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Research Article

On the Tumura-Clunie Theorem and Its Application

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We cast aside the restriction of the simple pole in the Tumura-Clunie type theorems for meromorphic functions and obtain a better result which improves the earlier results of Y. D. Ren. Furthermore, as an application, we improve a theorem given by B. Y. Su.

1. Introduction and Main Results

A meromorphic function will always mean meromorphic in the complex plane $\mathbb{C}$. We adopt the standard notation in the Nevanlinna value distribution theory of meromorphic functions such as $T(r,f)$, $m(r,f)$, $N(r,f)$, and $\overline{N}(r,f)$ as explained in [1, 2]. For any nonconstant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite linear measures that is not necessarily the same at each occurrence.

Definition 1 (see [1]). A meromorphic function "$a(z)$" is said to be a small function of f if $T(r,a(z)) = S(r,f)$.

Definition 2. Throughout this paper one denotes by $a_j(z)$ meromorphic functions satisfying $(r,a_j(z)) = S(r,f)(j = 0,1,\ldots,n)$. If $a_n \not\equiv 0$, we call $P[f] = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0$ a polynomial in f with degree n. If $n_0, n_1, \ldots, n_k$ are nonnegative integers, we call $M[f] = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$ a differential monomial in f of degree $\Upsilon_M = n_0+n_1+\cdots+n_k$ and of weight $\Gamma_M = n_0+2n_1+\cdots+(k+1)n_k$. If $M_1, M_2, \ldots, M_n$ are differential monomials in f, we call $Q[f] = \sum_{j=1}^{n} a_j(z) M_j[f]$ a differential polynomial in f and define the degree Υ_Q and the weight Γ_Q by $\Upsilon_Q = \max_{j=1}^{n}\Upsilon_{M_j}$ and $\Gamma_Q = \max_{j=1}^{n}\Gamma_{M_j}$, respectively.

Also $Q[f]$ is called a quasi-differential polynomial generated by f if, instead of assuming $T(r,a_j(z)) = S(r,f)$, we just assume that $m(r,a_j(z)) = S(r,f)$ for the coefficients $a_j(z)(j = 1,2,\ldots,n)$.

Definition 3. Let k be a positive integer; for any a in the complex plane, one denotes by $N_{k)}(r,1/(f-a))$ the counting function of a-points of f with multiplicity less than or equal to k, by $N_{(k}(r,1/(f-a))$ the counting function of a-points of f with multiplicity more than or equal to k, and by $N_k(r,1/(f-a))$ the counting function of a-points of f with multiplicity of k. Denote the reduced counting function by $\overline{N}_{k)}(r,1/(f-a))$, $\overline{N}_{(k}(r,1/(f-a))$, and $\overline{N}_k(r,(1/f-a))$, respectively.

Let f be a nonconstant meromorphic function and let

$$F = f^n + Q[f] \tag{1}$$

be a differential polynomial, where $Q[f]$ is also a differential polynomial and $\Upsilon_Q \leq n-1$.

Hua (see [3, page 69]) proved the following result.

Theorem A. *Let f be a nonconstant meromorphic function and let F be given by* (1) *with $\Upsilon_Q \leq n-1$. If*

$$N(r,f) + N\left(r,\frac{1}{F}\right) = S(r,f), \tag{2}$$

then

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{3}$$

where $a(z)$ is a small function of f.

Then $F = g^n$, $g = f + (a(z)/n)$, and $a(z)g^{n-1}$ is obtained by substituting g for f, g' for f', and so forth in the terms of degree $n-1$ in $Q[f]$.

Remark 4. The conclusion still holds good if condition (2) is replaced with

$$N(r, f) + N\left(r, \frac{1}{F}\right) = S_o(r, f), \tag{4}$$

where $S_o(r, f)$ denotes any quantity which satisfies $S_o(r, f) = o(T(r, f))$ as $r \to +\infty$ through a set of r of infinite measure.

Hua (see [3]) improved Theorem A and obtained the following result.

