Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 725781, 7 pages
http://dx.doi.org/10.1155/2014/725781

*Research Article*

# Existence of Nontrivial Solutions for Perturbed $p$-Laplacian Equation in $\mathbb{R}^N$ with Critical Nonlinearity

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Received 5 November 2013; Accepted 17 February 2014; Published 24 March 2014

Academic Editor: Shuangjie Peng



We consider a perturbed $p$-Laplacian equation with critical nonlinearity in $\mathbb{R}^N$. By using variational method, we show that it has at least one positive solution under the proper conditions.

## 1. Introduction and Main Results

In this paper, we are concerned with the existence of nontrivial solutions for the following nonlinear perturbed $p$-Laplacian equation with critical nonlinearity:

$$\begin{gathered} -\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u \\ = K(x)|u|^{p^*-2}u + f(x,u), \quad x \in \mathbb{R}^N, \\ u(x) > 0, \\ u(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \end{gathered} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator with $1 < p < N, N \geq 3$, $p^* = Np/(N-p)$ denotes the Sobolev critical exponent, $V(x)$ is a nonnegative potential, $K(x)$ is a bounded positive function, and $f(x,u)$ is a superlinear but subcritical function.

For $p = 2$, (1) turns into the following Schrödinger equation of the form

$$-\varepsilon^2 \Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u), \quad x \in \mathbb{R}^N. \tag{2}$$

The equation (2) has been studied extensively under various hypotheses on the potential and nonlinearity by many authors including Ambrosetti and Rabinowitz [1], Bartsch and Wang [2], Brézis and Lieb [3], Brėzis and Nirenberg [4], and Del Pino and Felmer [5] in bounded domains. Meanwhile, we recall some works in unbounded domains which contain Cingolani and Lazzo [6], Clapp and Ding [7], Ding and Lin [8], Floer and Weinstein [9], Grossi [10], Jeanjean and Tanaka [11], Kang and Wei [12], Oh [13], Pistoia [14], Rabinowitz [15], and Tang [16].

For general $p > 1$, most of the work (see [17–19] and the reference therein) dealt with (1) with $\varepsilon = 1$, $K(x) \equiv 0$ and a certain sign potential $V(x)$. Liu and Zheng [20] considered the above mentioned problem with sign-changing potential and subcritical $p$-superlinear nonlinearity. Cao et al. [21] also studied the similar problem. However, to our best knowledge, it seems that there is almost no work on the existence of semi-classical solutions to the equation in $\mathbb{R}^N$ with critical nonlinearities. This paper will study the critical nonlinearity case in whole space.

Throughout the paper, we make the following assumption:

$(H_1)$ $V \in C(\mathbb{R}^N)$, $V(0) = \inf_{x\in\mathbb{R}^N} V(x) = 0$ and there exists $b > 0$ such that the set $\nu^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;

$(H_2)$ $K(x) \in C(\mathbb{R}^N, \mathbb{R}^+)$, $0 < \inf K \leq \sup K < \infty$;

$(H_3)$ $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $f(x,t) = o(|t|^{p-2}t)$ uniformly in $x$ as $t \to 0$;

$(H_4)$ there are $a_0 > 0$ and $p < q < p^*$ such that $|f(x,t)| \le a_0(1 + |t|^{q-1})$ for all $(x,t)$;

$(H_5)$ there exist $b_0 > 0$, $\alpha > p$ and $\mu \in (p, p^*)$ such that $F(x,t) \ge b_0|t|^\alpha$ and $\mu F(x,t) \le f(x,t)t$ for all $(x,t)$, where $F(x,t) = \int_0^t f(x,s)ds$.

Our main result reads as follows.

