Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 624162, 12 pages
http://dx.doi.org/10.1155/2014/624162

Research Article

Stability and Hopf Bifurcation of Delayed Predator-Prey System Incorporating Harvesting

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Received 28 September 2013; Accepted 6 January 2014; Published 6 March 2014

Academic Editor: Chun-Lei Tang

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A kind of delayed predator-prey system with harvesting is considered in this paper. The influence of harvesting and delay is investigated. Our results show that Hopf bifurcations occur as the delay τ passes through critical values. By using of normal form theory and center manifold theorem, the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions are obtained. Finally, numerical simulations are given to support our theoretical predictions.

1. Introduction

The classical predator-prey systems have been extensively investigated in recent years, and they will continue to be one of the dominant themes in the future due to their universal existence and importance. Many biological phenomena are always described by differential equations, difference equations, and other type equations. In general, delay differential equations exhibit more complicated dynamical behaviors than ordinary ones; for example, the delay can induce the loss of stability, various oscillations, and periodic solutions. The dynamical behaviors of delay differential equations, stability, bifurcation and chaos, and so forth have been paid much attention by many researchers. Especially, the direction and stability of Hopf bifurcation to delay differential equations have been investigated extensively in recent work (see [1–7] and references therein).

After the classical predator-prey model was first proposed and discussed by May in [8], there were some similar topics, regarding persistence, local and global stabilities of equilibria, and other dynamical behaviors (see [5, 9, 10] and references therein). Recently, Song and Wei in [7] had considered a delayed predator-prey system as follows:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r_1 - a_{11}x(t-\tau) - a_{12}y(t)\right], \\ \dot{y}(t) &= y(t)\left[-r_2 + a_{21}x(t) - a_{22}y(t)\right], \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ were the densities of prey species and predator species at time t, respectively. The local Hopf bifurcation and the existence of the periodic solution bifurcating of system (1) was investigated in [7]. When selective harvesting was put into the predator-prey model similar to (1), Kar [11] studied two predator-prey models with selective harvesting; that is, in the first model, selective harvesting of predator species:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[g(x) - yp(x)\right], \\ \dot{y}(t) &= y(t)\left[-d + \alpha xp(x)\right] - qEy(t-\tau), \end{aligned} \tag{2}$$

and, in the second model, selective harvesting of prey species:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[g(x) - yp(x)\right] - qEx(t-\tau), \\ \dot{y}(t) &= y(t)\left[-d + \alpha xp(x)\right] \end{aligned} \tag{3}$$

had been considered by incorporating time delay on the harvesting term. They found that the delay for selective harvesting could induce the switching of stability and Hopf bifurcation occurred at $\tau = \tau_0$.

Recently, Kar and Ghorai [9] had investigated a predator-prey model with harvesting:

$$\dot{x}(t) = r_1 x(t) - b_1 x^2(t) - \frac{a_1 x(t) y(t)}{x(t) + k_1} - c_1 x(t),$$
$$\dot{y}(t) = y(t)\left[r_2 - \frac{a_2 y(t-\tau)}{x(t-\tau) + k_2}\right] - c_2 y(t). \quad (4)$$

They obtained the local stability, global stability, influence of the harvesting, direction of Hopf bifurcation and the stability to system (4). Motivated by models (1)–(4), we will consider a predator-prey system with delay incorporating harvests to predator and prey:

$$\dot{x}(t) = x(t)\left[1 - \frac{x(t)}{k_1 - ay(t)}\right] - h_1 x(t),$$
$$\dot{y}(t) = y(t)\left[1 - \frac{y(t-\tau)}{k_2 + bx(t-\tau)}\right] - h_2 y(t), \quad (5)$$

where $x(t)$ and $y(t)$ represent the population densities of prey species and predator species, respectively, at time t; a, b, h_1, h_2, k_1, and k_2 are model parameters assuming only positive values; k_1 measures the scale whose environment provides protection to prey x; k_2 denotes the scale whose environment provides protection to predator y; τ means the period of pregnancy; $x(t-\tau)$ represents the number of prey species which was born at time $t-\tau$ and still survived at time t; h_1 and h_2 represent the coefficients of prey species and predator species, respectively. We always assume that $0 \le h_1 \le h_2 < 1$ in this paper.

The organization of the paper is as follows. The stability of the positive equilibrium and the existence of the Hopf bifurcation are discussed in Section 2. The effect of harvesting to prey species and predator species is investigated in Section 3. The direction of Hopf bifurcation and stability of the corresponding periodic solution are obtained in Section 4. Numerical simulations are carried out to illustrate our results in Section 5.

2. Stability of Positive Equilibrium and Hopf Bifurcation

By simple computation, if $k_1 + k_2 a(h_2 - 1) > 0$ holds, system (5) admits a unique positive equilibrium $E^*(x^*, y^*)$:

$$x^* = \frac{(1-h_1)[k_1 + k_2 a(h_2 - 1)]}{1 + ab(1-h_1)(1-h_2)},$$
$$y^* = \frac{(1-h_2)[k_2 + k_1 b(1-h_1)]}{1 + ab(1-h_1)(1-h_2)}. \quad (6)$$

Let $x_1 = x - x^*$, $x_2 = y - y^*$, and then we get the linear system of (5):

$$\dot{x}_1(t) = -a_{11}x_1(t) - a_{12}x_2(t),$$
$$\dot{x}_2(t) = a_{21}x_1(t-\tau) - a_{22}x_2(t-\tau), \quad (7)$$

where $a_{11} = x^*/(k_1 - ay^*)$, $a_{12} = ax^{*2}/(k_1 - ay^*)^2$, $a_{21} = by^{*2}/(k_2 + bx^*)^2$, $a_{22} = y^*/(k_2 + bx^*)$. From linear system (5) the characteristic equation is as follows:

$$\lambda^2 + \lambda\left(a_{11} + a_{22}e^{-\lambda\tau}\right) + \left(a_{11}a_{22} + a_{12}a_{21}\right)e^{-\lambda\tau} = 0. \quad (8)$$

Roots of system (8) imply the stability of the equilibrium E^* and Hopf bifurcation of system (5). Obviously, $\lambda = 0$ is not a root of system (8). For $\tau = 0$, system (8) becomes

$$\lambda^2 + (a_{11} + a_{22})\lambda + (a_{11}a_{22} + a_{12}a_{21}) = 0. \quad (9)$$

It is obvious that the root of system (9) has negative real part. Now, for $\tau > 0$, if $\lambda = i\omega(\omega > 0)$ is a root of (8), then we have

$$-\omega^2 + i\omega\left(a_{11} + a_{22}e^{-i\omega\tau}\right) + \left(a_{11}a_{22} + a_{12}a_{21}\right)e^{-i\omega\tau} = 0. \quad (10)$$

Furthermore,

$$-\omega^2 + a_{22}\omega\sin\omega\tau + \left(a_{11}a_{22} + a_{12}a_{21}\right)\cos\omega\tau = 0,$$
$$a_{11}\omega + a_{22}\omega\cos\omega\tau - \left(a_{11}a_{22} + a_{12}a_{21}\right)\sin\omega\tau = 0, \quad (11)$$

which lead to polynomial equation

$$\omega^4 + \left(a_{11}^2 - a_{22}^2\right)\omega^2 - \left(a_{11}a_{22} + a_{12}a_{21}\right)^2 = 0. \quad (12)$$

