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Research Article

Conjugacy of a Discrete Semidynamical System in a Neighbourhood of the Nontrivial Invariant Manifold

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The conjugacy of a discrete semidynamical system and its partially decoupled discrete semidynamical system in a Banach space is proved in a neighbourhood of the nontrivial invariant manifold.

1. Introduction

The conjugacy for noninvertible mappings in a Banach space was considered by Aulbach and Garay [1–3]. For noninvertible mappings in a complete metric space it was extended and generalized by Reinfelds [4–9]. In the present paper we consider the case when the linear part of the noninvertible mapping depends on the behaviour of variables in a neighbourhood of the nontrivial invariant manifold.

2. Invariant Manifold

Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces, $\mathbf{B}(a) = \{r \in \mathbf{F} \mid |r| \le a\}$, and $a > 0$. Consider the following mapping $S : \mathbf{E} \times \mathbf{B}(a) \to \mathbf{E} \times \mathbf{B}(a)$ defined by

$$\begin{aligned} x_1 &= g(x) + \Psi(x, r) = X(x, r), \\ r_1 &= A(x) r + \Phi(x, r) = R(x, r), \end{aligned} \tag{1}$$

where the derivative of the diffeomorphism $g : \mathbf{E} \to \mathbf{E}$ is uniformly continuous $\|Dg(x) - Dg(x')\| \le \omega(|x - x'|)$, mappings A, Ψ, and Φ are Lipschitzian,

$$\begin{gathered} \left\|A(x) - A(x')\right\| \le \gamma \left|x - x'\right|, \\ \left|\Psi(x, r) - \Psi(x', r')\right| \le \varepsilon\left(\left|x - x'\right| + \left|r - r'\right|\right), \\ \left|\Phi(x, r) - \Phi(x', r')\right| \le \varepsilon\left(\left|x - x'\right| + \left|r - r'\right|\right), \\ \sup_x \|A(x)\| + 2\varepsilon < 1, \\ \sup_x |\Phi(x, 0)| \le a\left(1 - \sup_x \|A(x)\| - \varepsilon\right). \end{gathered} \tag{2}$$

At the beginning we will modify the previous results on the existence of invariant manifolds of Neĭmark and Sacker [10, 11] for (1).

Lemma 1. *If*

$$\begin{gathered} \left(\sup_x \|A(x)\| + 4\varepsilon + \gamma \frac{\sup_x |\Phi(x, 0)|}{1 - \sup_x \|A(x)\| - \varepsilon}\right) \\ \times \sup_x \left\|Dg(x)^{-1}\right\| \le 1 \end{gathered} \tag{3}$$

then there exists a continuous mapping $u : \mathbf{E} \to \mathbf{F}$ satisfying the following properties:

(i) $u(g(x) + \Psi(x, u(x))) = A(x)u(x) + \Phi(x, u(x))$;
(ii) $|u(x) - u(x')| \le |x - x'|$;
(iii) $\|u\| \le \sup_x |\Phi(x, 0)|/(1 - \sup_x \|A(x)\| - \varepsilon)$.

Proof. The set of continuous mappings $u : \mathbf{E} \to \mathbf{F}$,

$$\mathscr{K} = \left\{ u \in \mathbf{C}(\mathbf{E}, \mathbf{F}) \mid \sup_x |u(x)| < +\infty \right\} \tag{4}$$

equipped with the norm

$$\|u\| = \sup_x |u(x)| \tag{5}$$

is a Banach space. The set

$$\mathscr{K}_1 = \left\{ u \in \mathscr{K} \mid \|u\| \le a, \left|u(x) - u(x')\right| \le \left|x - x'\right| \right\} \tag{6}$$

is a closed subset of the Banach space $\mathscr{K}$.

Let us consider the mapping $u \mapsto \mathscr{L}u$, $u \in \mathscr{K}_1$ defined by the equality

$$\mathscr{L}u(x_1) = A(x)u(x) + \Phi(x, u(x)), \tag{7}$$

where

$$x_1 = g(x) + \Psi(x, u(x)). \tag{8}$$

If $u \in \mathscr{K}_1$ then

$$\|\mathscr{L}u\| \le \left(\sup_x \|A(x)\| + \varepsilon\right)\|u\| + \sup_x |\Phi(x, 0)| \le a. \tag{9}$$