**Theorem 1.** *Assume that $(H_1)$–$(H_5)$ hold. Then for any $\sigma > 0$, there exists $\varepsilon_\sigma > 0$ such that if $\varepsilon \le \varepsilon_\sigma$, (1) has at least one positive solution $u_\varepsilon$ of least energy which satisfied the following estimate:*

$$\frac{\mu - p}{p\mu} \int_{\mathbb{R}^N} \left(\varepsilon^p |\nabla u_\varepsilon|^p + V(x)|u_\varepsilon|^p\right) \le \sigma \varepsilon^N. \tag{3}$$

*The main tool used in the proof of Theorem 1 is variational method which was mainly developed in [8]. The main difficulty in the case is to overcome the loss of the compactness of the energy functional related to* (1) *because of unbounded domain $\mathbb{R}^N$ and critical nonlinearity. Although the energy functional does not satisfy the* $(PS)$ *condition, we can prove that it possesses* $(PS)_c$ *condition at some energy level c.*

This outline of the paper is organized as follows. In Section 2, we give the variational settings and preliminary results. In Section 3, we show that the corresponding energy functional satisfies $(\mathrm{PS})_c$ condition at the levels less than $\alpha_0 \lambda^{1-N/p}$ with some $\alpha_0 > 0$ independent of $\lambda$. Furthermore, it possesses the mountain geometry structure. Section 4 is devoted to the proof of the main result.

## 2. Preliminaries

Let $\lambda = \varepsilon^{-p}$ in (1). The equation (1) reads as

$$\begin{gathered} -\Delta_p u + \lambda V(x)|u|^{p-2}u \\ = \lambda K(x)|u|^{p^*-2}u + \lambda f(x,u), \quad x \in \mathbb{R}^N, \\ u(x) > 0, \\ u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty. \end{gathered} \tag{4}$$

In order to prove Theorem 1, we are going to prove the following result.

**Theorem 2.** *Assume that $(H_1)$–$(H_5)$ is satisfied. Then for any $\sigma > 0$, there exists $\lambda_\sigma > 0$ such that if $\lambda > \lambda_\sigma$, (4) has at least one positive solution $u_\lambda$ satisfying the following estimate:*

$$\frac{\mu - p}{p\mu} \int_{\mathbb{R}^N} \left(|\nabla u_\lambda|^p + \lambda V(x)|u_\lambda|^p\right) \le \sigma \lambda^{1-N/p}. \tag{5}$$

Next, we introduce the space

$$\begin{aligned} &E_\lambda\left(\mathbb{R}^N, V\right) \\ &= \left\{u \in W^{1,p}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} \lambda V(x)|u|^p < \infty, \lambda > 0\right\} \end{aligned} \tag{6}$$

equipped with the norm

$$\|u\|_{E_\lambda} = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + \lambda V(x)|u|^p\right)\right)^{1/p}. \tag{7}$$

Note that the norm $\|\cdot\|_{E_1}$ is equivalent to the one $\|\cdot\|_{E_\lambda}$ for any $\lambda > 0$. It follows from $(H_1)$ that $E_\lambda(\mathbb{R}^N, V)$ continuously is embedded in $W^{1,p}(\mathbb{R}^N)$. To prove Theorem 2, one considers the $C^1$ functional $I : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\int_{\mathbb{R}^N} \left(|\nabla u|^p + \lambda V(x)|u|^p\right) \\ &\quad - \frac{\lambda}{p^*}\int_{\mathbb{R}^N} K(x)|u|^{p^*} - \lambda \int_{\mathbb{R}^N} F(x,u) \\ &= \frac{1}{p}\|u\|_{E_\lambda}^p - \lambda \int_{\mathbb{R}^N} G(x,u), \end{aligned} \tag{8}$$

where $G(x,u) = (1/p^*)K(x)|u|^{p^*} + F(x,u)$.

Under the assumptions of Theorem 2, standard arguments [22] show that $I_\lambda \in C^1(E_\lambda, \mathbb{R})$ and its critical points are weak solutions of (4).

## 3. Necessary Lemmas

This section will show some lemmas which are important for the proof of the main result.

**Lemma 3.** *Assume that $(H_1)$–$(H_5)$ is satisfied. For the* $(PS)_c$ *sequence $\{u_n\} \subset E_\lambda$ for $I_\lambda$, we get that $c \ge 0$ and $\{u_n\}$ is bounded in the space $E_\lambda$.*

*Proof.* By direct computation and the assumptions $(H_2)$ and $(H_5)$, one has

$$\begin{aligned} &I_\lambda(u_n) - \frac{1}{\mu} I_\lambda'(u_n)u_n \\ &\quad = \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_{E_\lambda}^p + \left(\frac{1}{\mu} - \frac{1}{p^*}\right)\lambda \int_{\mathbb{R}^N} K(x)|u_n|^{p^*} \\ &\quad\quad + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} f(x,u)u - F(x,u)\right). \end{aligned} \tag{9}$$