It is easy to see that (12) has one positive root

$$\omega = \frac{\sqrt{2}}{2}\left(a_{22}^2 - a_{11}^2 + \sqrt{\Delta}\right)^{1/2}, \quad (13)$$

where $\Delta = (a_{11}^2 - a_{22}^2)^2 + 4(a_{11}a_{22} + a_{12}a_{21})$. By (11), one gets that

$$\tau_j = \frac{1}{\omega}\text{arc cos}\,\frac{a_{12}a_{21}\omega^2}{\omega^2 a_{22}^2 + \left(a_{11}a_{22} + a_{12}a_{21}\right)^2} + \frac{2\pi j}{\omega}, \quad j = 0, 1, \ldots. \quad (14)$$

Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau) \quad (15)$$

be a pair of purely imaginary roots of (8), such that

$$\alpha\left(\tau_j\right) = 0, \qquad \omega\left(\tau_j\right) = \omega. \quad (16)$$

Next, we will prove $\lambda(\tau_j)$ meets the transversality conditions; taking the derivative of system (8) with respect to τ, one derives that

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2\lambda + a_{11} + a_{22}e^{-\lambda\tau} - \tau e^{-\lambda\tau}\left[\lambda a_{22} + \left(a_{11}a_{22} + a_{12}a_{21}\right)\right]}{\lambda e^{-\lambda\tau}\left[\lambda a_{22} + \left(a_{11}a_{22} + a_{12}a_{21}\right)\right]}$$
$$= \frac{2\lambda + a_{11} + a_{22}e^{-\lambda\tau}}{\lambda e^{-\lambda\tau}\left[\lambda a_{22} + \left(a_{11}a_{22} + a_{12}a_{21}\right)\right]} - \frac{\tau}{\lambda}, \quad (17)$$

which, together with (11), leads to

$$\begin{aligned}
&\operatorname{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_j} \\
&= \operatorname{Re}\left\{\frac{2\lambda + a_{11} + a_{22}e^{-\lambda\tau}}{\lambda e^{-\lambda\tau}\left[\lambda a_{22} + \left(a_{11}a_{22} + a_{12}a_{21}\right)\right]} - \frac{\tau}{\lambda}\right\}_{\tau=\tau_j} \\
&= \operatorname{Re}\left\{\frac{2\lambda + a_{11} + a_{22}e^{-\lambda\tau}}{\lambda e^{-\lambda\tau}\left[\lambda a_{22} + \left(a_{11}a_{22} + a_{12}a_{21}\right)\right]}\right\}_{\tau=\tau_j} \\
&= \left(2a_{22}^2\omega^4 + \left[2\left(a_{11}a_{22} + a_{12}a_{21}\right)^2 + \left(a_{11}^2 - a_{22}^2\right)a_{22}^2\right]\omega^2 + \left(a_{11}^2 - a_{22}^2\right)\left(a_{11}a_{22} + a_{12}a_{21}\right)\right) \\
&\quad \times \left(\omega^2 a_{22}^2 + \left(a_{11}a_{22} + a_{12}a_{21}\right)^2\right)^{-1} > 0.
\end{aligned} \tag{18}$$

So, we have

$$\operatorname{sign}\left\{\operatorname{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]\right\}_{\tau=\tau_j} > 0. \tag{19}$$

Thus, we can obtain the following lemma.

Lemma 1. *If $k_1 + k_2a(h_2 - 1) > 0$ holds, then the following results are true:*

(i) *when $\tau = 0$, the positive equilibrium of E^* of system* (5) *is locally asymptotically stable;*

(ii) *when $0 < \tau < \tau_0$, the positive equilibrium of E^* of system* (5) *is locally asymptotically stable, and E^* is unstable when $\tau > \tau_0$, where ω, τ_j $(j = 0, 1, \ldots)$ can be defined in* (13), (14).

3. The Influence of Harvesting

Next, we will discuss the influence of the harvesting on system (5).

Case 1 (only predator species is harvested). For $h_1 = 0$, and the positive equilibrium of system (5) changes to $E_1^*(x_1^*, y_1^*)$, where

$$x_1^* = \frac{k_1 + k_2a\left(h_2 - 1\right)}{1 + ab\left(1 - h_2\right)}, \qquad y_1^* = \frac{\left(1 - h_2\right)\left(k_2 + k_1b\right)}{1 + ab\left(1 - h_2\right)}, \tag{20}$$

it is obvious that $y_1^* > 0$ and $x_1^* > 0$ if and only if $k_1 + k_2a(h_2 - 1) > 0$. Obviously, x_1^* and y_1^* are the continuous differentiable functions with respect to h_2; then, we have

$$\begin{aligned}
\frac{\mathrm{d}x_1^*}{\mathrm{d}h_2} &= \frac{k_2a + k_1ab}{\left[1 + ab\left(1 - h_2\right)\right]^2} > 0, \\
\frac{\mathrm{d}y_1^*}{\mathrm{d}h_2} &= \frac{-k_2 - k_1b}{\left[1 + ab\left(1 - h_2\right)\right]^2} < 0.
\end{aligned} \tag{21}$$

Theorem 2. *If $k_1 + k_2a(h_2 - 1) > 0$ holds, then x_1^* is the monotonic increasing function of h_2, y_1^* is the monotonic decreasing function of h_2; that is, when h_2 increases, the density of prey species will increase, the density of predator species will decrease.*

Case 2 (only prey species is harvested). For $h_2 = 0$, and the positive equilibrium of system (5) changes to $E_2^*(x_2^*, y_2^*)$, where

$$x_2^* = \frac{\left(1 - h_1\right)\left(k_1 - k_2a\right)}{1 + ab\left(1 - h_1\right)}, \qquad y_2^* = \frac{k_2 + k_1b\left(1 - h_1\right)}{1 + ab\left(1 - h_1\right)}, \tag{22}$$

it is obvious that $y_2^* > 0$ and $x_2^* > 0$ if and only if $k_1 - k_2a > 0$. Obviously, x_2^* and y_2^* are the continuous differentiable functions with respect to h_1; then, one get that

$$\begin{aligned}
\frac{\mathrm{d}x_2^*}{\mathrm{d}h_1} &= \frac{k_2a - k_1}{\left[1 + ab\left(1 - h_1\right)\right]^2} < 0, \\
\frac{\mathrm{d}y_2^*}{\mathrm{d}h_1} &= \frac{k_2ab - k_1b}{\left[1 + ab\left(1 - h_1\right)\right]^2} < 0.
\end{aligned} \tag{23}$$

Theorem 3. *If $k_1 - k_2a > 0$ holds, then x_2^* and y_2^* are the monotonic decreasing functions of h_1; that is, if h_1 increases, then the density of prey species and predator species will decrease; on the contrary, if h_1 decreases, the density of prey species and predator species will increase.*

Case 3 (predator species and prey species are harvested simultaneously). For $h_1h_2 \neq 0$, the mixed derivative of x^* and y^* are given by

$$\begin{aligned}
\frac{\partial x^*}{\partial h_1} &= \frac{-\left[k_1 + ak_2\left(h_2 - 1\right)\right]}{\left[1 + ab\left(1 - h_1\right)\left(1 - h_2\right)\right]^2} < 0, \\
\frac{\partial x^*}{\partial h_2} &= \frac{\left(1 - h_1\right)\left[k_2 + k_1b\left(1 - h_1\right)\right]a}{\left[1 + ab\left(1 - h_1\right)\left(1 - h_2\right)\right]^2} > 0, \\
\frac{\partial y^*}{\partial h_1} &= \frac{b\left(h_2 - 1\right)\left[k_1 + ak_2\left(h_2 - 1\right)\right]}{\left[1 + ab\left(1 - h_1\right)\left(1 - h_2\right)\right]^2} < 0, \\
\frac{\partial y^*}{\partial h_2} &= \frac{-\left[k_2 + k_1b\left(1 - h_1\right)\right]}{\left[1 + ab\left(1 - h_1\right)\left(1 - h_2\right)\right]^2} < 0.
\end{aligned} \tag{24}$$

Theorem 4. *If $k_1 + k_2a(h_2 - 1) > 0$ is valid, then the densities of prey species and predator species will both decrease when harvesting rate h_1 increases; on the contrary, the density of prey species will increase and predator species will decrease when harvesting rate h_2 increases.*

4. Direction and Stability of Hopf Bifurcation

Motivated by the ideas of Hassard et al. [12], by applying the normal form theory and the center manifold theorem, the properties of the Hopf bifurcation at the critical value $\tau = \tau_j$ are derived in this section.