We have

$$\begin{aligned}
&\left|\mathscr{L}u(x_1) - \mathscr{L}u'(x_1')\right| \\
&\quad \le \left(\sup_x \|A(x)\| + \varepsilon\right)\left\|u - u'\right\| \\
&\qquad + \left(\sup_x \|A(x)\| + 2\varepsilon + \gamma\|u\|\right)\left|x - x'\right|, \\
&\left|x - x'\right| \\
&\quad \le \sup_x \left\|Dg(x)^{-1}\right\| \left|g(x) - g(x')\right| \\
&\quad = \sup_x \left\|Dg(x)^{-1}\right\| \left|x_1 - x_1' - \Psi(x, u(x)) + \Psi(x', u'(x'))\right| \\
&\quad \le \sup_x \left\|Dg(x)^{-1}\right\| \left(\left|x_1 - x_1'\right| + 2\varepsilon\left|x - x'\right| + \varepsilon\left\|u - u'\right\|\right).
\end{aligned} \tag{10}$$

It follows

$$\begin{aligned}
\left|x - x'\right| \le{}& \frac{\sup_x \left\|Dg(x)^{-1}\right\|}{1 - 2\varepsilon \sup_x \left\|Dg(x)^{-1}\right\|}\left|x_1 - x_1'\right| \\
&+ \frac{\varepsilon \sup_x \left\|Dg(x)^{-1}\right\|}{1 - 2\varepsilon \sup_x \left\|Dg(x)^{-1}\right\|}\left\|u - u'\right\|.
\end{aligned} \tag{11}$$

Then

$$\begin{aligned}
&\left|\mathscr{L}u(x_1) - \mathscr{L}u'(x_1')\right| \\
&\quad \le \Bigg(\sup_x \|A(x)\| + \varepsilon \\
&\qquad + \frac{\left(\sup_x \|A(x)\| + 2\varepsilon + \gamma\|u\|\right)\sup_x \left\|Dg(x)^{-1}\right\|\varepsilon}{1 - 2\varepsilon \sup_x \left\|Dg(x)^{-1}\right\|}\Bigg) \\
&\qquad \times \left\|u - u'\right\| \\
&\qquad + \frac{\left(\sup_x \|A(x)\| + 2\varepsilon + \gamma\|u\|\right)\sup_x \left\|Dg(x)^{-1}\right\|}{1 - 2\varepsilon \sup_x \left\|Dg(x)^{-1}\right\|} \\
&\qquad \times \left|x_1 - x_1'\right|.
\end{aligned} \tag{12}$$

Let us note that

$$\frac{\left(\sup_x \|A(x)\| + 2\varepsilon + \gamma\|u\|\right)\sup_x \left\|Dg(x)^{-1}\right\|}{1 - 2\varepsilon \sup_x \left\|Dg(x)^{-1}\right\|} \le 1. \tag{13}$$

We obtain

$$\begin{aligned}
\left|\mathscr{L}u(x_1) - \mathscr{L}u'(x_1')\right| \le{}& \left(\sup_x \|A(x)\| + 2\varepsilon\right)\left\|u - u'\right\| \\
&+ \left|x_1 - x_1'\right|.
\end{aligned} \tag{14}$$

We get that $\mathscr{L} \in \mathscr{K}_1$ is contraction and consequently we have the invariant manifold $r = u(x)$. □

3. Conjugacy of Noninvertible Mappings

Definition 2. Two mappings $S, T : \mathbf{X} \to \mathbf{X}$ are *conjugate*, if there exists a homeomorphism $H : \mathbf{X} \to \mathbf{X}$ such that

$$S \circ H(x) = H \circ T(x). \tag{15}$$

Definition 3. Two discrete semidynamical systems $S^n, T^n : \mathbf{X} \to \mathbf{X}$ $(n \in \mathbb{N})$ are *conjugate*, if there exists a homeomorphism $H : \mathbf{X} \to \mathbf{X}$ such that

$$S^n \circ H(x) = H \circ T^n(x). \tag{16}$$

It is easily verified that two discrete semidynamical systems S^n and T^n, generated by mappings S and T, are conjugate if and only if the mappings S and T are conjugate.

Suppose that mapping (1) has an invariant manifold given by Lipschitzian mapping $u : \mathbf{E} \to \mathbf{F}$ such that

$$\begin{gathered}
\sup_x \|u(x)\| \le \delta, \\
\left|u(x) - u(x')\right| \le \left|x - x'\right|.
\end{gathered} \tag{17}$$

Our aim is to find a simpler mapping conjugated with (1).