Together with $I_\lambda(u_n) \to c$ and $I_\lambda'(u_n) \to 0$ as $n \to \infty$, we easily get that the $(\mathrm{PS})_c$ sequence is bounded in $E_\lambda$ and the energy level $c \ge 0$. □

By Lemma 3, there is $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in $E_\lambda$. Furthermore, passing to a subsequence, we have $u_n \to u$ in $L^b_{\mathrm{loc}}(\mathbb{R}^N)$ for any $b \in [p, p^*)$ and $u_n \to u$ *a.e.* in $\mathbb{R}^N$.

**Lemma 4.** *For any $s \in [p, p^*)$, there is a subsequence $\{u_{n_i}\}$ such that, for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ with*

$$\lim_{i \to \infty} \sup \int_{B_i \setminus B_r} \left|u_{n_i}\right|^s \le \varepsilon \quad \text{for any } r \ge r_\varepsilon, \tag{10}$$

*where $B_r := \{x \in \mathbb{R}^N : |x| \le r\}$.*

*Proof.* From $u_n \to u$ in $L^s_{\text{loc}}(\mathbb{R}^N)$, we have

$$\int_{B_i} |u_n|^s \longrightarrow \int_{B_i} |u|^s \quad \text{as } n \longrightarrow \infty. \tag{11}$$

Thus, there exists $\tilde{n}_i \in \mathbb{N}$ such that

$$\int_{B_i} \left(|u_n|^s - |u|^s\right) < \frac{1}{i}, \quad \forall n = \tilde{n}_i + j,\ j = 1, 2, \ldots. \tag{12}$$

In particular, for $n_i = \tilde{n}_i + i$, we have

$$\int_{B_i} \left(\left|u_{n_i}\right|^s - |u|^s\right) < \frac{1}{i}. \tag{13}$$

Note that there exists $r_\varepsilon > 0$ satisfying

$$\int_{\mathbb{R}^N \setminus B_r} \left(|u|^s\right) < \varepsilon \quad \forall r \geq r_\varepsilon. \tag{14}$$

Then

$$\begin{aligned}
\int_{B_i \setminus B_r} \left|u_{n_i}\right|^s &= \int_{B_i \setminus B_r} \left(\left|u_{n_i}\right|^s - |u|^s\right) + \int_{B_i \setminus B_r} |u|^s \\
&= \int_{B_i} \left(\left|u_{n_i}\right|^s - |u|^s\right) + \int_{B_r} \left(|u|^s - \left|u_{n_i}\right|^s\right) \\
&\quad + \int_{B_i \setminus B_r} |u|^s \\
&\leq \frac{1}{i} + \int_{\mathbb{R}^N \setminus B_r} |u|^s + \int_{B_r} \left(|u|^s - \left|u_{n_i}\right|^s\right) \\
&\leq \varepsilon, \quad \text{as } i \longrightarrow \infty.
\end{aligned} \tag{15}$$

This completes the proof of Lemma 4. □

Let $\eta \in C^\infty(\mathbb{R}^+)$ be a smooth function satisfying $0 \leq \eta(t) \leq 1$, $\eta(t) = 1$ if $t \leq 1$ and $\eta(t) = 0$ if $t \geq 2$. Define $\tilde{u}_i(x) = \eta(2|x|/i)u(x)$. It is clear that

$$\left\|u - \tilde{u}_i\right\|_{E_\lambda} \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{16}$$

**Lemma 5.** *One has*

$$\lim_{i \to \infty} \sup \left| \int_{\mathbb{R}^N} \left(f\left(x, u_{n_i}\right) - f\left(x, u_{n_i} - \tilde{u}_i\right) - f\left(x, \tilde{u}_i\right)\right) \varphi \right| = 0 \tag{17}$$

*uniformly in* $\varphi \in E_\lambda$ *with* $\|\varphi\|_{E_\lambda} \leq 1$.