Let $t = s\tau$, $x_i(s\tau) = \hat{x}_i(s)$, $i = 1, 2$, $\tau = \tau_0 + \mu$, $\mu \in R$; τ_0 is defined by (14), we still denote $\hat{x}_i(s) = u_i(s)$ and $s = t$, then system (5) is transformed into functional differential equations in $C([-1, 0], R^2)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{25}$$

where $u(t) = (u_1(t), u_2(t))^T \in R^2$, $u_t(\theta) = u(t+\theta)$, $\theta \in [-1, 0]$, and $L_\mu : C([-1, 0]; R^2) \to R$, $f : R \times C([-1, 0]; R^2) \to R$ are given by

$$L_\mu(\phi) = (\tau_0 + \mu)\begin{pmatrix} -a_{11} & -a_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_0 + \mu)\begin{pmatrix} 0 & 0 \\ a_{21} & -a_{22} \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}, \tag{26}$$

$$f(\mu, \phi) = (\tau_0 + \mu)\begin{pmatrix} \dfrac{c_1\phi_2^2(0) + c_2\phi_1(0)\phi_2(0) + c_3\phi_1^2(0)}{e_1 - e_2\phi_2(0)} \\ \dfrac{c_4\phi_1(-1)\phi_2(0) - c_5\phi_1^2(-1) + c_6\phi_1(-1)\phi_2(-1) - c_7\phi_2(0)\phi_2(-1)}{e_3 + e_4\phi_1(-1)} \end{pmatrix}, \tag{27}$$

where

$$\begin{gathered} c_1 = a^2(x^*)^2, \qquad c_2 = 2a^2x^*y^* - 2ak_1x^*, \\ c_3 = 2ak_1y^* - a^2ck_1^2y^*, \qquad c_4 = bk_2y^* + b^2x^*y^*, \\ c_5 = b^2(y^*)^2, \qquad c_6 = k_2by^* + b^2x^*y^*, \\ c_7 = 2k_2bx^* + k_2^2 + b^2(x^*)^2, \\ e_1 = (k_1 - ay^*)^3, \qquad e_2 = a(k_1 - ay^*)^2, \\ e_3 = (k_2 + bx^*)^3, \qquad e_4 = b(k_2 + bx^*)^2. \end{gathered} \tag{28}$$

By Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta). \tag{29}$$

We choose

$$\begin{aligned} \eta(\theta, \mu) &= (\tau_0 + \mu)\begin{pmatrix} -a_{11} & -a_{12} \\ 0 & 0 \end{pmatrix}\delta(\theta) \\ &\quad + (\tau_0 + \mu)\begin{pmatrix} 0 & 0 \\ a_{21} & -a_{22} \end{pmatrix}\delta(\theta + 1), \end{aligned} \tag{30}$$

where δ is the Dirac delta function. For $\phi \in C^1([-1, 0], R^2)$, we define

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0 \\ \int_{-1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases} \tag{31}$$

$$R(\mu)\phi(\theta) = \begin{cases} 0, & -1 \le \theta < 0 \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then, system (25) can be transformed into an operator differential equation of the form

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{32}$$

where $u_t(\theta) = u(t + \theta)$, for $\theta \in [-1, 0]$. For $\psi \in C^1([0, 1], (R^2)^*)$, we define

$$A^*(\mu)\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0 \end{cases} \tag{33}$$

and a bilinear inner product

$$\begin{aligned} \langle\psi(\theta), \phi(\theta)\rangle &= \overline{\psi}^T(0)\phi(0) \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}^T(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \end{aligned} \tag{34}$$

where $\eta(\theta) = \eta(\theta, 0)$; then, $A(0)$ and A^* are adjoint operators. Noting that $\pm i\omega\tau_0$ are eigenvalues of $A(0)$, thus, they are also eigenvalues of A^*. In order to calculate the eigenvector $q(\theta)$ of $A(0)$ corresponding to the eigenvalue $i\omega\tau_0$ and $p(s)$ of A^* corresponding to the eigenvalue $-i\omega\tau_0$, let $q(\theta) = (1, \alpha)^T e^{i\omega\tau_0\theta}$ be the eigenvector of $A(0)$ corresponding to $i\omega\tau_0$; then, $A(0)q(\theta) = i\omega\tau_0 q(\theta)$.

By the definition of $A(0)$ and (26), (30), then,

$$\tau_0\begin{pmatrix} -i\omega - a_{11} & -a_{12} \\ a_{21}e^{-i\omega\tau_0} & -i\omega - a_{22}e^{-i\omega\tau_0} \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{35}$$

Thus, we can get

$$q(0) = (1, \alpha)^T = \left(1, \frac{a_{11} + i\omega}{-a_{12}}\right)^T. \tag{36}$$

Similarly, let $p(s) = D(1, \beta)^T e^{i\omega\tau_0 s}$ be the eigenvector of A^* corresponding to $-i\omega\tau_0$; by similar discussion, we get $\beta = (a_{11} - i\omega)/a_{21}e^{i\omega\tau_0}$.

In view of standardization of $p(s)$ and $q(\theta)$; that is, $\langle p(s), q(\theta)\rangle = 1$, we have

$$\begin{aligned}
&\langle p(s), q(\theta)\rangle \\
&\quad = \overline{D}\left(1,\overline{\beta}\right)(1,\alpha)^T \\
&\qquad - \int_{-1}^{0}\int_{\xi=0}^{\theta}\overline{D}\left(1,\overline{\beta}\right)e^{-i\omega\tau_0(\xi-\theta)}\mathrm{d}\eta(\theta)(1,\alpha)^T e^{i\omega\tau_0\xi}d\xi \\
&\quad = \overline{D}\left\{1+\alpha\overline{\beta}-\int_{-1}^{0}\left(1,\overline{\beta}\right)\theta e^{i\omega\tau_0\theta}\mathrm{d}\eta(\theta)(1,\alpha)^T\right\} \\
&\quad = \overline{D}\left\{1+\alpha\overline{\beta}+\tau_0\overline{\beta}e^{-i\omega\tau_0}\left(a_{21}-\alpha a_{22}\right)\right\}.
\end{aligned} \tag{37}$$

Thus, choose $D = [1 + \beta\overline{\alpha} + \tau_0\beta e^{i\omega\tau_0}(a_{21} - \overline{\alpha}a_{22})]^{-1}$. Next, we will quote the same notation (see [13]), we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Define

$$z(t) = \langle p, u_t\rangle, \qquad W(t,\theta) = u_t(\theta) - 2\operatorname{Re}\{z(t)q(\theta)\}. \tag{38}$$