Theorem 4. *If* $\sup_x(\|(Dg(x))^{-1}\|\|A(x)\|) + 5\varepsilon \sup_x\|(Dg(x))^{-1}\| < 1$, *then there exists a continuous mapping* $v : \mathbf{E}\times\mathbf{B}(\delta) \to \mathbf{E}$ *which is Lipschitzian with respect to the second variable such that mappings* (1) *and*

$$x_1 = X(x, u(x)),\qquad r_1 = R(x+v(x,r), r) \tag{18}$$

are conjugated in a small neighbourhood of the invariant manifold $r = u(x)$.

We will seek the mapping establishing the conjugacy of (1) and (18) in the form

$$H(x,r) = (x + v(x,r), r). \tag{19}$$

We get the following functional equation:

$$X(x+v(x,r), r) = X(x,u(x)) + v(X(x,u(x)), R(x+v(x,r),r)) \tag{20}$$

or equivalently

$$v(x,r) = (Dg(x))^{-1}(Dg(x)v(x,r) - X(x+v(x,r),r) + X(x,u(x)) + v(X(x,u(x)), R(x+v(x,r),r))). \tag{21}$$

The proof of the theorem consists of four lemmas.

Lemma 5. *The functional equation* (20) *has a unique solution in* $\mathscr{M}_1$.

Proof. The set of continuous mappings $v : \mathbf{E}\times\mathbf{B}(\delta) \to \mathbf{E}$,

$$\mathscr{M} = \left\{v \in \mathbf{C}(\mathbf{E}\times\mathbf{B}(\delta), \mathbf{E}) \mid \sup_{x,r}\frac{|v(x,r)|}{|r-u(x)|} < +\infty\right\} \tag{22}$$

becomes a Banach space if we use the norm $\|v\| = \sup_{x,r}(|v(x,r)|/|r-u(x)|)$. The set

$$\mathscr{M}_1 = \left\{v\in\mathscr{M} \mid \|v\|\le 1, \left|v(x,r)-v(x,r')\right| \le \left|r-r'\right|\right\} \tag{23}$$

is a closed subset of the Banach space $\mathscr{M}$.

Let us consider the mapping $v \mapsto \mathscr{L}v$, $v\in\mathscr{M}_1$ defined by the equality

$$\begin{aligned}\mathscr{L}v(x,r) &= (Dg(x))^{-1}v(X(x,u(x)), R(x+v(x,r),r)) \\ &\quad + (Dg(x))^{-1}(Dg(x)v(x,r) - g(x+v(x,r)) + g(x) \\ &\quad - \Psi(x+v(x,r),r) + \Psi(x,u(x))).\end{aligned} \tag{24}$$

First we obtain

$$\begin{aligned}|Lv(x,r)| &\le \left\|(Dg(x))^{-1}\right\| |R(x+v(x,r),r) - u(X(x,u(x)))| \\ &\quad + \left\|(Dg(x))^{-1}\right\| \left|Dg(x)v(x,r) - g(x+v(x,r)) + g(x)\right| \\ &\quad + \left\|(Dg(x))^{-1}\right\| |\Psi(x+v(x,r),r) - \Psi(x,u(x))| \\ &\le \left\|(Dg(x))^{-1}\right\| \\ &\quad \times (\|A(x)\| + \gamma|r| + 2\varepsilon + \omega(|r-u(x)|) + 2\varepsilon) \\ &\quad \times |r-u(x)|.\end{aligned} \tag{25}$$

Here we used Hadamard lemma:

$$g(x') - g(x) = \int_0^1 Dg(x+\theta(x'-x))\,d\theta\,(x'-x). \tag{26}$$

Next we get

$$\begin{aligned}\left|Lv(x,r) - Lv(x,r')\right| &\le \left\|(Dg(x))^{-1}\right\| \left|R(x+v(x,r),r) - R(x+v(x,r'),r')\right| \\ &\quad + \left\|(Dg(x))^{-1}\right\| \Big|Dg(x)(v(x,r)-v(x,r')) \\ &\qquad - g(x+v(x,r)) + g(x+v(x,r'))\Big| \\ &\quad + \left\|(Dg(x))^{-1}\right\| \\ &\quad \times \left|\Psi(x+v(x,r),r) - \Psi(x+v(x,r'),r')\right| \\ &\le \left\|(Dg(x))^{-1}\right\| \left(\|A(x)\| + 2\gamma\max\{|r-u(x)|, |r'|\} + 2\varepsilon\right) \\ &\quad \times |r-r'| \\ &\quad + \left\|(Dg(x))^{-1}\right\| \\ &\quad \times \left(\omega\left(\max\{|r-u(x)|, |r'-u(x)|\}\right) + 2\varepsilon\right)|r-r'|.\end{aligned} \tag{27}$$