*Proof.* From (16) and the local compactness of Sobolev embedding, for any $r \geq 0$, we have

$$\lim_{i \to \infty} \sup \left| \int_{B_r} \left(f\left(x, u_{n_i}\right) - f\left(x, u_{n_i} - \tilde{u}_i\right) - f\left(x, \tilde{u}_i\right)\right) \varphi \right| = 0 \tag{18}$$

uniformly in $\|\varphi\|_{E_\lambda} \leq 1$. For any $\varepsilon > 0$, it follows from (14) that

$$\lim_{i \to \infty} \sup \int_{B_i \setminus B_r} \left|\tilde{u}_i\right|^s \leq \int_{\mathbb{R}^N \setminus B_r} |u|^s \leq \varepsilon \tag{19}$$

for all $r \geq r_\varepsilon$. By Lemma 4 and $(H_3)$-$(H_4)$, we obtain

$$\begin{aligned}
&\lim_{i \to \infty} \sup \left| \int_{\mathbb{R}^N} \left(f\left(x, u_{n_i}\right) - f\left(x, u_{n_i} - \tilde{u}_i\right) - f\left(x, \tilde{u}_i\right)\right) \varphi \right| \\
&\quad = \lim_{i \to \infty} \sup \left| \int_{B_i \setminus B_r} \left(f\left(x, u_{n_i}\right) - f\left(x, u_{n_i} - \tilde{u}_i\right) \right.\right. \\
&\qquad \left.\left. - f\left(x, \tilde{u}_i\right)\right) \varphi \right| \\
&\quad \leq a_1 \lim_{i \to \infty} \sup \int_{B_i \setminus B_r} \left(\left|u_{n_i}\right|^{p-1} + \left|\tilde{u}_i\right|^{p-1}\right) |\varphi| \\
&\qquad + a_2 \lim_{i \to \infty} \sup \int_{B_i \setminus B_r} \left(\left|u_{n_i}\right|^{q-1} + \left|\tilde{u}_i\right|^{q-1}\right) |\varphi| \\
&\quad \leq a_1 \lim_{i \to \infty} \sup \left(\left\|u_{n_i}\right\|^{p-1}_{L_p(B_i \setminus B_r)} \right. \\
&\qquad \left. + \left\|\tilde{u}_i\right\|^{p-1}_{L_p(B_i \setminus B_r)}\right) \|\varphi\|_{L_p(B_i \setminus B_r)} \\
&\qquad + a_2 \lim_{i \to \infty} \sup \left(\left\|u_{n_i}\right\|^{q-1}_{L_q(B_i \setminus B_r)} + \left\|\tilde{u}_i\right\|^{q-1}_{L_q(B_i \setminus B_r)}\right) \\
&\qquad \times \|\varphi\|_{L_q(B_i \setminus B_r)} \\
&\quad \leq a_3 \varepsilon^{(p-1)/p} + a_4 \varepsilon^{(q-1)/q}.
\end{aligned} \tag{20}$$

This shows that the desired conclusion holds. □

**Lemma 6.** *One has along a subsequence*

$$\begin{gathered}
I_\lambda\left(u_n - \tilde{u}_n\right) \longrightarrow c - I_\lambda(u), \\
I'_\lambda\left(u_n - \tilde{u}_n\right) \longrightarrow 0 \quad \textit{in } E^{-1}_\lambda \ (\textit{the dual space of } E_\lambda).
\end{gathered} \tag{21}$$

*Proof.* By Lemma 2.1 of [23] and the arguments of [24], we have

$$\begin{aligned}
&I_\lambda\left(u_n - \tilde{u}_n\right) \\
&\quad = \frac{1}{p} \int_{\mathbb{R}^N} \left(\left|\nabla u_n - \nabla \tilde{u}_n\right|^p + \lambda V(x) \left|u_n - \tilde{u}_n\right|^p\right) \\
&\qquad - \frac{\lambda}{p^*} \int_{\mathbb{R}^N} K(x) \left|u_n - \tilde{u}_n\right|^{p^*} - \lambda \int_{\mathbb{R}^N} F\left(x, u_n - \tilde{u}_n\right) \\
&\quad = I_\lambda\left(u_n\right) - I_\lambda\left(\tilde{u}_n\right) \\
&\qquad + \frac{\lambda}{p^*} \int_{\mathbb{R}^N} K(x) \left(\left|u_n\right|^{p^*} - \left|u_n - \tilde{u}_n\right|^{p^*} - \left|\tilde{u}_n\right|^{p^*}\right) \\
&\qquad + \lambda \int_{\mathbb{R}^N} \left(F\left(x, u_n\right) - F\left(x, u_n - \tilde{u}_n\right) - F\left(x, \tilde{u}_n\right)\right) \\
&\qquad + o(1).
\end{aligned} \tag{22}$$