Theorem B. *Let f be a nonconstant meromorphic function and let F be given by* (1) *with $\Upsilon_Q \leq n-1$. If*

$$N(r, f) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f), \tag{5}$$

then

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{6}$$

where $a(z)$ is a small function of f.

Another theorem is due to Zhang and Li (see [4]), which can be stated as follows.

Theorem C. *Let f be a nonconstant meromorphic function and let F be given by* (1)*, where $n(\geq \Upsilon_Q + 1)$ is an integer. Then one of the following occurs.*

(i) *If $\Gamma_Q > n-1$, then*

$$\begin{aligned} T(r, f) \leq{} & \{1 + 2(\Gamma_Q - n + 1)\}\overline{N}(r, f) \\ & + (\Gamma_Q - n + 2)\overline{N}\left(r, \frac{1}{F}\right) + S(r, f). \end{aligned} \tag{7}$$

Or there exists a small proximity function $a(z)$ of f such that

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{8}$$

and $N(r, a(z)) \leq (\Gamma_Q - n + 1)\{\overline{N}(r, f) + \overline{N}(r, 1/F)\} + S(r, f)$.

(ii) *If $\Gamma_Q \leq n-1$, then*

$$T(r, f) \leq 2\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f), \tag{9}$$

or

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{10}$$

where $a(z)$ is a small function of f.

(iii) *In the special case, if $Q[f] = a_{n-1}f^{n-1} + P[f]$, where $\Gamma_P \leq n-2$, then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f), \tag{11}$$

or

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{12}$$

where $a(z)$ is a small function of f.

Corollary 5. *From Theorem C we know that if condition* (2) *is replaced with "$\overline{N}(r, f) + \overline{N}(r, 1/F) = S(r, f)$" in Theorem A, then the conclusion remains valid.*

In this direction Ren (see [5]) also generalized Tumura-Clunie's theorem concerning differential polynomials.

Combining the methods used in their proofs we show the following theorem.

Theorem 6. *Let f be a nonconstant meromorphic function and let F be given by* (1)*, where $n(\geq \Upsilon_Q + 1)$ is an integer and $\Gamma_F(\neq 2)$ is the weight of F. If*

$$\overline{N}_{(2}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f), \tag{13}$$

then

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{14}$$

where $a(z)$ is a small function of f.

It is easily seen from the following example that $\Gamma_F \neq 2$ in Theorem 6 is necessary.

Example 7. Let $f = \tan z$ and $F = f^2 + 1$. Obviously, (13) is obtained but (14) does not hold.

2. Some Lemmas

To prove our results, we need some lemmas.

Lemma 8 (see [1]). *Let f_1 and f_2 be two nonzero meromorphic functions in the complex plane; then*

$$\begin{aligned} & N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) \\ & \quad = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right). \end{aligned} \tag{15}$$

Lemma 9. *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting functions of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$\begin{aligned} N\left(r, 0; f^{(k)} \mid f \neq 0\right) \leq{} & k\overline{N}(r, f) + N(r, 0; f \mid < k) \\ & + k\overline{N}(r, 0; f \mid \geq k) + S(r, f). \end{aligned} \tag{16}$$

Lemma 10. *Suppose that $Q[f]$ is given in Definition 2. Let z_0 be a pole of f of order p and neither a zero nor a pole of coefficients of $Q[f]$. Then z_0 is a pole of $Q[f]$ of order at most $p\Upsilon_Q + (\Gamma_Q - \Upsilon_Q)$.*

Lemma 11 (see [6]). *Let f be a nonconstant meromorphic function and let $Q[f]$ be given in Definition 2. Then*

$$\begin{aligned} m(r, Q[f]) & \leq \Upsilon_Q m(r, f) + \sum_{j=1}^{n} m\left(r, a_j\right) + S(r, f), \\ N(r, Q[f]) & \leq \Gamma_Q N(r, f) + \sum_{j=1}^{n} N\left(r, a_j\right) + S(r, f). \end{aligned} \tag{17}$$