On the center manifold C_0, we have

$$\begin{aligned}
W(t,\theta) &= W(z(t),\overline{z}(t),\theta) \\
&= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots
\end{aligned} \tag{39}$$

z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction p and $\overline{p}$; noting that W is real if u_t is real, we only consider real solution $u_t \in C_0$ of (25). Since $\mu = 0$, then we have

$$\begin{aligned}
\dot{z}(t) &= i\omega\tau_0 z + \overline{p}(0) f\left(0, W(z,\overline{z},\theta) + 2\operatorname{Re}\{z(t)q(0)\}\right) \\
&\overset{\text{def}}{=} i\omega\tau_0 z + \overline{p}(0) f_0(z,\overline{z}).
\end{aligned} \tag{40}$$

We rewrite this equation as

$$\dot{z}(t) = i\omega\tau_0 z + g(z,\overline{z}), \tag{41}$$

where

$$\begin{aligned}
g(z,\overline{z}) &= \overline{p}(0) f_0(z,\overline{z}) \\
&= g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots.
\end{aligned} \tag{42}$$

Noting $u_t(\theta) = (\phi_1(\theta), \phi_2(\theta))^T = W(t,\theta) + zq(\theta) + \overline{z}\overline{q}(\theta)$ and $q(\theta) = (1,\alpha)^T e^{i\omega\tau_0\theta}$, we have

$$\begin{aligned}
\phi_1(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} \\
&\quad + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
\phi_2(0) &= z\alpha + \overline{z}\overline{\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} \\
&\quad + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots, \\
\phi_1(-1) &= ze^{-i\omega\tau_0} + \overline{z}e^{i\omega\tau_0} + W_{20}^{(1)}(-1)\frac{z^2}{2} \\
&\quad + W_{11}^{(1)}(-1)z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots, \\
\phi_2(-1) &= z\alpha e^{-i\omega\tau_0} + \overline{z}\overline{\alpha}e^{i\omega\tau_0} + W_{20}^{(2)}(-1)\frac{z^2}{2} \\
&\quad + W_{11}^{(2)}(-1)z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + \cdots.
\end{aligned} \tag{43}$$

From (27), (42), we obtain that

$$\begin{aligned}
g_{20} &= 2\overline{D}\tau_0\left[\frac{1}{e_1}\left(c_3 + c_1\alpha^2 + c_2\alpha\right) + \frac{\overline{\beta}}{e_3}\right. \\
&\qquad \times\left(-c_5e^{-2i\omega\tau_0} + c_6\alpha e^{-2i\omega\tau_0}\right. \\
&\qquad \left.\left. + c_4\alpha e^{-i\omega\tau_0} - c_7\alpha^2 e^{-i\omega\tau_0}\right)\right], \\
g_{11} &= \overline{D}\tau_0\left\{\frac{1}{e_1}\left[2c_3 + 2c_1\alpha\overline{\alpha} + c_2(\alpha+\overline{\alpha})\right] + \frac{\overline{\beta}}{e_3}\right. \\
&\qquad \times\left[-2c_5 + c_6(\alpha+\overline{\alpha}) + c_4\left(\overline{\alpha}e^{-i\omega\tau_0} + \alpha e^{i\omega\tau_0}\right)\right. \\
&\qquad \left.\left. - c_7\left(\alpha\overline{\alpha}e^{-i\omega\tau_0} + \alpha\overline{\alpha}e^{i\omega\tau_0}\right)\right]\right\}, \\
g_{02} &= 2\overline{D}\tau_0\left[\frac{1}{e_1}\left(c_3 + c_1\overline{\alpha}^2 + c_2\overline{\alpha}\right) + \frac{\overline{\beta}}{e_3}\right. \\
&\qquad \times\left(-c_5e^{2i\omega\tau_0} + c_6\overline{\alpha}e^{2i\omega\tau_0}\right. \\
&\qquad \left.\left. + c_4\overline{\alpha}e^{i\omega\tau_0} - c_7\overline{\alpha}^2 e^{i\omega\tau_0}\right)\right],
\end{aligned}$$

$$\begin{aligned}
g_{21} = \overline{D}\tau_0 \Bigg\{ \frac{1}{e_1} \Bigg[& c_3 \left(2W_{20}^{(1)}(0) + 4W_{11}^{(1)}(0)\right) \\
& + c_1 \left(2\overline{\alpha}W_{20}^{(2)}(0) + 4\alpha W_{11}^{(2)}(0)\right) \\
& + c_2 \left(\overline{\alpha}W_{20}^{(1)}(0) + W_{20}^{(2)}(0) + 2\alpha W_{11}^{(1)}(0) \right.\\
& \left. + 2W_{11}^{(2)}(0)\right) + \frac{6c_1e_2}{e_1} + 2(\overline{\alpha} + 2\alpha) \\
& \times \frac{e_2c_3}{e_1} + 2\left(\alpha^2 + 2\alpha\overline{\alpha}\right)\frac{e_2c_2}{e_1}\Bigg] + \frac{\overline{\beta}}{e_3} \\
& \times \Big[-c_5\left(2e^{i\omega\tau_0}W_{20}^{(1)}(-1) + 4e^{-i\omega\tau_0}W_{11}^{(1)}(-1)\right) \\
& + c_6\left(\overline{\alpha}e^{i\omega\tau_0}W_{20}^{(1)}(-1) + e^{i\omega\tau_0}W_{20}^{(2)}(-1)\right.\\
& \left. + 2e^{-i\omega\tau_0}W_{11}^{(2)}(-1) + 2\alpha e^{-i\omega\tau_0}W_{11}^{(1)}(-1)\right) \\
& + c_4\left(\overline{\alpha}W_{20}^{(1)}(-1) + e^{i\omega\tau_0}W_{20}^{(2)}(0)\right.\\
& \left. + 2e^{-i\omega\tau_0}W_{11}^{(2)}(0) + 2\alpha W_{11}^{(1)}(-1)\right) \\
& - c_7\left(\overline{\alpha}W_{20}^{(2)}(-1) + \overline{\alpha}e^{i\omega\tau_0}W_{20}^{(2)}(0)\right.\\
& \left. + 2\alpha e^{-i\omega\tau_0}W_{11}^{(2)}(0) + 2\alpha W_{11}^{(2)}(-1)\right) \\
& + \frac{6e_4c_5}{e_3}e^{-i\omega\tau_0} - \frac{2e_4c_6}{e_3} \\
& \times \left(\overline{\alpha}e^{-i\omega\tau_0} + 2\alpha e^{-i\omega\tau_0}\right) - \frac{2e_4c_4}{e_3} \\
& \times \left(\overline{\alpha}e^{-2i\omega\tau_0} + 2\alpha\right) + \frac{2e_4c_7}{e_3} \\
& \times \left(\overline{\alpha}\alpha e^{-2i\omega\tau_0} + \alpha\overline{\alpha} + \alpha^2\right)\Big]\Bigg\}.
\end{aligned} \tag{44}$$

Because g_{21} contains W_{20} and W_{11}, from (32) and (38), we have

$$\begin{aligned}
\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\overline{z}}\overline{q} \\
&= \begin{cases} AW - 2\operatorname{Re}\{\overline{p}(0) f_0 q(\theta)\}, & -1 \le \theta < 0, \\ AW - 2\operatorname{Re}\{\overline{p}(0) f_0 q(0)\} + f_0, & \theta = 0, \end{cases} \\
&\overset{\text{def}}{=} AW + H(z, \overline{z}, \theta),
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
H(z(t), \overline{z}(t), \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\overline{z} \\
+ H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots.
\end{aligned} \tag{46}$$

Substituting the corresponding series into (45) and comparing the coefficients, we have

$$\begin{aligned}
(A - 2i\omega\tau_0) W_{20}(\theta) &= -H_{20}(\theta), \\
AW_{11}(\theta) &= -H_{11}(\theta), \ldots.
\end{aligned} \tag{47}$$

