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Review Article

Recent Advances in L^p-Theory of Homotopy Operator on Differential Forms

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The purpose of this survey paper is to present an up-to-date account of the recent advances made in the study of L^p-theory of the homotopy operator applied to differential forms. Specifically, we will discuss various local and global norm estimates for the homotopy operator T and its compositions with other operators, such as Green's operator and potential operator.

1. Introduction

The homotopy operator has been playing an important role in the study of L^p-theory of differential forms. We all know that any differential form u can be decomposed as $u = d(Tu) + T(du)$, where d is the differential operator and T is the homotopy operator. Hence, the homotopy operator provides an effective tool to study various properties of different norms and the related operators. As extensions of functions, differential forms have become invaluable tools for many fields of sciences and engineering, including theoretical physics, general relativity, potential theory, and electromagnetism. They can be used to describe various systems of PDEs and to express different geometrical structures on manifolds. In recent years, much progress has been made in the investigation of differential forms and the related operators; see [1–7]. The purpose of this survey paper is to present an up-to-date account of the recent advances made in the study of L^p-theory of the homotopy operator and its compositions applied to differential forms. We will first discuss the L^p-norm and L^φ-norm inequalities in Sections 2 and 3, respectively. Then, we present Lipschitz and BMO norm inequalities in Sections 4 and 5. We also give some global L^φ-inequalities in Section 6. Finally, we discuss the compositions of homotopy operator with the projection operator, potential operator, and Green's operator in Sections 7, 8, and 9. We will keep using the traditional symbols and notations in this survey paper. Specifically, we always assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, B and σB are the balls with the same center and $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$ throughout this paper. We use $|E|$ to denote the n-dimensional Lebesgue measure of a set $\subseteq \mathbb{R}^n$. For a function u, the average of u over B is defined by $u_B = (1/|B|)\int_B u\,dm$. All integrals involved in this paper are the Lebesgue integrals. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e.. Differential forms are extensions of differentiable functions in $\mathbb{R}^n$. For instance, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. A differential 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \ldots, x_n)dx_i$, where the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable. Similarly, a differential k-form $u(x)$ can be expressed as

$$u(x) = \sum_I u_I(x)\, dx_I = \sum u_{i_1 i_2 \ldots i_k}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \tag{1}$$

where $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, let $D'(\Omega, \wedge^l)$ be the space of all differential l-forms in Ω, and let $L^p(\Omega, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)dx_I$ in Ω satisfying $\int_\Omega |u_I(x)|^p dx < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by d and the Hodge star operator by $\star$. The

Hodge codifferential operator $d^\star$ is given by $d^\star = (-1)^{nl+1} \star d \star, l = 1, 2, \ldots, n$.

Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. The following operator K_y with the case $y = 0$ was first introduced by Cartan in [8]. Then, it was extended to the following general version in [9]. For each $y \in D$, there corresponds a linear operator $K_y : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ defined by $(K_y\omega)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})dt$ and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$. A homotopy operator $T : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ is defined by averaging K_y over all points y in D

$$T\omega = \int_D \varphi(y) K_y \omega dy, \tag{2}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)dy = 1$. For simplicity purpose, we write $\xi = (\xi_1, \ldots, \xi_{l-1})$. Then, $T\omega(x;\xi) = \int_0^1 t^{l-1} \int_D \varphi(y)\omega(tx + y - ty; x - y, \xi)dy\, dt$. By substituting $z = tx + y - ty$ and $t = s/(1 + s)$, we have

$$T\omega(x; \xi) = \int_D \omega(z, \zeta(z, x - z), \xi)\, dz, \tag{3}$$

where the vector function $\zeta : D \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $\zeta(z, h) = h \int_0^\infty s^{l-1}(1 + s)^{n-1}\varphi(z - sh)ds$. The integral (3) defines a bounded operator $T : L^s(D, \Lambda^l) \to W^{1,s}(D, \Lambda^{l-1}), l = 1, 2, \ldots, n$, and the decomposition

$$u = d(Tu) + T(du) \tag{4}$$

holds for any differential form u. The l-form $\omega_D \in D'(D, \Lambda^l)$ is defined by

$$\omega_D = \fint_D \omega(y)\, dy = |D|^{-1} \int_D \omega(y)\, dy, \quad l = 0,$$
$$\omega_D = d(T\omega), \quad l = 1, 2, \ldots, n, \tag{5}$$

for all $\omega \in L^p(D, \Lambda^l)$, $1 \le p < \infty$. Also, for any differential form u, we have

$$\|\nabla(Tu)\|_{p,D} \le C|D|\,\|u\|_{p,D},$$
$$\|Tu\|_{p,D} \le C|D| \operatorname{diam}(D)\,\|u\|_{p,D}. \tag{6}$$

