Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 691429, 7 pages
http://dx.doi.org/10.1155/2014/691429

Research Article

Global Existence of Solutions for a Nonstrictly Hyperbolic System

De-yin Zheng,[1] Yun-guang Lu,[1] Guo-qiang Song,[2] and Xue-zhou Lu[3]

[1] *Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China*
[2] *College of Health Administration, Anhui Medical University, Hefei 230032, China*
[3] *Laboratoire Ondes and Milieux Complexes UMR 6294, CNRS-Universite du Havre, 76058 Le Havre, France*

Correspondence should be addressed to Guo-qiang Song; gq_song2003@aliyun.com

Received 16 November 2013; Accepted 14 February 2014; Published 31 March 2014

Academic Editor: Adem Kilicman

We obtain the global existence of weak solutions for the Cauchy problem of the nonhomogeneous, resonant system. First, by using the technique given in Tsuge (2006), we obtain the uniformly bounded L^∞ estimates $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le B(x)$ and $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le \beta$ when $a(x)$ is increasing (similarly, $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le B(x)$ and $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le \beta$ when $a(x)$ is decreasing) for the ε-viscosity and δ-flux approximation solutions of nonhomogeneous, resonant system without the restriction $z_0(x) \le 0$ or $w_0(x) \le 0$ as given in Klingenberg and Lu (1997), where z and w are Riemann invariants of nonhomogeneous, resonant system; $B(x) > 0$ is a uniformly bounded function of x depending only on the function $a(x)$ given in nonhomogeneous, resonant system, and β is the bound of $B(x)$. Second, we use the compensated compactness theory, Murat (1978) and Tartar (1979), to prove the convergence of the approximation solutions.

1. Introduction

The following system

$$\begin{gathered}(a\rho)_t + (a\rho u)_x = 0,\\ (a\rho u)_t + \left(a\left(\rho u^2 + P(\rho)\right)\right)_x = a_x P(\rho), \qquad (1)\\ a_t = 0\end{gathered}$$

describes the evolution of an isothermal fluid in a nozzle with discontinuous cross-sectional area $a(x) > 0$, where ρ and u stand for the density and the particle velocity of the fluid under consideration, respectively, and $P(\rho)$ denotes the pressure function (See [1]). The existence of global weak solutions for the Cauchy problem or the initial boundary value problem of system (1) has been studied in [1–3]. In [4–6], the authors showed the global existence of BV entropy solutions to a more general class of nonhomogeneous, resonant system by the generalized Glimm scheme.

The Riemann problem for a more general resonant system of $n + 1$ equations,

$$\begin{gathered}a_t = 0,\\ u_t + F(u, a)_x = 0,\end{gathered} \qquad (2)$$

was resolved in [7], where $u \in R^n$ and $F : R^n \times R \to R^n$ is a smooth function.

To study the existence of entropy solutions of the Cauchy problem (1), the main difficulty is to establish boundedness of solutions because the equations are not in conservative form and the Conley-Chuey-Smoller principle of invariant regions does not apply (See [1] for the details about the physical background of system (1) and its difficulty in analysis). For the polytropic gas and the adiabatic exponent $\gamma \in (1, 5/3]$, the definition of a finite energy solution (unbounded) is given and its existence is obtained by using the compensated compactness method in [1].

For smooth solution, system (1) is equivalent to the following conservation laws of three equations:

$$\begin{aligned} a_t &= 0, \\ (a\rho)_t + (a\rho u)_x &= 0, \\ u_t + \left(\frac{1}{2}u^2 + \int_0^\rho \frac{P'(s)}{s}ds\right)_x &= 0 \end{aligned} \tag{3}$$

or the system of two equations

$$\begin{aligned} \rho_t + (\rho u)_x &= A(x)\rho u, \\ u_t + \left(\frac{1}{2}u^2 + \int_0^\rho \frac{P'(s)}{s}ds\right)_x &= 0, \end{aligned} \tag{4}$$

where $A(x) = -(a'(x)/a(x))$. When a is a constant, system (3) or system (4) itself has many different physical backgrounds. For instance, it is a scaling limit system of Newtonian dynamics with long-range interaction for a continuous distribution of mass in R (cf. [8, 9]) and also a hydrodynamic limit for the Vlasov equation (cf. [10]). Its global weak solution was obtained by using the random choice method [2] in [11] and by using the compensated compactness theory in [12, 13].