From (45), we know that for $\theta \in [-1, 0)$, we have

$$\begin{aligned}
H(z(t), \overline{z}(t), \theta) &= -\overline{p}(0) f_0 q(\theta) - p(0)\overline{f}_0\overline{q}(\theta) \\
&= -g(z, \overline{z}) q(\theta) - \overline{g}(z, \overline{z})\overline{q}(\theta).
\end{aligned} \tag{48}$$

Comparing the coefficient with (46) yields that for $\theta \in [-1, 0)$

$$H_{20}(\theta) = -g_{20}(\theta) - \overline{g}_{02}\overline{q}(\theta), \tag{49}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{50}$$

From (47), (49) and the definition of A, it follows that

$$\dot{W}_{20}(\theta) = 2i\omega\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta), \tag{51}$$

taking notice of $q(\theta) = (1, \alpha)^T e^{i\omega\tau_0\theta}$; hence,

$$W_{20}(\theta) = \frac{ig_{20}}{\omega\tau_0}q(0)e^{i\omega\tau_0\theta} - \frac{\overline{g_{02}}}{3i\omega\tau_0}\overline{q}(0)e^{-i\omega\tau_0\theta} + E_1e^{2i\omega\tau_0\theta}, \tag{52}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in R^2$ is a constant vector. By the similar way, we have

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega\tau_0}q(0)e^{i\omega\tau_0\theta} - \frac{\overline{g}_{11}}{i\omega\tau_0}\overline{q}(0)e^{-i\omega\tau_0\theta} + E_2, \tag{53}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in R^2$ is a constant vector.

Next, computing E_1 and E_2, from the definition of A and (47), one then obtains

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega\tau_0 W_{20}(0) - H_{20}(0), \tag{54}$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{55}$$

where $\eta(\theta) = \eta(0, \theta)$. Furthermore, we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_0 \begin{pmatrix} \frac{1}{e_1}\left(c_3 + c_1\alpha^2 + c_2\alpha\right) \\ \frac{1}{e_3}\left(-c_5 e^{-2i\omega\tau_0} + c_6\alpha e^{-2i\omega\tau_0} + c_4\alpha e^{-i\omega\tau_0} - c_7\alpha^2 e^{-i\omega\tau_0}\right) \end{pmatrix}, \tag{56}$$

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + 2\tau_0 \begin{pmatrix} \frac{1}{e_1}\left(c_3 + c_1\alpha\overline{\alpha} + c_2 \operatorname{Re}\{\alpha\}\right) \\ \frac{1}{e_3}\left(-c_5 + c_6 \operatorname{Re}\{\alpha\} + c_4 \operatorname{Re}\left\{\alpha e^{i\omega\tau_0}\right\} - c_7 \operatorname{Re}\left\{\alpha\overline{\alpha} e^{i\omega\tau_0}\right\}\right) \end{pmatrix}. \tag{57}$$

Substituting (52) and (56) into (54) and noting that

$$\begin{gathered} \left(i\omega\tau_0 I - \int_{-1}^{0} e^{i\omega\tau_0\theta} \mathrm{d}\eta(\theta)\right) q(0) = 0, \\ \left(-i\omega\tau_0 I - \int_{-1}^{0} e^{-i\omega\tau_0\theta} \mathrm{d}\eta(\theta)\right) \overline{q}(0) = 0, \end{gathered} \tag{58}$$

it implies that

$$\begin{aligned} &\left(2i\omega\tau_0 I - \int_{-1}^{0} e^{2i\omega\tau_0\theta} \mathrm{d}\eta(\theta)\right) E_1 \\ &\quad = 2\begin{pmatrix} \frac{1}{e_1}\left(c_3 + c_1\alpha^2 + c_2\alpha\right) \\ \frac{1}{e_3}\left(-c_5 e^{-2i\omega\tau_0} + c_6\alpha e^{-2i\omega\tau_0} + c_4\alpha e^{-i\omega\tau_0} - c_7\alpha^2 e^{-i\omega\tau_0}\right) \end{pmatrix}, \end{aligned} \tag{59}$$

Namely,

$$\begin{aligned} &\begin{pmatrix} 2i\omega + a_{11} & a_{12} \\ -a_{21}e^{-2i\omega\tau_0} & 2i\omega + a_{22}e^{-2i\omega\tau_0} \end{pmatrix} E_1 \\ &\quad = 2\begin{pmatrix} \frac{1}{e_1}\left(c_3 + c_1\alpha^2 + c_2\alpha\right) \\ \frac{1}{e_3}\left(-c_5 e^{-2i\omega\tau_0} + c_6\alpha e^{-2i\omega\tau_0} + c_4\alpha e^{-i\omega\tau_0} - c_7\alpha^2 e^{-i\omega\tau_0}\right) \end{pmatrix}. \end{aligned} \tag{60}$$

Then it yields that

$$\begin{aligned} E_1^{(1)} &= \frac{2}{A_1}\begin{vmatrix} \frac{1}{e_1}\left(c_3 + c_1\alpha^2 + c_2\alpha\right) & a_{12} \\ \frac{1}{e_3}\left[\left(-c_5 + c_6\alpha\right)e^{-2i\omega\tau_0} + \left(c_4\alpha - c_7\alpha^2\right)e^{-i\omega\tau_0}\right] & 2i\omega + a_{22}e^{-2i\omega\tau_0} \end{vmatrix}, \\ E_1^{(2)} &= \frac{2}{A_1}\begin{vmatrix} 2i\omega + a_{11} & \frac{1}{e_1}\left(c_3 + c_1\alpha^2 + c_2\alpha\right) \\ -a_{21}e^{-2i\omega\tau_0} & \frac{1}{e_3}\left[\left(-c_5 + c_6\alpha\right)e^{-2i\omega\tau_0} + \left(c_4\alpha - c_7\alpha^2\right)e^{-i\omega\tau_0}\right] \end{vmatrix}, \end{aligned} \tag{61}$$

where

$$A_1 = \begin{vmatrix} 2i\omega + a_{11} & a_{12} \\ -a_{21}e^{-2i\omega\tau_0} & 2i\omega + a_{22}e^{-2i\omega\tau_0} \end{vmatrix}. \tag{62}$$

Similarly, we get

$$\begin{aligned} E_2^{(1)} &= \frac{2}{A_2}\begin{vmatrix} \frac{1}{e_1}\left(c_3 + c_1\alpha\overline{\alpha} + c_2 \operatorname{Re}\{\alpha\}\right) & a_{12} \\ \frac{1}{e_3}\left(-c_5 + c_6 \operatorname{Re}\{\alpha\} + c_4 \operatorname{Re}\left\{\alpha e^{i\omega\tau_0}\right\} - c_7 \operatorname{Re}\left\{\alpha\overline{\alpha} e^{i\omega\tau_0}\right\}\right) & a_{22} \end{vmatrix}, \\ E_2^{(2)} &= \frac{2}{A_2}\begin{vmatrix} a_{11} & \frac{1}{e_1}\left(c_3 + c_1\alpha\overline{\alpha} + c_2 \operatorname{Re}\{\alpha\}\right) \\ -a_{21} & \frac{1}{e_3}\left(-c_5 + c_6 \operatorname{Re}\{\alpha\} + c_4 \operatorname{Re}\left\{\alpha e^{i\omega\tau_0}\right\} - c_7 \operatorname{Re}\left\{\alpha\overline{\alpha} e^{i\omega\tau_0}\right\}\right) \end{vmatrix}, \end{aligned} \tag{63}$$