Lemma 12. *Suppose that f is a nonconstant meromorphic function and $Q[f]$ is given in Definition 2. Then $S(r,Q) = S(r,f)$.*

Proof. It is straightforward by Lemma 11. □

Lemma 13 (see [7]). *Let f be a nonconstant meromorphic function in the complex plane and let $Q_1[f]$ and $Q_2[f]$ be quasi-differential polynomials in f. If $\Upsilon_{Q_2} \leq n$ and $f^n Q_1[f] = Q_2[f]$, then $m(r, Q_1[f]) = S(r,f)$.*

Lemma 14. *Let f be a nonconstant meromorphic function and let $\mathcal{F}$ be given by (1). Then*

$$\left(\Gamma_{\mathcal{F}} - 2\right) N_1(r,f) \leq 2\overline{N}_{(2}(r,f) + 2\overline{N}\left(r, \frac{1}{\mathcal{F}}\right) + S(r,f). \tag{18}$$

Proof. If $\Gamma_{\mathcal{F}} \leq 2$, the conclusion of Lemma 14 holds obviously. In the following we suppose that $\Gamma_{\mathcal{F}} > 2$.

With $\mathcal{F} = f^n + Q[f]$, we set

$$g(z) = \frac{\{\mathcal{F}'\}^{\Gamma_{\mathcal{F}}}}{\{\mathcal{F}\}^{\Gamma_{\mathcal{F}}+1}}. \tag{19}$$

Let z_0 be a simple pole of f and not a zero of coefficients of $Q[f]$; then

$$f(z) = \frac{a}{z - z_0} + O(1), \quad a \neq 0 \text{ as } z \longrightarrow z_0. \tag{20}$$

From Lemma 10 we know that z_0 is a pole of $\mathcal{F}$ of order at most $\Gamma_{\mathcal{F}}$; then we have

$$\mathcal{F}(z) = \frac{b}{(z - z_0)^{\Gamma_{\mathcal{F}}}} + O(1), \qquad \mathcal{F}'(z) = -\frac{b\Gamma_{\mathcal{F}}}{(z - z_0)^{\Gamma_{\mathcal{F}}+1}} + O(1), \tag{21}$$

where $b \neq 0$.

Then

$$\begin{aligned} \mathcal{F}(z) &= \frac{b}{(z - z_0)^{\Gamma_{\mathcal{F}}}} \left\{1 + O(z - z_0)^{\Gamma_{\mathcal{F}}}\right\}, \\ \mathcal{F}'(z) &= -\frac{b\Gamma_{\mathcal{F}}}{(z - z_0)^{\Gamma_{\mathcal{F}}+1}} \left\{1 + O(z - z_0)^{\Gamma_{\mathcal{F}}+1}\right\}, \\ g(z) &= \frac{(-1)^{\Gamma_{\mathcal{F}}} \Gamma_{\mathcal{F}}^{\Gamma_{\mathcal{F}}}}{b} \left\{1 + O(z - z_0)^{\Gamma_{\mathcal{F}}}\right\}. \end{aligned} \tag{22}$$

So $g(z_0) \neq 0, \infty$. But z_0 is a zero of $g'(z)$ of order at least $\Gamma_{\mathcal{F}} - 1$. Then

$$\left(\Gamma_{\mathcal{F}} - 1\right) N_1(r,f) \leq N_0\left(r, \frac{1}{g'}\right), \tag{23}$$

where $N_0(r, 1/g')$ denotes the counting function of the zeros of g', not of g.