By (16) and the similar idea of proving the Brézis-Lieb Lemma [3], we easily get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left(|u_n|^{p^*}-|u_n-\tilde{u}_n|^{p^*}-|\tilde{u}_n|^{p^*}\right)=0,$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(F(x,u_n)-F(x,u_n-\tilde{u}_n)-F(x,\tilde{u}_n)\right)=0. \tag{23}$$

Together with the fact $I_\lambda(u_n)\to c$ and $I_\lambda(\tilde{u}_n)\to I_\lambda(u)$, one has

$$I_\lambda(u_n-\tilde{u}_n)\longrightarrow c-I_\lambda(u). \tag{24}$$

Next, we will check the fact $I'_\lambda(u_n-\tilde{u}_n)\to 0$ in $E_\lambda^{-1}$. For any $\varphi\in E_\lambda$, we have

$$\begin{aligned}
&I'_\lambda(u_n-\tilde{u}_n)\varphi\\
&\quad=I'_\lambda(u_n)\varphi-I'_\lambda(\tilde{u}_n)\varphi\\
&\qquad+\lambda\int_{\mathbb{R}^N}K(x)\left(|u_n|^{p^*-2}u_n-|u_n-\tilde{u}_n|^{p^*-2}(u_n-\tilde{u}_n)\right.\\
&\qquad\qquad\left.-|\tilde{u}_n|^{p^*-2}\tilde{u}_n\right)\varphi\\
&\qquad+\lambda\int_{\mathbb{R}^N}\left(f(x,u_n)-f(x,u_n-\tilde{u}_n)-f(x,\tilde{u}_n)\right)\varphi\\
&\qquad\qquad+o(1).
\end{aligned} \tag{25}$$

By the standard argument, it follows that

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)&\left(|u_n|^{p^*-2}u_n-|u_n-\tilde{u}_n|^{p^*-2}(u_n-\tilde{u}_n)\right.\\
&\left.-|\tilde{u}_n|^{p^*-2}\tilde{u}_n\right)\varphi=0
\end{aligned} \tag{26}$$

uniformly in $\|\varphi\|_{E_\lambda}\le 1$. Together with Lemma 5, we get the desired conclusion. □

Set $u_n^1=u_n-\tilde{u}_n$; then, $u_n-u=u_n^1+(\tilde{u}_n-u)$. From (16), it shows that $u_n\to u$ in $E_\lambda$ if and only if $u_n^1\to 0$ in $E_\lambda$.

Furthermore, we have

$$\begin{aligned}
&I_\lambda(u_n^1)-\frac{1}{p}I'_\lambda(u_n^1)u_n^1\\
&\quad=\left(\frac{1}{p}-\frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N}K(x)|u_n^1|^{p^*}\\
&\qquad+\lambda\int_{\mathbb{R}^N}\left(\frac{1}{p}u_n^1f(x,u_n^1)-F(x,u_n^1)\right)\\
&\quad\ge\frac{\lambda}{N}\int_{\mathbb{R}^N}K(x)|u_n^1|^{p^*}\\
&\quad\ge\frac{\lambda}{N}K_{\min}\|u_n^1\|_{p^*}^{p^*},
\end{aligned} \tag{27}$$

where $K_{\min}=\inf_{x\in\mathbb{R}^N}K(x)>0$.