By simple calculations, two eigenvalues of system (4) are

$$\lambda_1 = u - \sqrt{P'(\rho)}, \qquad \lambda_2 = u + \sqrt{P'(\rho)}, \tag{5}$$

with corresponding Riemann invariants

$$\begin{aligned} z(u,\rho) &= \int_c^\rho \frac{\sqrt{P'(s)}}{s}ds - u, \\ w(u,\rho) &= \int_c^\rho \frac{\sqrt{P'(s)}}{s}ds + u, \end{aligned} \tag{6}$$

where c is a constant.

The existence of weak solutions for the Cauchy problem (4) with bounded initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \qquad \rho_0(x) \ge 0 \tag{7}$$

was first studied in [12], where $a(x)$ is a smooth function and the technical condition $z(u_0(x), \rho_0(x)) \le 0$ or $w(u_0(x), \rho_0(x)) \le 0$ on the initial data is imposed for obtaining the *a-priori*, uniform L^∞ estimate of z or w.

Without the condition $z_0(x) \le 0$ or $w_0(x) \le 0$, the reasonable estimate, depending on the variable $x : z(x,t) \le Cx^{-(2(\gamma-1)/(\gamma+1))}$, $C > 0$ was first obtained in [2] for system (1) when $P(\rho) = \rho^\gamma$, $1 < \gamma < 5/3$ by using a modified Godunov scheme, and in [14] for general pressure function $P(\rho)$ and smooth function $a(x)$ by using the compensated compactness.

In this paper, using the vanishing viscosity method and the maximum principle coupled with the flux approximation proposed in [15] for the homogeneous system of isentropic gas dynamics, we extend the results in [2, 12] to the Cauchy problem (4)–(7) for any bounded initial data and for the function $a(x)$ satisfying the conditions $0 < c_1 \le a(x) \le c_2$, $a'(x) \in L^1(R)$.

We first construct the sequence of hyperbolic systems

$$\begin{aligned} \rho_t + (u(\rho-\delta))_x &= A^{\varepsilon_1}(x)u(\rho-\delta), \\ u_t + \left(\frac{1}{2}u^2 + P_1(\rho,\delta)\right)_x &= 0 \end{aligned} \tag{8}$$

to approximate system (4), where $\delta > 0$ denotes the flux approximation constant and the approximation pressure

$$P_1(\rho,\delta) = \int_\delta^\rho \frac{t-\delta}{t^2}P'(t)\,dt, \tag{9}$$

$A^{\varepsilon_1}(x) = -(a^{\varepsilon_1}(x)'/a^{\varepsilon_1}(x))$, and $a^{\varepsilon_1} = a(x) * G^{\varepsilon_1}$ is the smooth approximation of $a(x)$, G^{ε_1} being a mollifier. If $a(x)$ is a monotonic function, $0 < c_1 \le a(x) \le c_2$ as required in Theorem 2, and ε and δ converge to zero much faster than ε_1, then it is easy to prove that $A^{\varepsilon_1}(x)$ and a^{ε_1} satisfy

$$\begin{gathered} \lim_{\varepsilon_1\to 0} A^{\varepsilon_1}(x) = A(x), \quad a.e. \text{ on } R, \\ \lim_{\varepsilon_1,\delta\to 0} \delta A^{\varepsilon_1}(x) = 0, \quad a.e. \text{ on } R, \\ A^{\varepsilon_1}(x) \text{ is uniformly bounded in } L^1(R), \\ \lim_{\varepsilon_1,\varepsilon\to 0} \left(\varepsilon a^{\varepsilon_1}(x)', \varepsilon a^{\varepsilon_1}(x)''\right) = (0,0), \quad a.e. \text{ on } R. \end{gathered} \tag{10}$$

Second, we add the viscosity terms to the right-hand side of (8) to obtain the following parabolic system:

$$\begin{aligned} \rho_t + (u(\rho-\delta))_x &= A^{\varepsilon_1}(x)u(\rho-\delta) + \varepsilon\rho_{xx}, \\ u_t + \left(\frac{1}{2}u^2 + P_1(\rho,\delta)\right)_x &= \varepsilon u_{xx}, \end{aligned} \tag{11}$$

with initial data

$$\left(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)\right) = (\rho_0(x)+\delta, u_0(x)), \tag{12}$$

where $(\rho_0(x), u_0(x))$ are given in (7).