By Lemma 8 and Nevanlinna first fundamental theorem, we get

$$\begin{aligned} N\left(r, \frac{g}{g'}\right) - N\left(r, \frac{g'}{g}\right) &= N\left(r, \frac{1}{g'}\right) + N(r,g) - N(r,g') - N\left(r, \frac{1}{g}\right) \\ &= N_0\left(r, \frac{1}{g'}\right) - \overline{N}(r,g) - \overline{N}\left(r, \frac{1}{g}\right), \\ N\left(r, \frac{g}{g'}\right) - N\left(r, \frac{g'}{g}\right) &= m\left(r, \frac{g'}{g}\right) - m\left(r, \frac{g}{g'}\right) + O(1). \end{aligned} \tag{24}$$

From (24), we have

$$\begin{aligned} N_0\left(r, \frac{1}{g'}\right) &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r,g) + m\left(r, \frac{g'}{g}\right) + O(1) \\ &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r,g) + S(r,f). \end{aligned} \tag{25}$$

From (19), we know that the poles and zeros of $g(z)$ can only occur at the multiple zeros of $f(z)$, the zeros of $\mathcal{F}$, and the zeros of $\mathcal{F}'$. Hence

$$\overline{N}(r,g) + \overline{N}\left(r, \frac{1}{g}\right) \leq \overline{N}_{(2}(r,f) + \overline{N}\left(r, \frac{1}{\mathcal{F}}\right) + N_0\left(r, \frac{1}{\mathcal{F}'}\right) + S(r,f), \tag{26}$$

where $N_0(r, 1/\mathcal{F}')$ denotes the counting function of the zeros of $\mathcal{F}'$, not of $\mathcal{F}$.

By Lemmas 9 and 12, we obtain

$$\begin{aligned} N_0\left(r, \frac{1}{\mathcal{F}'}\right) &\leq \overline{N}(r,\mathcal{F}) + \overline{N}\left(r, \frac{1}{\mathcal{F}}\right) + S(r,\mathcal{F}) \\ &\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{\mathcal{F}}\right) + S(r,f), \\ \overline{N}(r,f) &= N_1(r,f) + \overline{N}_{(2}(r,f). \end{aligned} \tag{27}$$

Combining (23), (25), (26), and (27), we obtain (18).

This completes the proof of Lemma 14. □

Proof of Theorem 6. We consider two cases.

Case 1. If $\Gamma_{\mathcal{F}} = 1$, (14) holds obviously.

Case 2. If $\Gamma_{\mathcal{F}} > 2$, by Lemma 14 and (13) we have

$$\begin{aligned} \overline{N}(r,f) &= N_1(r,f) + \overline{N}_{(2}(r,f) \\ &\leq \frac{\Gamma_{\mathcal{F}}}{\Gamma_{\mathcal{F}} - 2} \overline{N}_{(2}(r,f) + \frac{2}{\Gamma_{\mathcal{F}} - 2} \overline{N}\left(r, \frac{1}{\mathcal{F}}\right) + S(r,f) \\ &\leq S(r,f). \end{aligned} \tag{28}$$

This shows that

$$\overline{N}(r,f)=S(r,f). \quad (29)$$

Suppose that $\mathcal{F}\equiv 0$.

So we have $f^n=-Q[f]$ and $Q[f]\not\equiv 0$; moreover $T(r,Q[f])=nT(r,f)+S(r,f)$.

By Lemma 11 we get $m(r,Q[f])\le \Upsilon_Q m(r,f)+S(r,f)$.

On the other hand, we have

$$\begin{aligned} nm(r,f)&=m(r,f^n)=m(r,\mathcal{F}-Q[f])\\ &\le m(r,\mathcal{F})+m(r,Q[f])+S(r,f)\\ &\le \Upsilon_Q m(r,f)+S(r,f). \end{aligned} \quad (30)$$

It follows that $m(r,f)=S(r,f)$, which is impossible.

Therefore, $\mathcal{F}\not\equiv 0$.

Then

$$\begin{aligned} T\left(r,\frac{\mathcal{F}'}{\mathcal{F}}\right)&\le \overline{N}(r,\mathcal{F})+\overline{N}\left(r,\frac{1}{\mathcal{F}}\right)+m\left(r,\frac{\mathcal{F}'}{\mathcal{F}}\right)+S(r,f)\\ &\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{\mathcal{F}}\right)+S(r,f). \end{aligned} \quad (31)$$

From (29) and the condition of the theorem, we know $T(r,\mathcal{F}'/\mathcal{F})=S(r,f)$.