In addition,

$$\begin{aligned}\left|Lv(x,r) - Lv'(x,r)\right| &\le \left\|(Dg(x))^{-1}\right\| \left|R(x+v(x,r),r) - R(x+v'(x,r),r)\right| \\ &\quad + \left\|(Dg(x))^{-1}\right\| \Big|v(X(x,u(x)), R(x+v(x,r),r)) \\ &\qquad - v'(X(x,u(x)), R(x+v(x,r),r))\Big| \\ &\quad + \left\|(Dg(x))^{-1}\right\| \Big|Dg(x)(v(x,r) - v'(x,r)) \\ &\qquad - g(x+v(x,r)) + g(x+v'(x,r))\Big|\end{aligned}$$

$$
\begin{aligned}
&+\left\|(Dg(x))^{-1}\right\| \\
&\times\left|\Psi(x+v(x,r),r)-\Psi\left(x+v'(x,r),r\right)\right| \\
&\leq\left\|(Dg(x))^{-1}\right\|(\|A(x)\|+2\gamma|r|+3\varepsilon)\left\|v-v'\right\||r-u(x)| \\
&+\left\|(Dg(x))^{-1}\right\|(\omega(|r-u(x)|)+\varepsilon) \\
&\times\left\|v-v'\right\||r-u(x)|.
\end{aligned} \tag{28}
$$

We choose $\delta>0$, where $\max\{|r|,|r'|\}=\delta\leq a$, such that

$$
\begin{aligned}
&\sup_x\left(\left\|(Dg(x))^{-1}\right\|\|A(x)\|\right) \\
&\quad+(5\varepsilon+\omega(8\delta)+4\gamma\delta)\sup_x\left\|(Dg(x))^{-1}\right\|<1.
\end{aligned} \tag{29}
$$

Then $\|\mathscr{L}v\|\leq 1$, $|\mathscr{L}v(x,r)-\mathscr{L}v(x,r')|\leq|r-r'|$, the mapping $\mathscr{L}$ is a contraction, and consequently the functional equation (20) has unique solution in $\mathscr{M}_1$. □

Next we will prove that the mapping H is a homeomorphism in the small neighbourhood of the invariant manifold $r=u(x)$. Let us consider the functional equation

$$
\begin{aligned}
&X(x+v_1(x,r),u(x+v_1(x,r))) \\
&\quad=X(x,r)+v_1(X(x,r),R(x,r))
\end{aligned} \tag{30}
$$

or equivalently

$$
\begin{aligned}
&v_1(x,r) \\
&=(Dg(x))^{-1}\left(Dg(x)v_1(x,r)\right. \\
&\qquad-X(x+v_1(x,r),u(x+v_1(x,r))) \\
&\qquad\left.+X(x,r)+v_1(X(x,r),R(x,r))\right).
\end{aligned} \tag{31}
$$

Lemma 6. *The functional equation* (30) *has a unique solution in* $\mathscr{M}_2$.

Proof. The set

$$
\mathscr{M}_2=\{v\in\mathscr{M}\mid\|v\|\leq 1\} \tag{32}
$$

is a closed subset of the Banach space $\mathscr{M}$.

Let us consider the mapping $v_1\mapsto\mathscr{L}v_1$, $v_1\in\mathscr{M}_2$ defined by the equality

$$
\begin{aligned}
&\mathscr{L}v_1(x,r) \\
&=(Dg(x))^{-1}v_1(X(x,r),R(x,r))+(Dg(x))^{-1} \\
&\quad\times\left(Dg(x)v_1(x,r)-g(x+v_1(x,r))+g(x)\right. \\
&\qquad\left.-\Psi(x+v_1(x,r),u(x+v_1(x,r)))+\Psi(x,r)\right).
\end{aligned} \tag{33}
$$

We have

$$
\begin{aligned}
&|Lv_1(x,r)| \\
&\quad\leq\left\|(Dg(x))^{-1}\right\||R(x,r)-u(X(x,r))|+\left\|(Dg(x))^{-1}\right\| \\
&\quad\times|Dg(x)v_1(x,r)-g(x+v_1(x,r))+g(x)| \\
&\quad+\left\|(Dg(x))^{-1}\right\| \\
&\quad\times|\Psi(x+v_1(x,r),u(x+v_1(x,r)))-\Psi(x,r)| \\
&\quad\leq\left\|(Dg(x))^{-1}\right\|(\|A(x)\|+2\varepsilon+\omega(|r-u(x)|)+3\varepsilon) \\
&\quad\times|r-u(x)|.
\end{aligned} \tag{34}
$$