Lemma 1. *Let* $0 < c_1 \le a(x) \le c_2$ *or equivalently* $0 < c_1 \le a^{\varepsilon_1}(x) \le c_2$ *for two constants* c_1 *and* c_2*. If* $a(x)$ *is increasing or equivalently* $a^{\varepsilon_1}(x)' \ge 0$*, then we can choose a function* $B(x)$ *satisfying* $B'(x) \le 0$, $0 < \beta_0 \le B(x) \le \beta$*, and*

$$2A^{\varepsilon_1}(x)B(x) \le B'(x) \le A^{\varepsilon_1}(x)B(x), \tag{13}$$

where the positive constants β_0*, and* β *depend on* c_1*, and* c_2*, but are independent of* ε_1*.*

The proof of Lemma 1 is trivial.

By applying the maximum principle to the Cauchy problem (11)-(12), we first obtain the L^∞ estimates $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le B(x)$ and $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le \beta$ when $a(x)$ is increasing (similarly $w(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le B(x)$ and $z(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \le \beta$ when $a(x)$ is decreasing) for a suitable positive, bounded function $B(x)$ given in Lemma 1; then by using the compensated compactness theory and the already existed compact frameworks given in [12, 13], we give the following global existence theorem of weak solutions.

Theorem 2. *Let* $(\sqrt{P'(\rho)}/\rho)' \geq 0$ *and* $0 < c_1 \leq a(x) \leq c_2$ *for two positive constants* c_1 *and* c_2.

(A) Let $a(x)$ *be increasing and we choose* $B(x) \in C^2(R)$ *satisfying all conditions in Lemma 1 and, moreover,* $B''(x) \leq 0$ *or* $B''(x) = B_1 + B_2$, *where* $B_1 \leq 0$ *and* $|2cB_2| \leq |B(x)B'(x)|$ *for a sufficiently small constant* c. *Then the Riemann invariants* z *and* w *of system* (4) *with respect to the approximated solutions of the Cauchy problem* (11)-(12) *satisfy the estimate*

$$z\left(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}\right) = \int_{\delta}^{\rho^{\delta,\varepsilon}} \frac{\sqrt{P'(s)}}{s} ds - u^{\delta,\varepsilon} \leq B(x), \tag{14}$$

if $z(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) \leq B(x)$ *and*

$$w\left(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t)\right) = \int_{\delta}^{\rho^{\delta,\varepsilon}} \frac{\sqrt{P'(s)}}{s} ds + u^{\delta,\varepsilon} \leq \beta, \tag{15}$$

if $w(\rho^{\delta,\varepsilon}(x,0), u^{\delta,\varepsilon}(x,0)) \leq \beta$.

(B) For such function $A(x)$ *and the initial data satisfying the conditions in Part (A), if either* $P(\rho) = (1/\gamma)\rho^{\gamma}$, $\gamma > 3$, *or* $(\sqrt{P'(\rho)}/\rho)' \geq d$, *where* d *is a positive constant, then there exists a subsequence of* $(\rho^{\delta,\varepsilon}(x,t), u^{\delta,\varepsilon}(x,t))$, *which converges pointwise to a pair of bounded functions* $(\rho(x,t), u(x,t))$ *as* δ, *and* ε *tend to a zero, and the limit is a weak entropy solution of the Cauchy problem* (4)–(7).