From [10, Page 16], we know that any open subset Ω in $\mathbb{R}^n$ is the union of a sequence of cubes Q_k, whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from F. Specifically, (i) $\Omega = \cup_{k=1}^\infty Q_k$, (ii) $Q_j^0 \cap Q_k^0 = \phi$ if $j \neq k$, and (iii) there exist two constants $c_1, c_2 > 0$ (we can take $c_1 = 1$ and $c_2 = 4$), so that $c_1 \operatorname{diam}(Q_k) \le$ distance Q_k from $F \le c_2 \operatorname{diam}(Q_k)$. Thus, the definition of the homotopy operator T can be generalized to any domain Ω in $\mathbb{R}^n$: for any $x \in \Omega$, $x \in Q_k$ for some k. Let T_{Q_k} be the homotopy operator defined on Q_k (each cube is bounded and convex). Thus, we can define the homotopy operator T_Ω on any domain Ω by $T_\Omega = \sum_{k=1}^\infty T_{Q_k}\chi_{Q_k(x)}$.

The nonlinear partial differential equation

$$d^\star A(x, du) = B(x, du) \tag{7}$$

is called nonhomogeneous A-harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1}, \qquad A(x, \xi) \cdot \xi \ge |\xi|^p,$$
$$|B(x, \xi)| \le b|\xi|^{p-1} \tag{8}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (7). A solution to (7) is an element of the Sobolev space $W^{1,p}_{\text{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0 \tag{9}$$

for all $\varphi \in W^{1,p}_{\text{loc}}(\Omega, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, (7) reduces to

$$\operatorname{div} A(x, \nabla u) = B(x, \nabla u). \tag{10}$$

If the operator $B = 0$, (7) becomes

$$d^\star A(x, du) = 0 \tag{11}$$

which is called the (homogeneous) A-harmonic equation. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and (11) becomes the p-harmonic equation $d^\star(du|du|^{p-2}) = 0$ for differential forms. See [1, 11–18] for recent results on the A-harmonic equations and related topics.

Lemma 1 (see [12]). *Let u be a solution of the nonhomogeneous A-harmonic* (7) *in a domain Ω and $0 < s, t < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \le C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{12}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

A continuously increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ is called an Orlicz function. The Orlicz space $L^\varphi(\Omega)$ consists of all measurable functions f on Ω such that $\int_\Omega \varphi(|f|/\lambda)dx < \infty$ for some $\lambda = \lambda(f) > 0$. $L^\varphi(\Omega)$ is equipped with the nonlinear Luxemburg functional

$$\|f\|_{\varphi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx \le 1\right\}. \tag{13}$$

A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_\varphi$ defines a norm in $L^\varphi(\Omega)$, which is called the Luxemburg norm or Orlicz norm.

Definition 2 (see [19]). We say a Young function φ lies in the class $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$, if (i) $1/C \le \varphi(t^{1/p})/g(t) \le C$ and (ii) $1/C \le \varphi(t^{1/q})/h(t) \le C$ for all $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0, \infty)$.

From [19], each of φ, g, and h in the above definition is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$C_1 t^q \le h^{-1}(\varphi(t)) \le C_2 t^q, \qquad C_1 t^p \le g^{-1}(\varphi(t)) \le C_2 t^p, \tag{14}$$

where C_1 and C_2 are constants. Also, for all $1 \le p_1 < p < p_2$ and $\alpha \in \mathbb{R}$, the function $\varphi(t) = t^p \log_+^{\alpha} t$ belongs to $G(p_1, p_2, C)$ for some constant $C = C(p, \alpha, p_1, p_2)$. Here $\log_+(t)$ is defined by $\log_+(t) = 1$ for $t \le e$, and $\log_+(t) = \log(t)$ for $t > e$. Particularly, if $\alpha = 0$, we see that $\varphi(t) = t^p$ lies in $G(p_1, p_2, C)$, $1 \le p_1 < p < p_2$.