By $\mathcal{F}=f^n+Q[f]$, we have

$$\mathcal{F}'=\frac{\mathcal{F}'}{\mathcal{F}}f^n+\frac{\mathcal{F}'}{\mathcal{F}}Q[f], \qquad \mathcal{F}'=nf^{n-1}f'+Q'[f]. \quad (32)$$

And hence

$$f^{n-1}\left(f\frac{\mathcal{F}'}{\mathcal{F}}-nf'\right)=Q[f]\left(\frac{Q'[f]}{Q[f]}-\frac{\mathcal{F}'}{\mathcal{F}}\right). \quad (33)$$

Let

$$\begin{aligned} \Omega_1[f]&=f\frac{\mathcal{F}'}{\mathcal{F}}-nf',\\ \Omega_2[f]&=Q[f]\left(\frac{Q'[f]}{Q[f]}-\frac{\mathcal{F}'}{\mathcal{F}}\right). \end{aligned} \quad (34)$$

Then

$$f^{n-1}\Omega_1[f]=\Omega_2[f], \quad (35)$$

where $\Omega_1[f]$ and $\Omega_2[f]$ are quasi-differential polynomials.

By Lemma 13 we have

$$m(r,\Omega_1[f])=S(r,f). \quad (36)$$

By Lemma 10 and (35) we obtain

$$\begin{aligned} N(r,\Omega_1[f])&=N(r,\Omega_2[f])-(n-1)N(r,f)+S(r,f)\\ &\le \Upsilon_Q N(r,f)+(\Gamma_Q-\Upsilon_Q+1)\overline{N}(r,f)\\ &\quad -(n-1)N(r,f)+S(r,f)\\ &\le (\Gamma_Q-\Upsilon_Q+1)\overline{N}(r,f)+S(r,f). \end{aligned} \quad (37)$$

Note that $\overline{N}(r,f)=S(r,f)$.

So $T(r,\Omega_1[f])=S(r,f)$.

From (34) we know that $Q[f]$ is a polynomial and $\Upsilon_Q\le n-1$.

Set

$$Q[f]=b(z)f^{n-1}+P[f], \quad (38)$$

where $P[f]$ is a polynomial and $b(z)$ is a small function of f; moreover $\Upsilon_P\le n-2$.

Set $g=f+(b(z)/n)$; we have

$$\mathcal{F}=g^n+R[g], \quad (39)$$

where $R[g]$ is a polynomial and $\Upsilon_R\le n-2$.

Now proceeding as the above proof, we get

$$g^{n-1}\left(g\frac{\mathcal{F}'}{\mathcal{F}}-ng'\right)=R[g]\left(\frac{R'[g]}{R[g]}-\frac{\mathcal{F}'}{\mathcal{F}}\right). \quad (40)$$

By Lemma 13 we obtain

$$\begin{aligned} m\left(r,\left(g\frac{\mathcal{F}'}{\mathcal{F}}-ng'\right)g\right)&=S(r,f),\\ m\left(r,g\frac{\mathcal{F}'}{\mathcal{F}}-ng'\right)&=S(r,f). \end{aligned} \quad (41)$$

Therefore we have

$$\begin{aligned} T\left(r,\left(g\frac{\mathcal{F}'}{\mathcal{F}}-ng'\right)g\right)&=S(r,f),\\ T\left(r,g\frac{\mathcal{F}'}{\mathcal{F}}-ng'\right)&=S(r,f). \end{aligned} \quad (42)$$

Notice that $T(r,g)=T(r,f)+S(r,f)\ne S(r,f)$.

We can get $g(\mathcal{F}'/\mathcal{F})-ng'\equiv 0$.

So $\mathcal{F}\equiv cg^n$, where c is a constant. Obviously $c=1$.