Definition 3. For integrable function $a'(x)/a(x) \in L^1(R)$, a pair of bounded measurable functions $(\rho(x,t), u(x,t))$ is called a weak entropy solution of the Cauchy problem (4)–(7), if

$$\begin{gathered}
\int_0^{\infty}\int_{-\infty}^{\infty} \rho\phi_t + (\rho u)\phi_x - \frac{a'(x)}{a(x)}(\rho u)\phi \, dx\, dt \\
+ \int_{-\infty}^{\infty} \rho_0(x)\phi(x,0)\, dx = 0, \\
\int_0^{\infty}\int_{-\infty}^{\infty} u\phi_t + \left(\frac{1}{2}u^2 + \int_0^{\rho}\frac{P'(s)}{s}ds\right)\phi_x dx\, dt \\
+ \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\, dx = 0
\end{gathered} \tag{16}$$

hold for all test function $\phi \in C_0^1(R \times R^+)$ and

$$\begin{gathered}
\int_0^{\infty}\int_{-\infty}^{\infty} \eta(\rho,u)\phi_t + q(\rho,u)\phi_x \\
- \frac{a'(x)}{a(x)}\eta(\rho,u)_{\rho}\rho u\phi\, dx\, dt \geq 0
\end{gathered} \tag{17}$$

holds for any nonnegative test function $\phi \in C_0^{\infty}(R \times R^+ - \{t = 0\})$, where (η, q) is a pair of convex entropy-entropy flux of system (4).

We can easily construct many functions $A(x)$, and $B(x)$ satisfying the conditions in Theorem 2.

Example 4. Let

$$A(x) = -\frac{1}{1+x^2}, \tag{18}$$

and choose

$$B(x) = e^{-\int_{-\infty}^{x}(1/(1+s^2))ds}. \tag{19}$$

Then (13) is satisfied since $B'(x) = A(x)B(x)$ and $B(x)$ satisfies all the conditions in Theorem 2. In fact, we may choose $B_1 = 0$ and then $B_2 = B''(x)$ and

$$\begin{aligned}
\left|B''(x)\right| &= \left|B'(x)A(x) + B(x)A'(x)\right| \\
&= \left|B'(x)A(x) + \frac{A'(x)}{A(x)}B'(x)\right| = \left|\frac{1+2x}{1+x^2}B'(x)\right| \\
&\leq M\left|B'(x)\right|
\end{aligned} \tag{20}$$

for a positive constant M.

We are going to prove Theorem 2 in the next section.

2. Proof of Theorem 2

By simple calculations, two eigenvalues of system (8) are

$$\lambda_1^{\delta} = u - \frac{\rho-\delta}{\rho}\sqrt{P'(\rho)}, \qquad \lambda_2^{\delta} = u + \frac{\rho-\delta}{\rho}\sqrt{P'(\rho)}, \tag{21}$$

with corresponding right eigenvectors

$$r_1 = \left(1, u - \sqrt{P'(\rho)}\right)^T, \qquad r_2 = \left(1, u + \sqrt{P'(\rho)}\right)^T, \tag{22}$$

and Riemann invariants

$$\begin{aligned}
z(u,\rho) &= \int_{\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds - u, \\
w(u,\rho) &= \int_{\delta}^{\rho}\frac{\sqrt{P'(s)}}{s}ds + u
\end{aligned} \tag{23}$$

which are similar to the Riemann invariants of system (4) given by (6).

We multiply (11) by (w_{ρ}, w_u) and (z_{ρ}, z_u), respectively, to obtain

$$\begin{aligned}
w_t + \lambda_2^{\delta} w_x &= \varepsilon w_{xx} - \varepsilon\left(\frac{\sqrt{P'(\rho)}}{\rho}\right)'\rho_x^2 \\
&\quad + A^{\varepsilon_1}(x)(\rho-\delta)u\frac{\sqrt{P'(\rho)}}{\rho},
\end{aligned} \tag{24}$$

$$\begin{aligned}
z_t + \lambda_1^{\delta} z_x &= \varepsilon z_{xx} - \varepsilon\left(\frac{\sqrt{P'(\rho)}}{\rho}\right)'\rho_x^2 \\
&\quad + A^{\varepsilon_1}(x)(\rho-\delta)u\frac{\sqrt{P'(\rho)}}{\rho}.
\end{aligned} \tag{25}$$