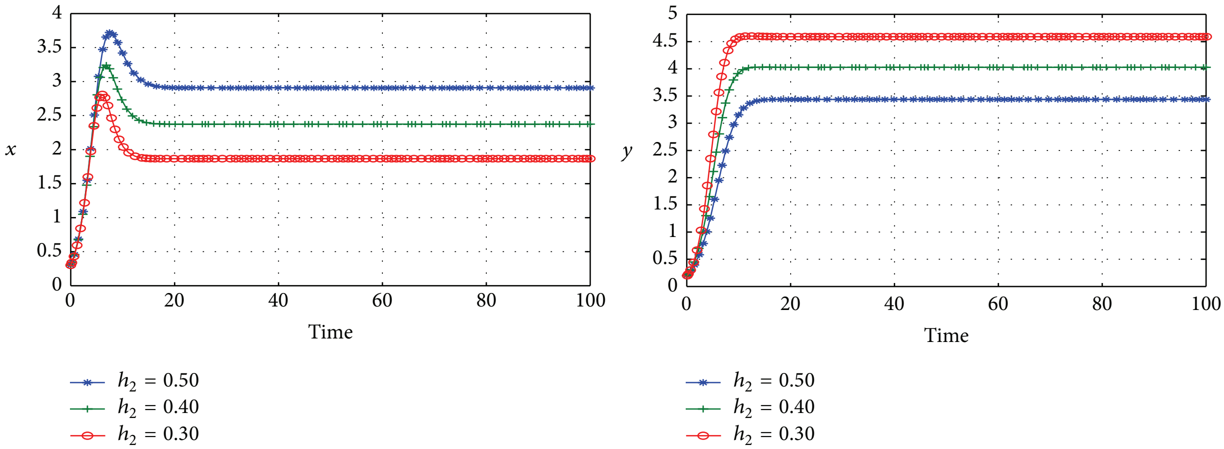

FIGURE 1: When $h_1 = 0.4$ and h_2 decreases, prey species x decreases and predator species y increases.

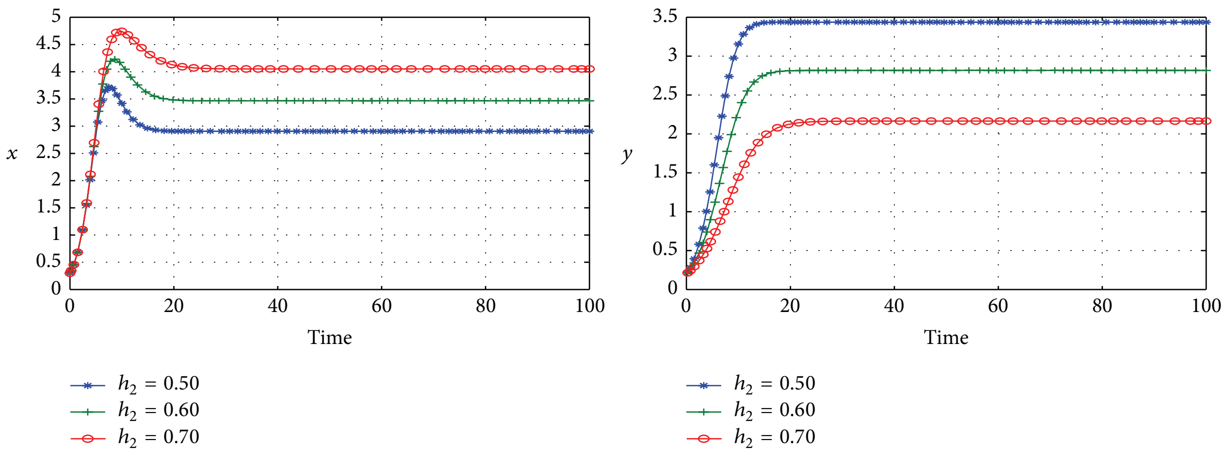

FIGURE 2: When $h_1 = 0.4$ and h_2 increases, prey species x increases and predator species y decreases.

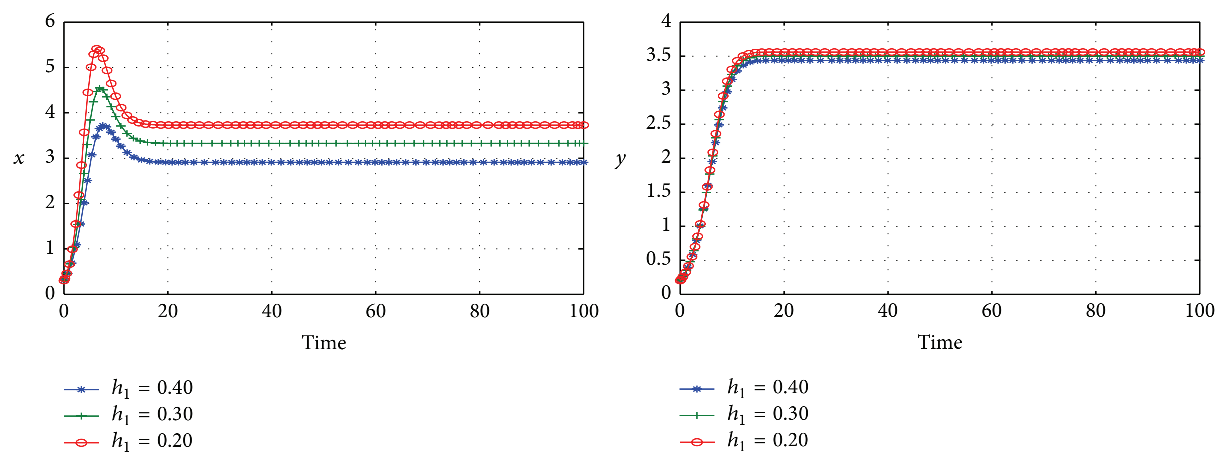

FIGURE 3: When $h_2 = 0.5$ and h_1 decreases, prey species x and predator species y increase.

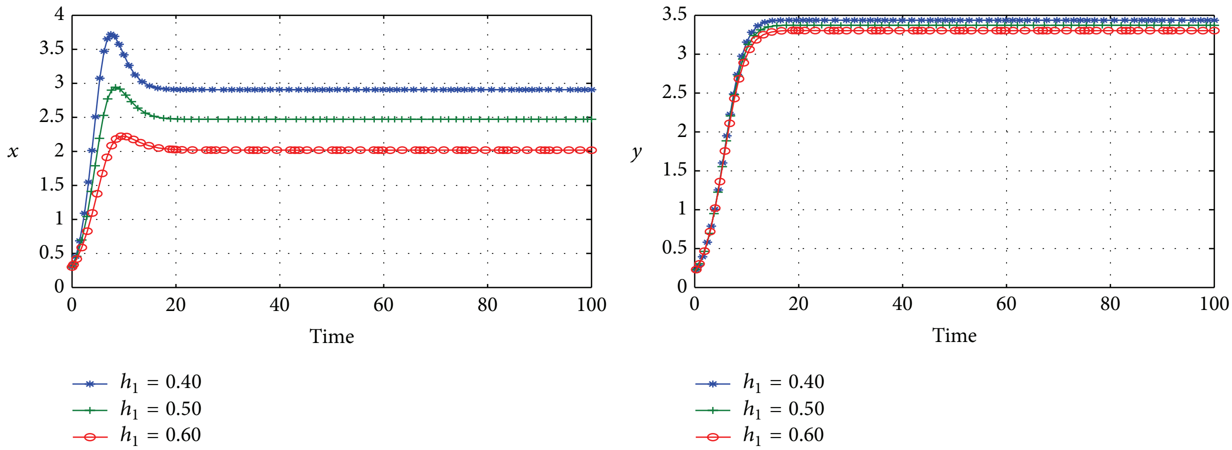

FIGURE 4: When $h_2 = 0.5$ and h_1 increases, prey species x and predator species y decrease.