This proves Theorem 6. □

3. Application

Very recently, Yi (see [8, 9]) proved the following result.

Theorem D. *Let f be a transcendental meromorphic function and let $p(z)$ be a polynomial, $p(z)\not\equiv 0$. If f and f' share 0 in $\mathbb{C}$, then $f'-p(z)$ has infinitely many zeros.*

Remark 15. From the hypothesis of Theorem E, it can be easily seen that all zeros of f have multiplicity at least two.

Ren and Yang 2013 (see [10]) obtained the following result.

Theorem E. *Let f be a transcendental meromorphic function and let R be a rational function, $R\not\equiv 0$. Suppose that, with the exception of possibly finitely many, all zeros and poles of f are multiple. Then $f'-R$ has infinitely many zeros.*

It is natural to ask the following question: what can we say if f' is replaced by $f^{(k)}$ and $p(z)$ and R are replaced by a small function relative to f in Theorems D and E?

Later, Yang (see [11]) answered the above question and obtained the following result.

Theorem F. *Let f be a transcendental meromorphic function satisfying*

$$N\left(r, \frac{1}{f}\right) = S(r, f). \tag{43}$$

Then, for any $k \geq 1$ and any small function $a(z)(\not\equiv 0, \infty)$ of f,

$$N\left(r, \frac{1}{f^{(k)} - a(z)}\right) \neq S(r, f). \tag{44}$$

We supplement Theorems D and E, improve Theorem F, and obtain the following result.

Theorem 16. *Let h be a transcendental meromorphic function satisfying*

$$\overline{N}_{(2}\left(r, \frac{1}{h}\right) = S(r, h). \tag{45}$$

Then, for any $n \geq 2$ and any small function $a(z)(\not\equiv 0, \infty)$ of h,

$$N\left(r, \frac{1}{h^{(n)} - a(z)}\right) \neq S(r, h). \tag{46}$$

The method of our proof essentially belongs to Yang. For the completeness, we give the proof here.

Proof. Set

$$h = \frac{1}{f}. \tag{47}$$

Then

$$\begin{gathered} T(r, f) = T(r, h) + O(1), \\ \overline{N}_{(2}\left(r, \frac{1}{h}\right) = \overline{N}_{(2}(r, f). \end{gathered} \tag{48}$$

Obviously

$$S(r, f) = S(r, h). \tag{49}$$

Now

$$\begin{gathered} h'' = \frac{-ff' + 2(f')^2}{f^3}, \\ h''' = \frac{-6(f')^3 - f^2 f'' + 2f(f')^2 + 4ff'f''}{f^4} \cdots. \end{gathered} \tag{50}$$

Thus, in general,

$$h^{(n)} = \frac{Q_n(f)}{f^{n+1}}, \tag{51}$$

where $Q_n(f)$ denotes a homogeneous differential polynomial in f of degree n. So

$$h^{(n)} - a(z) = \frac{Q_n(f) - a(z) f^{n+1}}{f^{n+1}}. \tag{52}$$

If the assertion of the theorem was false, that is,

$$N\left(r, \frac{1}{h^{(n)} - a(z)}\right) = S(r, f), \tag{53}$$

then from (52) we have

$$F = f^{n+1} - \frac{Q_n(f)}{a(z)}. \tag{54}$$

Thus from (48), (53), and (54), we obtain

$$\overline{N}_{(2}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f). \tag{55}$$

Combining Theorem 6, (55) gives

$$F = \left(f + \frac{c}{n+1}\right)^{n+1}, \tag{56}$$

where c (a small function of f) is determined by the two equations: $g = f + (c/(n+1))$ and $cg^n = -(Q_n(g)/a(z))$.

We may claim that

(i) $S(r, f) = S(r, g)$;

(ii) $\overline{N}(r, g) = S(r, g)$;

(iii) $T(r, g^{(k)}/g) = S(r, g)$ for all $k \in \mathbb{N}$.

In fact, from the definition of g we know that the claim (i) above holds.