Let $v = w - B(x)$. Then

$$\begin{aligned} v_t &+ \left(u + \frac{\rho-\delta}{\rho}\sqrt{P'(\rho)}\right)\left(v_x + B'(x)\right) \\ &= \varepsilon v_{xx} + \varepsilon B''(x) - \varepsilon\left(\frac{\sqrt{P'(\rho)}}{\rho}\right)' \rho_x^2 \\ &\quad + A^{\varepsilon_1}(x)\frac{\rho-\delta}{\rho}\sqrt{P'(\rho)} \\ &\quad \times\left(B(x) + v - \int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho\right), \end{aligned} \tag{26}$$

or

$$\begin{aligned} v_t &+ \left(u + \frac{\rho-\delta}{\rho}\sqrt{P'(\rho)}\right)v_x \\ &+ B'(x)\left(B(x) + v - \int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho\right) \\ &+ (\rho-\delta)B'(x)\frac{\sqrt{P'(\rho)}}{\rho} \\ &= \varepsilon v_{xx} + \varepsilon B''(x) - \varepsilon\left(\frac{\sqrt{P'(\rho)}}{\rho}\right)' \rho_x^2 \\ &+ A^{\varepsilon_1}(x)B(x)(\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\ &+ A^{\varepsilon_1}(x)(\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho}v \\ &- A^{\varepsilon_1}(x)(\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho, \end{aligned} \tag{27}$$

or

$$\begin{aligned} v_t &+ a(x,t)v_x + b(x,t)v \\ &+ \left(\frac{1}{2}B'(x)B(x) - \varepsilon B''(x)\right) \\ &- B'(x)\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho + \frac{1}{2}B(x)B'(x) \\ &+ \left[B'(x) - A^{\varepsilon_1}(x)B(x)\right](\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\ &+ A^{\varepsilon_1}(x)(\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \\ &= \varepsilon v_{xx} - \varepsilon\left(\frac{\sqrt{P'(\rho)}}{\rho}\right)' \rho_x^2, \end{aligned} \tag{28}$$

where $a(x,t) = u + ((\rho-\delta)/\rho)\sqrt{P'(\rho)}$ and $b(x,t) = B'(x) - A^{\varepsilon_1}(x)(\rho-\delta)(\sqrt{P'(\rho)}/\rho)$.

Using the first equation of (11), we have the *a-priori* estimate $\rho \geq \delta$. Since the conditions on $B''(x)$ in Theorem 2, the following two terms on the left-hand side of (28):

$$\frac{1}{2}B'(x)B(x) - \varepsilon B''(x) \geq 0. \tag{29}$$

Now, we consider the other terms

$$\begin{aligned} L = &-B'(x)\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho + \frac{1}{2}B(x)B'(x) \\ &+ \left[B'(x) - A^{\varepsilon_1}(x)B(x)\right](\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho} \\ &+ A^{\varepsilon_1}(x)(\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \end{aligned} \tag{30}$$

on the left-hand side of (28).

First, we have from $B'(x) \geq B(x)A^{\varepsilon_1}(x)$ that

$$\begin{aligned} L \geq &-B'(x)\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho + \frac{1}{2}B(x)B'(x) \\ &+ A^{\varepsilon_1}(x)(\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho. \end{aligned} \tag{31}$$

Second, we have from $(\sqrt{P'(\rho)}/\rho)' \geq 0$, $A^{\varepsilon_1}(x) \geq 0$ that

$$\begin{aligned} A^{\varepsilon_1}(x)(\rho-\delta)\frac{\sqrt{P'(\rho)}}{\rho}\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho \\ \geq A^{\varepsilon_1}(x)\left(\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho\right)^2 \end{aligned} \tag{32}$$

and so

$$\begin{aligned} L &\geq -B'(x)\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho + \frac{1}{2}B(x)B'(x) \\ &\quad + A^{\varepsilon_1}(x)\left(\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho\right)^2 \\ &= A^{\varepsilon_1}(x)\left(\int_\delta^\rho \frac{\sqrt{P'(\rho)}}{\rho}d\rho - \frac{B'(x)}{2A^{\varepsilon_1}(x)}\right)^2 \\ &\quad + \frac{1}{2}B(x)B'(x) - \frac{\left(B'(x)\right)^2}{4A^{\varepsilon_1}(x)} \geq 0, \end{aligned} \tag{33}$$

since $B'(x) \leq 2B(x)A(x)$.