By the facts that $I_\lambda(u_n^1)\to c-I_\lambda(u)$ and $I'_\lambda(u_n^1)\to 0$ in $E_\lambda^{-1}$, one has

$$\|u_n^1\|_{p^*}^{p^*}\le\frac{N(c-I_\lambda(u))}{\lambda K_{\min}}+o(1). \tag{28}$$

Let $V_b(x):=\max\{V(x),b\}$, where $b$ is the positive constant in the assumption $(H_1)$. Since the set $\nu_b$ has finite measure and $u_n^1\to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N}V(x)|u_n^1|^p=\int_{\mathbb{R}^N}V_b(x)|u_n^1|^p+o(1). \tag{29}$$

From $(H_2)$–$(H_5)$ and Young inequality, there exists $C_b>0$ such that

$$\int_{\mathbb{R}^N}\left(K(x)|u_n^1|^{p^*}+u_n^1f(x,u_n^1)\right)\le b\|u_n^1\|_p^p+C_b\|u_n^1\|_{p^*}^{p^*}. \tag{30}$$

Next, we consider the energy level of the functional $I_\lambda$ below which the $(\mathrm{PS})_c$ condition held.

**Lemma 7.** *Assume that the assumptions of Theorem 2 are satisfied. There exists $\alpha_0>0$ (independent of $\lambda$) such that, for any $(PS)_c$ sequence $\{u_n\}\subset E_\lambda$ for $I_\lambda$ with $u_n\rightharpoonup u$, either $u_n\to u$ in $E_\lambda$ or $c-I_\lambda(u)\ge\alpha_0\lambda^{1-(N/p)}$.*

*Proof.* Assume that $u_n\nrightarrow u$; then,

$$\liminf_{n\to\infty}\|u_n^1\|_{E_\lambda}>0,$$
$$c-I_\lambda(u)>0. \tag{31}$$

By the Sobolev inequality, (29), and (30), we get

$$\begin{aligned}
&S\|u_n^1\|_{p^*}^p\\
&\quad\le\int_{\mathbb{R}^N}|\nabla u_n^1|^p\\
&\quad=\int_{\mathbb{R}^N}\left(|\nabla u_n^1|^p+\lambda V(x)|u_n^1|^p\right)-\lambda\int_{\mathbb{R}^N}V(x)|u_n^1|^p\\
&\quad=\lambda\int_{\mathbb{R}^N}K(x)|u_n^1|^{p^*}+u_n^1f(x,u_n^1)-\lambda\\
&\qquad\times\int_{\mathbb{R}^N}V_b(x)|u_n^1|^p+o(1)\\
&\quad\le\lambda b\|u_n^1\|_p^p+\lambda C_b\|u_n^1\|_{p^*}^{p^*}-\lambda b\|u_n^1\|_p^p+o(1)\\
&\quad=\lambda C_b\|u_n^1\|_{p^*}^{p^*}+o(1),
\end{aligned} \tag{32}$$

where $S$ is the best Sobolev constant of the immersion

$$S\|u\|_{p^*}^p\le\int_{\mathbb{R}^N}|\nabla u|^p\quad\forall u\in W^{1,p}(\mathbb{R}^N). \tag{33}$$

This gives

$$\begin{aligned} S &\le \lambda C_b \left\| u_n^1 \right\|_{p^*}^{p^*-p} + o(1) \\ &\le \lambda C_b \left( \frac{N(c - I_\lambda(u))}{\lambda K_{\min}} \right)^{p/N} + o(1) \\ &= \lambda^{1-p/N} C_b \left( \frac{N}{K_{\min}} \right)^{p/N} (c - I_\lambda(u))^{p/N} + o(1). \end{aligned} \tag{34}$$

Set $\alpha_0 = S^{N/p} C_b^{-N/p} N^{-1} K_{\min}$; then,

$$\alpha_0 \lambda^{1-N/p} \le c - I_\lambda(u) + o(1). \tag{35}$$

This proof is completed. □

From Lemma 7, we will show that $I_\lambda$ satisfies the following local $(\mathrm{PS})_c$ condition.

**Lemma 8.** *Assume that $(H_1)$–$(H_5)$ is satisfied. There exists a constant $\alpha_0 > 0$ (independent of $\lambda$) such that, if a $(PS)_c$ sequence $\{u_n\} \subset E_\lambda$ for $I_\lambda$ satisfies $c \le \alpha_0 \lambda^{1-N/p}$, the sequence $\{u_n\}$ has a strongly convergent subsequence in $E_\lambda$.*

*Proof.* By Lemma 7, we easily obtain the required conclusion. □

Now, we consider $\lambda \ge 1$. The following standard arguments show that the energy functional $I_\lambda$ possesses the mountain-pass structure.