Lemma 3 (see [1]). *Let $u \in D'(M, \Lambda^l)$ be a solution to the nonhomogeneous A-harmonic (7) on M and $\sigma > 1$ be a constant. Then there exists a constant C, independent of u, such that*

$$\|du\|_{p,B} \le C \operatorname{diam}(B)^{-1} \|u - c\|_{p,\sigma B} \tag{15}$$

for all balls or cubes B with $\sigma B \subset M$ and all closed forms c. Here $1 < p < \infty$.

Lemma 4 (see [1]). *Suppose that u is a solution to the nonhomogeneous A-harmonic (7) on M, $\sigma > 1$ and $q > 0$. There exists a constant C, depending only on σ, n, p, a, b, and q, such that*

$$\|du\|_{p,Q} \le C|Q|^{(q-p)/pq} \|du\|_{q,\sigma Q} \tag{16}$$

for all Q with $\sigma Q \subset M$.

The following Hölder inequality will be used in this paper.

Lemma 5. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s,E} \le \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E} \tag{17}$$

for any $E \subset \mathbb{R}^n$.

2. L^p-Norm Inequalities

The following L^s-norm Poincaré-type inequality for T was proved in [13].

Theorem 6. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be any differential form in a bounded, convex domain Ω and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Then, there exists a constant C, independent of u, such that*

$$\|T(u) - (T(u))_B\|_{s,B} \le C\,|B| \operatorname{diam}(B) \|u\|_{s,B} \tag{18}$$

for all balls B with $B \subset \Omega$.

Proof. Using (4), (5), and (6), we have

$$\begin{aligned}\|T(u) - (T(u))_B\|_{s,B} &= \|Td(T(u))\|_{s,B} \\ &\le C_1 |B| \operatorname{diam}(B) \|d(Tu)\|_{s,B} \\ &= C_1 |B| \operatorname{diam}(B) \|u_B\|_{s,B} \\ &\le C_2 |B| \operatorname{diam}(B) \|u\|_{s,B}.\end{aligned} \tag{19}$$

We have completed the proof of Theorem 6. □

The basic L^s-norm inequality (18) can be extended into different weighted cases. Let us recall some weight classes as follows. We first introduce the Muckenhoupt weights.

Definition 7. We say the weight $w(x)$ satisfies the $A_r(M)$ condition, $r > 1$, and write $w \in A_r(M)$, if $w(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty \tag{20}$$

for any ball $B \subset M$.

Definition 8. A weight w is called a doubling weight and write $w \in D(\Omega)$ if there exists a constant C such that

$$\mu(2B) \le C\mu(B) \tag{21}$$

for all balls B with $2B \subset \Omega$. Here the measure μ is defined by $d\mu = w(x)dx$. If this condition holds only for all balls B with $4B \subset \Omega$, then w is weak doubling and we write $w \in WD(\Omega)$.

Definition 9. Let $\sigma > 1$. It is said that w satisfies a weak reverse Hölder inequality and write $w \in WRH(\Omega)$ when there exist constants $\beta > 1$ and $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w^\beta dx\right)^{1/\beta} \le C\frac{1}{|B|}\int_{\sigma B} w\,dx \tag{22}$$

for all balls B with $\sigma B \subset \Omega$. We say that w satisfies a reverse Hölder inequality when (22) holds with $\sigma = 1$, and write $w \in RH(\Omega)$. In fact the space $WRH(\Omega)$ is independent of $\sigma > 1$.

If w satisfies the A_r-condition for all balls B with $2B \subset E$, we write $w \in A_r^{\mathrm{loc}}(E)$. Also we write $A_\infty(E) = \cup_{r>1} A_r(E)$ and $A_\infty^{\mathrm{loc}}(E) = \cup_{r>1} A_r^{\mathrm{loc}}(E)$.