We obtain

$$
\begin{aligned}
&\left|Lv_1(x,r)-Lv_1'(x,r)\right| \\
&\leq\left\|(Dg(x))^{-1}\right\| \\
&\quad\times\left|v_1(X(x,r),R(x,r))-v_1'(X(x,r),R(x,r))\right| \\
&\quad+\left\|(Dg(x))^{-1}\right\| \\
&\quad\times\left|Dg(x)\left(v_1(x,r)-v_1'(x,r)\right)\right. \\
&\qquad\left.-g(x+v_1(x,r))+g\left(x+v_1'(x,r)\right)\right|+\left\|(Dg(x))^{-1}\right\| \\
&\quad\times\left|\Psi(x+v_1(x,r),u(x+v_1(x,r)))\right. \\
&\qquad\left.-\Psi\left(x+v_1'(x,r),u\left(x+v_1'(x,r)\right)\right)\right| \\
&\leq\left\|(Dg(x))^{-1}\right\|(\|A(x)\|+2\varepsilon+\omega(|r-u(x)|)+2\varepsilon) \\
&\quad\times\left\|v_1-v_1'\right\||r-u(x)|.
\end{aligned} \tag{35}
$$

We get that $\mathscr{L}$ is a contraction and consequently the functional equation (30) has a unique solution in $\mathscr{M}_2$. □

Consider the mapping G defined by equality $G(x,r)=(x+v_1(x,r),r)$.

Lemma 7. *One has* $G\circ H=id$.

Proof. Let us consider the functional equation

$$
\begin{aligned}
&X(x+v_2(x,r),u(x+v_2(x,r))) \\
&\quad=X(x,u(x))+v_2(X(x,u(x)),R(x+v(x,r),r))
\end{aligned} \tag{36}
$$

or equivalently

$$
\begin{aligned}
&v_2(x,r) \\
&=(Dg(x))^{-1} \\
&\quad\times\left(Dg(x)v_2(x,r)-X(x+v_2(x,r),u(x+v_2(x,r)))\right. \\
&\qquad\left.+X(x,u(x))+v_2(X(x,u(x)),R(x+v(x,r),r))\right).
\end{aligned} \tag{37}
$$

It is easily verified that the functional equation (36) has the trivial solution. Let us prove the uniqueness of the solution in $\mathscr{M}_3$, where

$$\mathscr{M}_3 = \{v_2 \in \mathscr{M} \mid \|v_2\| \le 3\} \tag{38}$$

is a closed subset of the Banach space $\mathscr{M}$. We get

$$\begin{aligned}
&|v_2(x,r)| \\
&\quad \le \left\|(Dg(x))^{-1}\right\| \|v_2\| \, |R(x+v(x,r),r) - u(X(x,u(x)))| \\
&\qquad \times \left\|(Dg(x))^{-1}\right\| \\
&\qquad \times |Dg(x)v_2(x,r) - g(x+v_2(x,r)) + g(x)| \\
&\qquad + \left\|(Dg(x))^{-1}\right\| \\
&\qquad \times |\Psi(x+v_2(x,r),u(x+v_2(x,r))) - \Psi(x,u(x))| \\
&\quad \le \left\|(Dg(x))^{-1}\right\| \\
&\qquad \times (\|A(x)\| + \gamma|r| + 2\varepsilon + \omega(3|r-u(x)|) + 2\varepsilon) \\
&\qquad \times \|v_2\| \, |r-u(x)|.
\end{aligned} \tag{39}$$

It follows that $v_2(x,r) \equiv 0$. The mapping w_1, where

$$w_1(x,r) = v(x,r) + v_1(x+v(x,r),r), \tag{40}$$

also satisfies the functional equation (36). Using the change of variables $x \mapsto x + v(x,r)$ in (30) we get

$$\begin{aligned}
&X(x+w_1(x,r),u(x+w_1(x,r))) \\
&\quad = X(x+v(x,r),r) \\
&\qquad + v_1(X(x+v(x,r),r),R(x+v(x,r),r)).
\end{aligned} \tag{41}$$