**Lemma 9.** *Under the assumptions of Theorem 2, there exist $\alpha_\lambda, \rho_\lambda > 0$ such that*

$$\begin{gathered} I_\lambda(u) > 0 \quad \text{if } 0 < \|u\|_{E_\lambda} < \rho_\lambda, \\ I_\lambda(u) \ge \alpha_\lambda \quad \text{if } \|u\|_{E_\lambda} = \rho_\lambda. \end{gathered} \tag{36}$$

*Proof.* By (30) and $(H_5)$, for any $\delta > 0$, there is $C_\delta > 0$ such that

$$\int_{\mathbb{R}^N} G(x,u) \le \delta \|u\|_p^p + C_\delta \|u\|_{p^*}^{p^*}. \tag{37}$$

Thus

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p} \|u\|_{E_\lambda}^p - \lambda \int_{\mathbb{R}^N} G(x,u) \\ &\ge \frac{1}{p} \|u\|_{E_\lambda}^p - \lambda \delta \|u\|_p^p - \lambda C_\delta \|u\|_{p^*}^{p^*}. \end{aligned} \tag{38}$$

Observe that $\|u\|_p^p \le a_5 \|u\|_{E_\lambda}^p$. Choosing $\delta \le (2p\lambda a_5)^{-1}$,

$$I_\lambda(u) \ge \frac{1}{2p} \|u\|_{E_\lambda}^p - \lambda C_\delta \|u\|_{p^*}^{p^*}. \tag{39}$$

The fact $p^* > p$ implies the desired conclusion. □

**Lemma 10.** *For any finite dimensional subspace $F \subset E_\lambda$, we have*

$$I_\lambda(u) \longrightarrow -\infty, \quad u \in E_\lambda \text{ as } \|u\|_{E_\lambda} \longrightarrow \infty. \tag{40}$$

*Proof.* By the assumption $(H_5)$, one has

$$I_\lambda(u) \le \frac{1}{p} \|u\|_\alpha^p - \lambda b_0 \|u\|_\alpha^\alpha \quad \forall u \in E_\lambda. \tag{41}$$

Since all norms in a finite-dimensional space are equivalent and $\alpha > p$, this implies the desired conclusion. □

Lemma 8 shows that $I_\lambda$ satisfies $(\mathrm{PS})_{c_\lambda}$ condition for $\lambda$ large enough and $c_\lambda$ small sufficiently. In the following, we will find special finite-dimensional subspaces by which we establish sufficiently small minimax levels.

Define the functional

$$\Phi_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left( |\nabla u|^p + \lambda V(x) |u|^p \right) - \lambda b_0 \int_{\mathbb{R}^N} |u|^\alpha. \tag{42}$$

It is apparent that $\Phi_\lambda \in C^1(E_\lambda)$ and $I_\lambda(u) \le \Phi_\lambda(u)$ for all $u \in E_\lambda$.

Note that

$$\inf \left\{ \int_{\mathbb{R}^N} |\nabla \phi|^p : \phi \in C_0^\infty \left( \mathbb{R}^N, \mathbb{R} \right), \|\phi\|_{L^\alpha(\mathbb{R}^N)} = 1 \right\} = 0. \tag{43}$$

For any $\delta > 0$, there is $\phi_\delta \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ with $\|\phi_\delta\|_{L^\alpha(\mathbb{R}^N)} = 1$ and $\operatorname{supp} \phi_\delta \subset B_{r_\delta}(0)$ such that $\|\nabla \phi_\delta\|_p^p < \delta$. Let $e_\lambda(x) = \phi_\delta(\sqrt[p]{\lambda} x)$, then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-1/p} r_\delta}(0)$. For any $t \ge 0$, we have

$$\begin{aligned} \Phi_\lambda(t e_\lambda) &= \frac{t^p}{p} \|e_\lambda\|_{E_\lambda}^p - b_0 \lambda t^\alpha \int_{\mathbb{R}^N} \left| \phi_\delta \left( \sqrt[p]{\lambda} x \right) \right|^\alpha \\ &= \lambda^{1-N/p} J_\lambda(t \phi_\delta), \end{aligned} \tag{44}$$

where

$$J_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} \left( |\nabla u|^p + V\left( \lambda^{-(1/p)} x \right) |u|^p \right) - b_0 \int_{\mathbb{R}^N} |u|^\alpha. \tag{45}$$