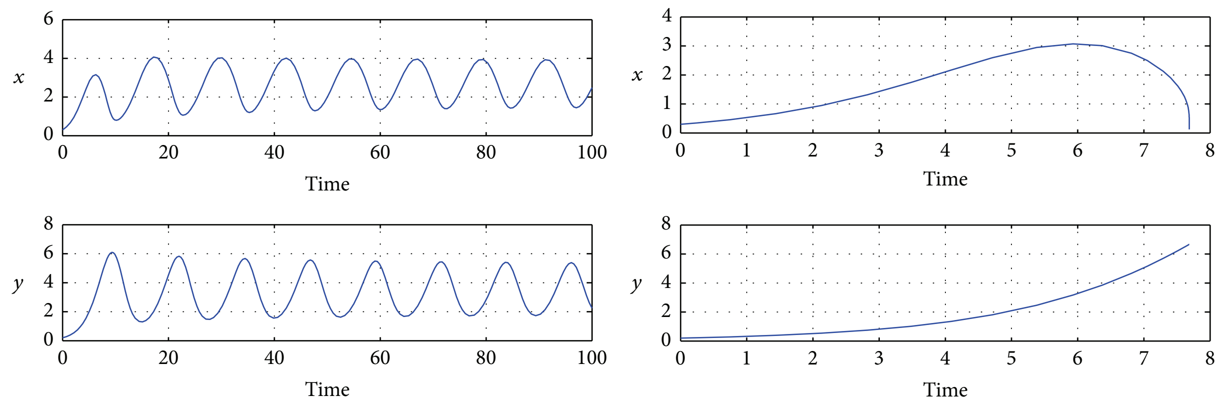

FIGURE 5: When $\tau = 2.9 > \tau_0 \doteq 2.8015$, prey species x and predator species y coexist; when $\tau = 5 > \tau_0 \doteq 2.8015$, prey species x goes to extinct.

where

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{vmatrix}. \tag{64}$$

Through simple computation, we determine W_{20}, W_{11} from (52) and (53); further, we can determine g_{21}. Therefore, g_{ij} in (44) can be expressed by the parameter and delay; hence,

$$\begin{gathered} C_1(0) = \frac{i}{2\omega\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \qquad \zeta = 2\operatorname{Re}\{C_1(0)\}, \\ T = -\frac{\operatorname{Im}\{C_1(0)\} + \mu \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega\tau_0}, \end{gathered} \tag{65}$$

which determine the qualities of bifurcation periodic solution of the critical value τ_0.

Theorem 5. *(i) μ_2 determines the direction of Hopf bifurcation: if $\mu_2 > 0$ (<0), then Hopf bifurcation is supercritical (subcritical), and the bifurcating periodic solutions exist for $\tau > \tau_0$ ($\tau < \tau_0$).*

(ii) ζ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\zeta < 0$ ($\zeta > 0$). T determines the period of the bifurcating periodic solution: the period increases (decrease) if $T > 0$ ($T < 0$).

5. Numerical Simulations

In this section, we consider a delayed predator-prey system with harvesting as follows:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(1 - \frac{x(t)}{10 - 1.5y(t)}\right) - 0.4x(t), \\ \dot{y}(t) &= y(t)\left(1 - \frac{y(t-\tau)}{6 + 0.3x(t-\tau)}\right) - 0.5y(t). \end{aligned} \tag{66}$$

Because (H_1) holds, from (14), we obtain that

$$\operatorname{sign}\left\{\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]\right\}_{\tau=\tau_j} > 0, \quad \tau_0 \approx 2.8015. \tag{67}$$

The unique positive equilibrium is $E^* = (2.907, 3.436)$.

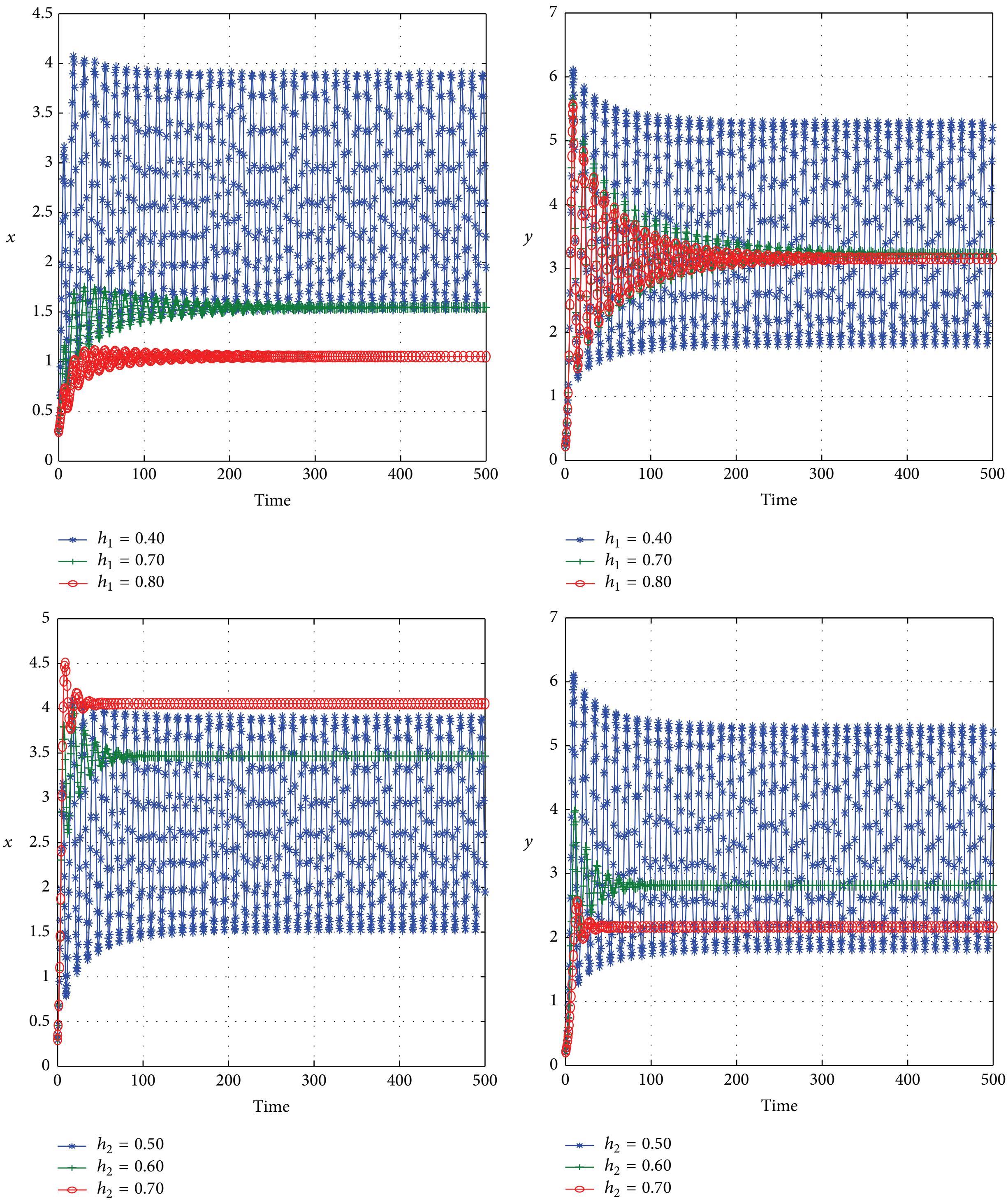

FIGURE 6: When $\tau = 2.9 > \tau_0 \approx 2.8015$ and h_1 increases, prey species x and predator species y become stable; when h_2 increases, prey species x and predator species y also become stable.