Using (20), we obtain

$$\begin{aligned}
&X(x+w_1(x,r),u(x+w_1(x,r))) \\
&\quad = X(x,u(x)) + v(X(x,u(x)),R(x+v(x,r),r)) \\
&\qquad + v_1(X(x,u(x)) + v(X(x,u(x)),R(x+v(x,r),r)), \\
&\qquad\qquad R(x+v(x,r),r)) \\
&\quad = X(x,u(x)) + w_1(X(x,u(x)),R(x+v(x,r),r)).
\end{aligned} \tag{42}$$

Let us note that

$$|w_1(x,r)| \le |r-u(x)| + |r-u(x+v(x,r))| \le 3|r-u(x)|. \tag{43}$$

Therefore $\|w_1\| \le 3$ and we have

$$v(x,r) + v_1(x+v(x,r),r) = 0. \tag{44}$$

We obtain that $G \circ H = \mathrm{id}$. □

Lemma 8. *One has $H \circ G = id$.*

Proof. The set of continuous mappings $v_3 : \mathbf{E}\times\mathbf{B}(\delta)\times\mathbf{B}(\delta) \to \mathbf{E}$,

$$\begin{aligned}
\mathscr{N} = \Bigg\{ & v_3 \in \mathbf{C}(\mathbf{E}\times\mathbf{B}(\delta)\times\mathbf{B}(\delta),\mathbf{E}) \mid \\
& \sup_{x,r,z} \frac{|v_3(x,r,z)|}{\max(|r-u(x)|,|z-r|)} < \infty \Bigg\}
\end{aligned} \tag{45}$$

becomes a Banach space if we use the norm $\|v_3\| = \sup_{x,r,z}(|v_3(x,r,z)|/\max(|r-u(x)|,|z-r|))$. The set

$$\begin{aligned}
\mathscr{N}_1 = \{ & v_3 \in \mathscr{N} \mid \|v_3\| \le 1, \\
& \left|v_3(x,r,z) - v_3(x,r,z')\right| \le |z-z'| \}
\end{aligned} \tag{46}$$

is a closed subset of the Banach space $\mathscr{N}$.

Let us consider the functional equation

$$\begin{aligned}
&X(x,r) + v_3(X(x,r),R(x,r),R(x+v_3(x,r,z),z)) \\
&\quad = X(x+v_3(x,r,z),z)
\end{aligned} \tag{47}$$

or equivalently

$$\begin{aligned}
&v_3(x,r,z) \\
&\quad = (Dg(x))^{-1} \\
&\qquad \times (Dg(x)v_3(x,r,z) - g(x+v_3(x,r,z)) + g(x) \\
&\qquad\quad + \Psi(x,r) - \Psi(x+v_3(x,r,z),z) \\
&\qquad\quad + v_3(X(x,r),R(x,r),R(x+v_3(x,r,z),z))).
\end{aligned} \tag{48}$$

Let us consider the mapping $v_3 \mapsto \mathscr{L}v_3$, $v_3 \in \mathscr{N}_1$ defined by the equality

$$\begin{aligned}
&\mathscr{L}v_3(x,r,z) \\
&\quad = (Dg(x))^{-1} \\
&\qquad \times (Dg(x)v_3(x,r,z) - g(x+v_3(x,r,z)) + g(x) \\
&\qquad\quad - \Psi(x+v_3(x,r,z),z) + \Psi(x,r) \\
&\qquad\quad + v_3(X(x,r),R(x,r),R(x+v_3(x,r,z),z))).
\end{aligned} \tag{49}$$

We obtain

$$\begin{aligned}
&|Lv_3(x,r,z)| \\
&\quad \le \left\|(Dg(x))^{-1}\right\| \max\{|R(x,r) - u(X(x,r))|, \\
&\qquad\qquad |R(x+v_3(x,r,z),z) - R(x,r)|\}
\end{aligned}$$

$$\begin{aligned}&+\left\|(Dg(x))^{-1}\right\|\\&\times\left|Dg(x)v_3(x,r,z)-g(x+v_3(x,r,z))+g(x)\right|\\&+\left\|(Dg(x))^{-1}\right\|\left|\Psi(x,r)-\Psi(x+v_3(x,r,z),z)\right|\\&\le\left\|(Dg(x))^{-1}\right\|(\|A(x)\|+\gamma|z|+2\varepsilon)\\&\times\max\{|r-u(x)|,|z-r|\}+\left\|(Dg(x))^{-1}\right\|\\&\times(\omega(\max\{|r-u(x)|,|z-r|\})+2\varepsilon)\\&\times\max\{|r-u(x)|,|z-r|\}.\end{aligned}\tag{50}$$