Thus, (28) is reduced to the following inequality about v:

$$v_t + a(x,t)v_x + b(x,t)v \leq \varepsilon v_{xx} \tag{34}$$

and we can prove that $v \leq 0$ or $w \leq B(x)$ if applying for the maximum principle to (34).

To prove the estimate of z, we have from (25) that

$$z_t + a_1(x,t)z_x \leq \varepsilon z_{xx} + A(x)(\rho-\delta)u\frac{\sqrt{P'(\rho)}}{\rho}, \tag{35}$$

where $a_1(x,t) = u - (\rho-\delta)(\sqrt{P'(\rho)}/\rho)$.

Let

$$z = X + \beta + \frac{N\left(x^2 + Lhe^{3t}\right)}{L^2}, \tag{36}$$

where $\beta \geq B(x) > 0$ is the upper bound of $z_0(x)$ and N, and h are the bounds of $|z|$ and $|a_1(x,t)|$ obtained from the local solution. Then

$$\begin{aligned} X(x,0) &= z_0(x) - \beta - \frac{N\left(x^2 + Lh\right)}{L^2} < 0, \\ X(\pm L,t) &= z(\pm L,t) - \beta - \frac{N\left(L^2 + Lhe^{3t}\right)}{L^2} < 0. \end{aligned} \tag{37}$$

We have from (35)–(37) that

$$X(x,t) < 0, \quad \text{on } (-L,L)\times(0,T). \tag{38}$$

We argue by assuming that (38) is violated for X at a point (x,t) in $(-L,L)\times(0,T)$. Let $\bar{t}$ be the least upper bound of values of t at which $X < 0$. Then, by the continuity we see that $X = 0$ at some points $(\bar{x},\bar{t}) \in (-L,L)\times(0,T)$. So $X_t \geq 0$, $X_x = 0$, and $-X_{xx} \geq 0$ at $(\bar{x},\bar{t})$; that is,

$$X_t + a_1(x,t)X_x - \varepsilon X_{xx} \geq 0 \quad \text{at } (\bar{x},\bar{t}). \tag{39}$$

But from (35) and (36),

$$\begin{aligned} &X_t + a_1(x,t)X_x - \varepsilon X_{xx} \\ &\quad\leq A(x)(\rho-\delta)u\frac{\sqrt{P'(\rho)}}{\rho} \\ &\qquad - \left(\frac{3NhLe^{3t}}{L^2} + a_1(x,t)\frac{2Nx}{L^2} - \varepsilon\frac{2N}{L^2}\right). \end{aligned} \tag{40}$$

Since $X = 0$ on $(\bar{x},\bar{t})$; then

$$z = \beta + \frac{N\left(x^2 + Lhe^{3t}\right)}{L^2} > \beta \geq w \quad \text{at } (\bar{x},\bar{t}), \tag{41}$$

Thus, $u < 0$ at $(\bar{x},\bar{t})$ from the relation of w, and z given by (23). So the right-hand side of (40) is negative, which yields a conclusion contradicting (39). So (38) is proved. Therefore for any point (x_0,t_0) in $(-L,L)\times(0,T)$,

$$z(x_0,t_0) < \beta + \frac{N\left(x_0^2 + Lhe^{3t_0}\right)}{L^2}, \tag{42}$$

which yields the desired estimate

$$z(x,t) \leq \beta, \tag{43}$$

if we let $L \uparrow \infty$ in (42), and, hence, complete the proof of Part (A) in Theorem 2.

For the homogeneous case ($a(x) = 0$), the convergence of $(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \to (\rho, u)$ as δ, and ε tend to zero in the Part (B) was given in [13] when $P(\rho) = (1/\gamma)\rho^\gamma$, $\gamma > 3$, and given in [12] when $(\sqrt{P'(\rho)}/\rho)' \geq d > 0$ by using the compensated compactness theory [16, 17] coupled with some basic ideas of the kinetic formulation [18, 19].

Now, we are going to prove the convergence of $(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \to (\rho, u)$ as δ, and ε tend to zero for the inhomogeneous system (11).

