hence, (10) can be obtained by (31)–(34). □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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