In addition,

$$\begin{aligned}&\left|Lv_3(x,r,z)-Lv_3(x,r,z')\right|\\&\le\left\|(Dg(x))^{-1}\right\|\\&\times\left|Dg(x)(v_3(x,r,z)-v_3(x,r,z'))\right.\\&\left.-g(x+v_3(x,r,z))+g(x+v_3(x,r,z'))\right|\\&+\left\|(Dg(x))^{-1}\right\|\\&\times\left|\Psi(x+v_3(x,r,z),z)-\Psi(x+v_3(x,r,z'),z')\right|\\&+\left\|(Dg(x))^{-1}\right\|\\&\times\left|R(x+v_3(x,r,z),z)-R(x+v_3(x,r,z'),z')\right|\\&\le\left\|(Dg(x))^{-1}\right\|\\&\times\left(\omega\left(\max\left\{|r-u(x)|,|z-r|,|z'-r|\right\}\right)+2\varepsilon\right)|z-z'|\\&+\left\|(Dg(x))^{-1}\right\|\\&\times\left(\|A\|+2\varepsilon+2\gamma\max\left\{|r-u(x)|,|z|,|z'-r|\right\}\right)\\&\times|z-z'|.\end{aligned}\tag{51}$$

Let $v_3\in\mathcal{N}_1$ and $v_3'\in\mathcal{N}_1\cup\mathcal{N}_2$ where

$$\begin{aligned}\mathcal{N}_2=&\left\{v_3'\in\mathcal{N}\mid\sup_{x,|r|\le\delta,|z|\le\delta}\left|v_3'(x,r,z)\right|\le 8\delta,\right.\\&\left.\left|v_3'(x,r,z)-v_3'(x,r,z')\right|\le|z-z'|\right\}.\end{aligned}\tag{52}$$

We have

$$\begin{aligned}&\left|Lv_3(x,r,z)-Lv_3'(x,r,z)\right|\\&\le\left\|(Dg(x))^{-1}\right\|\\&\times\left|Dg(x)(v_3(x,r,z)-v_3'(x,r,z))\right.\\&\left.-g(x+v_3(x,r,z))+g(x+v_3'(x,r,z))\right|\\&+\left\|(Dg(x))^{-1}\right\|\\&\times\left|\Psi(x+v_3(x,r,z),z)-\Psi(x+v_3'(x,r,z),z)\right|\\&+\left\|(Dg(x))^{-1}\right\|\\&\times\left|v_3(X(x,r),R(x,r),R(x+v_3(x,r,z),z))\right.\\&\left.-v_3'(X(x,r),R(x,r),R(x+v_3(x,r,z),z))\right|\\&+\left\|(Dg(x))^{-1}\right\|\\&\times\left|v_3'(X(x,r),R(x,r),R(x+v_3(x,r,z),z))\right.\\&\left.-v_3'(X(x,r),R(x,r),R(x+v_3'(x,r,z),z))\right|\\&\le\left\|(Dg(x))^{-1}\right\|(\omega(\max\{|r-u(x)|,|z-r|,8\delta\})+\varepsilon)\\&\times\left\|v_3-v_3'\right\|\max\{|r-u(x)|,|z-r|\}+\left\|(Dg(x))^{-1}\right\|\\&\times\max\left\{|R(x,r)-u(X(x,r))|,\right.\\&\left.\qquad\left|R(x+v_3'(x,r,z),z)-R(x,r)\right|\right\}\left\|v_3-v_3'\right\|\\&+\left\|(Dg(x))^{-1}\right\|\\&\times\left|R(x+v_3(x,r,z),z)-R(x+v_3'(x,r,z),z)\right|\\&\le\left\|(Dg(x))^{-1}\right\|(\omega(8\delta)+\varepsilon)\left\|v_3-v_3'\right\|\\&\times\max\{|r-u(x)|,|z-r|\}\\&+\left\|(Dg(x))^{-1}\right\|(\|A\|+3\varepsilon+\gamma|z|)\\&\times\max\{|r-u(x)|,|z-r|\}\left\|v_3-v_3'\right\|\\&=\left\|(Dg(x))^{-1}\right\|(\|A(x)\|+4\varepsilon+\omega(8\delta)+\gamma|z|)\\&\times\left\|v_3-v_3'\right\|\max\{|r-u(x)|,|z-r|\}.\end{aligned}\tag{53}$$