Any entropy-entropy flux pair $(\eta(\rho,u), q(\rho,u))$ of the original hyperbolic system (4) satisfies the additional system. Consider

$$q_\rho = u\eta_\rho + \frac{P'(\rho)}{\rho}\eta_u, \qquad q_u = \rho\eta_\rho + u\eta_u. \tag{44}$$

Eliminating the q from (44), we have

$$\eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2}\eta_{uu}. \tag{45}$$

Similarly, any entropy-entropy flux pair $(\eta(\rho,u), q(\rho,u))$ of the approximated hyperbolic system (8) satisfies

$$q_\rho = u\eta_\rho + \frac{(\rho-\delta)P'(\rho)}{\rho^2}\eta_u, \qquad q_u = (\rho-\delta)\eta_\rho + u\eta_u. \tag{46}$$

By eliminating the q from (46), we have also the same entropy equation (45). Therefore, system (4) and system (8) have the same entropies.

For any entropy-entropy flux pair $(\eta_0(\rho,u), q_0(\rho,u))$ of system (4), by multiplying $(\eta_{0\rho}, \eta_{0u})$ to system (11), we have

$$\begin{aligned}\eta_{0t} + q_{0x} + \delta q_{1x} &= \varepsilon \eta_{0xx} - \varepsilon \left(\rho_x^{\varepsilon,\delta}, u_x^{\varepsilon,\delta}\right) \cdot \nabla^2 \eta_0 \left(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}\right) \\ &\quad \cdot \left(\rho_x^{\varepsilon,\delta}, u_x^{\varepsilon,\delta}\right)^T + \eta_{0\rho} A^{\varepsilon_1}(x) u(\rho - \delta),\end{aligned} \tag{47}$$

where $q_0 + \delta q_1$ is the entropy flux of the approximated system (8) corresponding to entropy η_0. Since $A^{\varepsilon_1}(x)$ is uniformly integrable, then the last term on the right-hand side of system (47) is compact in $W_{\text{loc}}^{-1,\alpha}(R \times R^+)$, for some $\alpha \in (1,2)$, by the Sobolev embedding theorems. It is obvious that the term δq_{1x} on the left-hand side is compact in $H_{\text{loc}}^{-1}(R \times R^+)$. Therefore, using the same techniques given in [12, 13] for the homogeneous system, we may prove that $\eta_{0t} + q_{0x}$ is compact in $H_{\text{loc}}^{-1}(R \times R^+)$ and so the convergence of $(\rho^{\delta,\varepsilon}, u^{\delta,\varepsilon}) \to (\rho, u)$ as δ, and ε tend to zero. Furthermore, the limit (ρ,u) satisfies (16).

If precisely using (10), we can prove that the limit (ρ,u) satisfies the following conservation form:

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty a(x)\rho\phi_t + (a(x)\rho u)\phi_x \\ + \int_{-\infty}^\infty a(x)\rho_0(x)\phi(x,0)\,dx = 0.\end{aligned} \tag{48}$$

In fact, we multiply the first equation in (11) by $a^{\varepsilon_1}(x)$ to obtain

$$\begin{aligned}\left(a^{\varepsilon_1}(x)\rho\right)_t + \left(a^{\varepsilon_1}(x)u(\rho-\delta)\right)_x &= -\delta a^{\varepsilon_1}(x)' + \varepsilon a^{\varepsilon_1}(x)\rho_{xx} \\ &= -\delta a^{\varepsilon_1}(x)' + \varepsilon\left(a^{\varepsilon_1}(x)\rho\right)_{xx} \\ &\quad - 2\varepsilon\left(a^{\varepsilon_1}(x)'\rho\right)_x + \varepsilon a^{\varepsilon_1}(x)''\rho,\end{aligned} \tag{49}$$

which yields (48) when ε_1 goes to zero.

Since both systems (4) and (8) have the same entropies, we can easily prove that the limit (ρ,u) satisfies the entropy condition (17). So we complete the proof of Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Yun-guang Lu and De-yin Zheng's work was partially supported by the Natural Science Foundation of Zhejiang Province of China (Grant no. LY12A01030 and no. LZ13A010002) and the National Natural Science Foundation of China (Grant no. 11271105) and Guo-qiang Song's work was partially supported by the Natural Science Foundation of Anhui Education Committee (Grant no. KJ2012A171).