By (54) we have $\Gamma_F > 2$.

From $g = f + (c/(n+1))$, $\Gamma_F > 2$, and (29) we get

$$\overline{N}(r, g) = \overline{N}(r, f) + \overline{N}(r, c) = S(r, f) = S(r, g). \tag{57}$$

That is, the claim (ii) above holds.

Combining (53) and the claims (i) and (ii), we may deduce

$$\begin{aligned} T\left(r, \frac{g^{(k)}}{g}\right) &= N\left(r, \frac{g^{(k)}}{g}\right) + m\left(r, \frac{g^{(k)}}{g}\right) \\ &\leq k\overline{N}(r, g) + N\left(r, \frac{1}{g}\right) + S(r, g) \\ &\leq k\overline{N}(r, g) + N\left(r, \frac{1}{F}\right) + S(r, g) \\ &\leq S(r, g). \end{aligned} \tag{58}$$

Then the claim (iii) is true also.

Thus, by (54) and (56), we obtain

$$\begin{aligned} &\left(f + \frac{c}{n+1}\right)^{n+1} \\ &\quad = f^{n+1} + cf^n + \sum_{k=2}^{n+1} C_{n+1}^k \left(\frac{c}{n+1}\right)^k f^{n+1-k} \\ &\quad = f^{n+1} - \frac{Q_n(f)}{a(z)}. \end{aligned} \tag{59}$$

Since $cf^n \equiv -(Q_n(f)/a(z))$, it follows that

$$\sum_{k=2}^{n+1} C_{n+1}^{k}\left(\frac{c}{n+1}\right)^{k} f^{n+1-k} \equiv 0, \tag{60}$$

which is impossible unless $c \equiv 0$.

But then, from (59), $-(Q_n(f)/a(z)) \equiv 0$ and we have $h^{(n)} \equiv 0$ which contradicts the fact that h is a transcendental meromorphic function.

This completes the proof of Theorem 16. □

Remark 17. For $n = 1$, from the proof of Theorem 16 and Corollary 5, we know that if the condition "$\overline{N}_{(2}(r, 1/h) = S(r,h)$" is replaced with "$\overline{N}(r, 1/h) = S(r,h)$" in Theorem 16, then the conclusion still holds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1964.

[2] C. C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, vol. 557 of *Mathematics and Its Applications*, Kluwer Academic, New York, NY, USA, 2003.

[3] X. H. Hua, "Some extensions of the Tumura-Clunie theorem," *Complex Variables: Theory and Application*, vol. 16, no. 1, pp. 69–77, 1991.

[4] Z. L. Zhang and W. Li, "Tumura-Clunie's theorem for differential polynomials," *Complex Variables: Theory and Application*, vol. 25, no. 2, pp. 97–105, 1994.

[5] Y. D. Ren, "Solving integral representations problems for the stationary Schrödinger equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 715252, 5 pages, 2013.

[6] W. Doeringer, "Exceptional values of differential polynomials," *Pacific Journal of Mathematics*, vol. 98, no. 1, pp. 55–62, 1982.

[7] J. Clunie, "On integral and meromorphic functions," *Journal of the London Mathematical Society*, vol. 37, pp. 17–27, 1962.

[8] H. X. Yi, "On the theorem of Tumura-Clunie," *Kodai Mathematical Journal*, vol. 12, no. 1, pp. 49–55, 1989.

[9] H. X. Yi, "On a theorem of Tumura and Clunie for a differential polynomial," *Bulletin of the London Mathematical Society*, vol. 20, no. 6, pp. 593–596, 1988.

[10] Y. D. Ren and P. Yang, "Growth estimates for modified Neumann integrals in a half space," *Journal of Inequalities and Applications*, vol. 2013, article 572, 2013.

[11] C. C. Yang, "On the value distribution of a transcendental meromorphic function and its derivatives," *Indian Journal of Pure and Applied Mathematics*, vol. 35, no. 8, pp. 1027–1031, 2004.