Then $\|\mathcal{L}v_3\|\le 1$, $|\mathcal{L}v_3(x,r,z)-\mathcal{L}v_3(x,r,z')|\le|z-z'|$, the mapping $\mathcal{L}$ is a contraction, and consequently the functional equation (47) has a unique solution in $\mathcal{N}_1$. Moreover, this solution is also unique in the closed subset $\mathcal{N}_2$. Let us note that

$$v_3(x,r,r)=0.\tag{54}$$

The mapping w_2, where

$$w_2(x,r,z)=v_1(x,r)+v(x+v_1(x,r),z),\tag{55}$$

satisfies (47). Using the change of variables $(x,r)\mapsto(x+v_1(x,r),z)$ in (20) we get

$$\begin{aligned}&X(x+w_2(x,r,z),z)\\&=X(x+v_1(x,r),0)\\&\quad+v(X(x+v_1(x,r),0),R(x+w_2(x,r,z),z)).\end{aligned}\tag{56}$$

Using (30) we obtain

$$\begin{aligned} X & \left(x+w_2(x,r,z),z\right) \\ &= X(x,r)+v_1(X(x,r),R(x,r)) \\ &\quad + v\left(X(x,r)+v_1(X(x,r),R(x,r)),\right. \\ &\qquad \left. R\left(x+w_2(x,r,z),z\right)\right) \\ &= X(x,r) \\ &\quad + w_2\left(X(x,r),R(x,r),R\left(x+w_2(x,r,z),z\right)\right). \end{aligned} \tag{57}$$

Let us note that

$$\left|w_2(x,r,z)-w_2\left(x,r,z'\right)\right| \le \left|z-z'\right|, \tag{58}$$

$$\begin{aligned} \left|w_2(x,r,z)\right| &\le |r-u(x)|+|z-r| \\ &\quad + \left|r-u\left(x+v_1(x,r)\right)\right| \\ &\le 4\max\{|r-u(x)|,|z-r|\}. \end{aligned} \tag{59}$$

Therefore $w_2 \in \mathcal{N}_2$ and we have

$$v_1(x,r)+v\left(x+v_1(x,r),r\right)=0. \tag{60}$$

It follows that $H \circ G = \text{id}$.

Finally we conclude that the mapping H is a homeomorphism establishing a conjugacy of the noninvertible mappings (1) and (18). □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] B. Aulbach and B. M. Garay, "Linearization and decoupling of dynamical and semidynamical systems," in *The Second Colloquium on Differential Equations*, D. Bainov and V. Covachov, Eds., pp. 15–27, World Scientific, Singapore, 1992.

[2] B. Aulbach and B. M. Garay, "Linearizing the expanding part of noninvertible mappings," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 44, no. 3, pp. 469–494, 1993.

[3] B. Aulbach and B. M. Garay, "Partial linearization for noninvertible mappings," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 45, no. 4, pp. 505–542, 1994.

[4] A. Reinfelds, "Partial decoupling for noninvertible mappings," *Differential Equations and Dynamical Systems*, vol. 2, no. 3, pp. 205–215, 1994.

[5] A. Reinfelds, "The reduction principle for discrete dynamical and semidynamical systems in metric spaces," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 45, no. 6, pp. 933–955, 1994.

[6] A. Reinfelds, "Partial decoupling of semidynamical system," in *Mathematics*, vol. 593, pp. 54–61, Latvia University, Riga, Latvia, 1994.

[7] A. Reinfelds, "The reduction of discrete dynamical and semidynamical systems in metric spaces," in *Six Lectures on Dynamical Systems*, B. Aulbach and F. Colonius, Eds., pp. 267–312, World Scientiffc, Singapore, 1996.

[8] A. Reinfelds, "Partial decoupling of semidynamical system in metric space," *Journal of the Technical University at Plovdiv*, vol. 5, pp. 33–40, 1997.

[9] A. Reinfelds, "Conjugacy of discrete semidynamical systems in the neighbourhood of invariant manifold," in *Differential and Difference Equations With Applications*, S. Pinelas, M. Chipot, and Z. Dosla, Eds., vol. 47 of *Springer Proceedings in Mathematics and Statistics*, pp. 571–578, Springer, New York, NY, USA, 2013.

[10] Ju. I. Neĭmark, "Existence and structural stability of invariant manifolds of pointwise mappings," *Izvestiya Vysshikh Uchebnykh Zavedeniĭ*, vol. 10, pp. 311–320, 1967 (Russian).

[11] R. J. Sacker, *On Invariant Surface and Bifurcation of Periodic Solution of Ordinary Differential Equations*, vol. 333, New York University, 1964.