References

[1] P. G. LeFloch and M. Westdickenberg, "Finite energy solutions to the isentropic Euler equations with geometric effects," *Journal de Mathématiques Pures et Appliquées*, vol. 88, no. 5, pp. 389–429, 2007.

[2] N. Tsuge, "Global L^∞ solutions of the compressible Euler equations with spherical symmetry," *Journal of Mathematics of Kyoto University*, vol. 46, no. 3, pp. 457–524, 2006.

[3] G. Q. Chen and J. Glimm, "Global solutions to the compressible Euler equations with geometrical structure," *Communications in Mathematical Physics*, vol. 180, no. 1, pp. 153–193, 1996.

[4] J. M. Hong and B. Temple, "A bound on the total variation of the conserved quantities for solutions of a general resonant nonlinear balance law," *SIAM Journal on Applied Mathematics*, vol. 64, no. 3, pp. 819–857, 2004.

[5] J. M. Hong, "An extension of Glimms method to inhomogeneous strictly hyperbolic systems of conservation laws by weaker than weak solutions of the Riemann problem," *Journal of Differential Equations*, vol. 222, no. 2, pp. 515–549, 2006.

[6] Y. C. Su, J. M. Hong, and S. W. Chou, "An extension of Glimm's method to the gas dynamical model of transonic flows," *Nonlinearity*, vol. 26, no. 6, pp. 1581–1597, 2013.

[7] E. Isaacson and B. Temple, "Nonlinear resonance in systems of conservation laws," *SIAM Journal on Applied Mathematics*, vol. 52, no. 5, pp. 1260–1278, 1992.

[8] K. Oelschläger, "On the connection between Hamiltonian many-particle systems and the hydrodynamical equations," *Archive for Rational Mechanics and Analysis*, vol. 115, no. 4, pp. 297–310, 1991.

[9] K. Oelschläger, "An integro-differential equation modelling a Newtonian dynamics and its scaling limit," *Archive for Rational Mechanics and Analysis*, vol. 137, no. 2, pp. 99–134, 1997.

[10] S. Caprino, R. Esposito, R. Marra, and M. Pulvirenti, "Hydrodynamic limits of the Vlasov equation," *Communications in Partial Differential Equations*, vol. 18, no. 5-6, pp. 805–820, 1993.

[11] R. J. DiPerna, "Global solutions to a class of nonlinear hyperbolic systems of equations," *Communications on Pure and Applied Mathematics*, vol. 26, pp. 1–28, 1973.

[12] C. Klingenberg and Y. G. Lu, "Existence of solutions to hyperbolic conservation laws with a source," *Communications in Mathematical Physics*, vol. 187, no. 2, pp. 327–340, 1997.

[13] Y. G. Lu, "Existence of global entropy solutions of a nonstrictly hyperbolic system," *Archive for Rational Mechanics and Analysis*, vol. 178, no. 2, pp. 287–299, 2005.

[14] Y. G. Lu, "Global existence of solutions to resonant system of isentropic gas dynamics," *Nonlinear Analysis*, vol. 12, no. 5, pp. 2802–2810, 2011.

[15] Y. G. Lu, "Some results for general systems of isentropic gas dynamics," *Differential Equations*, vol. 43, no. 1, pp. 130–138, 2007.

[16] F. Murat, "Compacité par compensation," *Annali della Scuola Normale Superiore di Pisa*, vol. 5, no. 3, pp. 489–507, 1978.

[17] L. Tartar, "Compensated compactness and applications to partial differential equations," in *Research Notes in Mathematics, Nonlinear Analysis and Mechanics, Heriot-Watt Symposium*, R. J. Knops, Ed., vol. 4, Pitman Press, London, UK, 1979.

[18] P. L. Lions, B. Perthame, and P. E. Souganidis, "Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates," *Communications on Pure and Applied Mathematics*, vol. 49, no. 6, pp. 599–638, 1996.

[19] P. L. Lions, B. Perthame, and E. Tadmor, "Kinetic formulation of the isentropic gas dynamics and p-systems," *Communications in Mathematical Physics*, vol. 163, no. 2, pp. 415–431, 1994.