By direct computation, we easily get

$$\begin{aligned} &\max_{t \ge 0} J_\lambda(t\phi_\delta) \\ &\quad \le \frac{\alpha - p}{p\alpha (\alpha b_0)^{p/(\alpha-p)}} \left( \int_{\mathbb{R}^N} \left( |\nabla \phi_\delta|^p \right.\right. \\ &\qquad \left.\left. + V\left( \lambda^{-1/p} x \right) |\phi_\delta|^p \right) \right)^{\alpha/(\alpha-p)}. \end{aligned} \tag{46}$$

In connection with $V(0) = 0$ and $\|\nabla \phi_\delta\|_p^p < \delta$, it shows that there exists $\Lambda_\delta > 0$ such that for all $\lambda \ge \Lambda_\delta$, we have

$$\max_{t \ge 0} I_\lambda(t\phi_\delta) \le \left( \frac{\alpha - p}{p\alpha (\alpha b_0)^{p/(\alpha-p)}} (2\delta)^{\alpha/(\alpha-p)} \right) \lambda^{1-N/p}. \tag{47}$$

It follows from (47) that

**Lemma 11.** *Assume that $(H_1)$–$(H_5)$ is satisfied. For any $\sigma > 0$, there is $\Lambda_\sigma > 0$ such that $\lambda \geq \Lambda_\sigma$; there exists $\bar{e}_\lambda \in E_\lambda$ with $\|\bar{e}_\lambda\|_{E_\lambda} > \rho_\lambda$; we have $I_\lambda(\bar{e}_\lambda) \leq 0$ and*

$$\max_{t\geq 0} I_\lambda (t\bar{e}_\lambda) \leq \sigma\lambda^{1-N/p}, \tag{48}$$

*where $\rho_\lambda$ is defined in Lemma 9.*

*Proof.* This proof is similar to the one of Lemma 4.3 in [8], so we omit it. □

## 4. Proof of Theorem 2

In the following, we will give the proof of Theorem 2.

*Proof.* By Lemma 11, for any $\sigma > 0$ with $0 < \sigma < \alpha_0$, there is $\Lambda_\sigma > 0$ such that for $\lambda \geq \Lambda_\sigma$, we obtain

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda (\gamma (t)) \leq \sigma\lambda^{1-N/p}, \tag{49}$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0,\ \gamma(1) = \bar{e}_\lambda\}$.

It follows from Lemma 8 that $I_\lambda$ satisfies $(PS)_{c_\lambda}$ condition. Hence, by the mountain-pass theorem, there exists $u_\lambda \in E_\lambda$ which satisfies $I_\lambda(u_\lambda) = c_\lambda$ and $I'_\lambda(u_\lambda) = 0$. Actually, $u_\lambda$ is a weak solution of (4). Similar to the argument in [8], we also get that $u_\lambda$ is a positive least energy solution.

In the end, we show that the solution $u_\lambda$ satisfies the estimate (5). We easily get

$$\begin{aligned} I_\lambda (u_\lambda) &= I_\lambda (u_\lambda) - \frac{1}{\mu} I'_\lambda (u_\lambda)(u_\lambda) \\ &= \left(\frac{1}{p} - \frac{1}{\mu}\right) \|u_\lambda\|^p_{E_\lambda} + \left(\frac{1}{\mu} - \frac{1}{p^*}\right) \lambda \int_{\mathbb{R}^N} K(x) |u_\lambda|^{p^*} \\ &\quad + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\mu} u_\lambda f(x, u_\lambda) - F(x, u_\lambda)\right) \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) \|u_\lambda\|^p_{E_\lambda}. \end{aligned} \tag{50}$$

Note that $I_\lambda(u_\lambda) = c_\lambda$ and $c_\lambda \leq \sigma\lambda^{1-N/p}$ and it implies the required conclusion. The proof is complete. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

The authors would like to appreciate the referees for their precious comments and suggestions about the original manuscript. This research was supported by the Fundamental Research Funds for the Central Universities (2013XK03) and the National Training Programs of Innovation and Entrepreneurship for Undergraduates (201310290049).

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