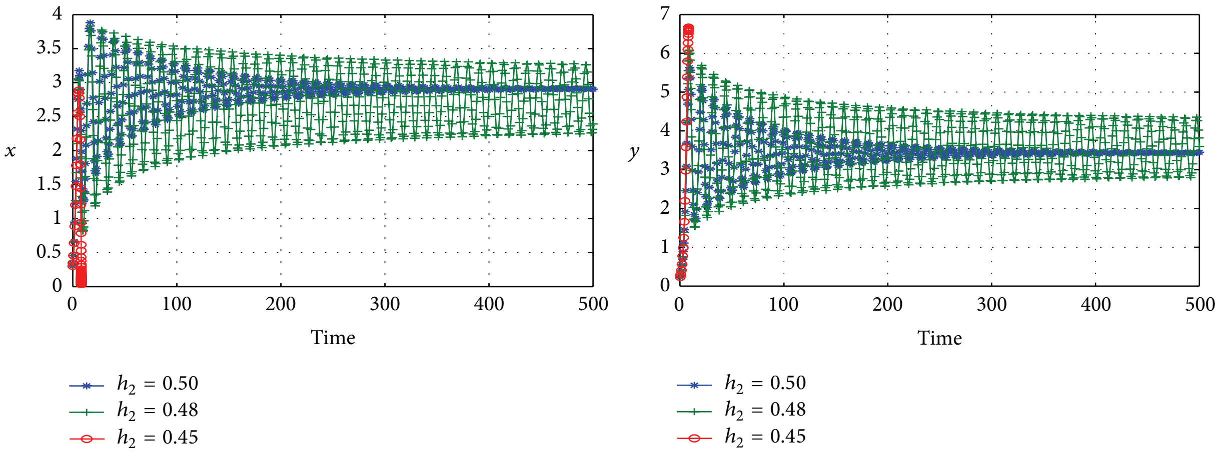

Figure 7: When $\tau = 2.7 < \tau_0 \approx 2.8015$ and h_2 decreases, prey species x and predator species y become unstable.

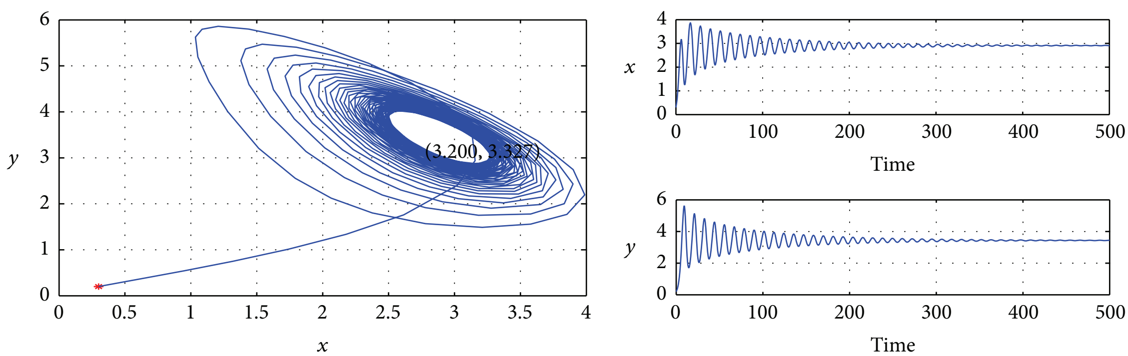

Figure 8: When $\tau = 2.7 < \tau_0 \doteq 2.8015$, the positive equilibrium E^* of system (5) is asymptotically stable.

If $h_1 = 0.4$, when h_2 decreases, then prey species decreases and predator species increases (see Figure 1); when h_2 increases, prey species increases and predator species decreases (see Figure 2); If $h_2 = 0.5$, when the values of harvesting h_1 decreases, then both predator species and prey species will increase (see Figure 3); on the other hand, when h_1 increases, then both predator species and prey species will decrease (see Figure 4).

When parameter τ is little bigger than the critical value τ_0, system (5) will become unstable and predator species and prey species can coexist; when τ increases much more, prey species will go to extinct (see Figure 5). Moreover, from Figure 6, we can see that system (5) is unstable when τ passes through the critical value τ_0. By controlling the harvesting rates h_1 and h_2, respectively, the stability of positive equilibrium to system (5) can been changed. Similarly, when $\tau < \tau_0$, system (5) is stable; if we decrease the harvesting rate h_2, then the stable system becomes unstable one (see Figure 7).

Since $\mu_2 < 0$, $\zeta < 0$, Hopf bifurcation is subcritical and the positive equilibrium E^* is asymptotically stable for $0 < \tau < \tau_0$ (see Figure 8); when $\tau > \tau_0$, E^* loses its stability and Hopf bifurcation occurs; that is, a family of periodic solutions bifurcate from E^* (see Figure 9).

As discussed, our results show that the delay τ affects the stability of system (5) and harvesting rates h_1 and h_2 also affect the stability of system (5).

6. Conclusion

In our model, the harvesting term has been introduced into the model (5); by applying the normal form theorem and the center manifold theorem, we investigate the dynamical behaviors of the delayed predator-prey model with harvesting term and obtain the influence of harvesting term on the prey species and predator species. Further, we prove that the influence of the harvesting rates h_1 and h_2 to the stability of system (5), by controlling harvesting rates h_1 and h_2 of prey species and predator species, which makes the unstable (stable) system become stable (unstable).

Our results show that Hopf bifurcations occur as the delay τ passes through critical values $\tau_0 \approx 2.8015$. When $\tau < \tau_0$, the positive equilibrium E^* of system (5) is asymptotically stable; when $\tau > \tau_0$, the positive equilibrium E^* of system (5) loses its stability and Hopf bifurcations occur.

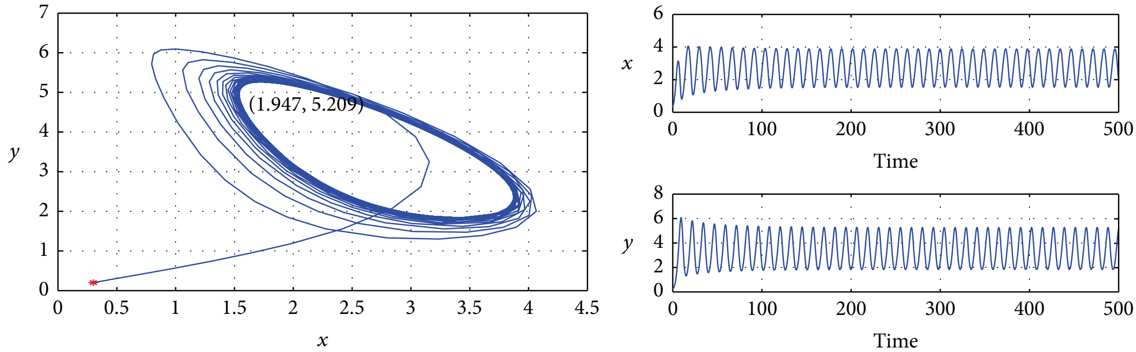

Figure 9: When $\tau = 2.9 > \tau_0 \doteq 2.8015$, the positive equilibrium E^* of system (5) loses its stability and a Hopf bifurcations occurs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by National Natural Science Foundation of China (no. 11201075) and Natural Science Foundation of Fujian Province of China (no. 2010J01005).

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