It is well known that $w \in A_\infty(\Omega)$ if and only if $w \in RH(\Omega)$. This is also true for $A_\infty^{\mathrm{loc}}(\Omega)$ and $WRH(\Omega)$. Moreover, $A_\infty^{\mathrm{loc}}(\Omega) \subset WD(\Omega)$.

Definition 10. Let w be a locally integrable nonnegative function in $E \subset \mathbb{R}^n$ and assume that $0 < w < \infty$ a.e.. We say that w belongs to the $A_r(\lambda, E)$ class, $1 < r < \infty$, and $0 < \lambda < \infty$ or that w is an $A_r(\lambda, E)$-weight, and write

$w \in A_r(\lambda, E)$ or $w \in A_r(\lambda)$ when it will not cause any confusion, if

$$\sup_B \left(\frac{1}{|B|}\int_B w^\lambda dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1} < \infty. \quad (23)$$

for all balls $B \subset \mathbb{R}^n$.

It is clear that $A_r(1)$ is the usual A_r-class; see [1] for more properties of A_r-weights. We prove some properties of the $A_r(\lambda)$-weights. The following theorem says that $A_r(\lambda)$ is an increasing class with respect to r.

The following result shows that $A_r(\lambda)$-weights have the property similar to the strong doubling property of A_r-weights: if $w \in A_r(\lambda)$, $\lambda \geq 1$, and the measure μ is defined by $d\mu = w(x)dx$, then

$$\frac{|E|^r}{|B|^{\lambda+r-1}} \leq C_{r,\lambda,w}\frac{\mu(E)}{\mu(B)^\lambda}, \quad (24)$$

where B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B.

If we put $\lambda = 1$ (24), then we have

$$\frac{|E|^r}{|B|^r} \leq C_{r,w}\frac{\mu(E)}{\mu(B)} \quad (25)$$

which is called the strong doubling property of A_r-weights. It is well known that an A_r-weight w satisfies the following reverse Hölder inequality.

The definitions of the following several weight classes can be found in [1] and these weight classes have been widely used recently in the study of the integral properties of differential forms.

Definition 11. We say that the weight $w(x) > 0$ satisfies the $A_r^\lambda(E)$-condition, $r > 1$ and $\lambda > 0$, and write $w \in A_r^\lambda(E)$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-r)} dx\right)^{\lambda(r-1)} < \infty \quad (26)$$

for any ball $B \subset E$. Here E is a subset of $\mathbb{R}^n$.

Definition 12. A pair of weights (w_1, w_2) satisfies the $A_{r,\lambda}(E)$-condition in a set $E \subset \mathbb{R}^n$, and write $(w_1, w_2) \in A_{r,\lambda}(E)$, for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$, if

$$\sup_{B\subset E}\left(\frac{1}{|B|}\int_B w_1^\lambda dx\right)^{1/\lambda r}\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'} < \infty. \quad (27)$$

Definition 13. A pair of weights (w_1, w_2) satisfies the $A_r^\lambda(E)$-condition in a set $E \subset \mathbb{R}^n$, and write $(w_1, w_2) \in A_r^\lambda(E)$ for some $r > 1$ and $\lambda > 0$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w_1 dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{\lambda(r-1)} < \infty \quad (28)$$

for any ball $B \subset E$.

Definition 14. A pair of weights (w_1, w_2) satisfies the $A_r(\lambda, E)$-condition in a set $E \subset \mathbb{R}^n$, and write $(w_1, w_2) \in A_r(\lambda, E)$ for some $r > 1$ and $\lambda > 0$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w_1^\lambda dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w_2}\right)^{1/(r-1)} dx\right)^{r-1} < \infty \quad (29)$$

for any ball $B \subset E$.

Using the basic Poincaré-type estimate for the homotopy operator T established in Theorem 6, we have the following $A_r(\Omega)$-weighted inequality.

Theorem 15. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic* (7) *in a bounded domain Ω and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in* (2). *Assume that $\rho > 1$ and $w(x) \in A_r(\Omega)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|T(u) - (T(u))_B\|_{s,B,w^\alpha} \leq C|B| \operatorname{diam}(B)\|u\|_{s,\rho B,w^\alpha} \quad (30)$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha \leq 1$.

The above L^s-norm inequality can also be written in the integral form as

$$\left(\int_B |T(u) - (T(u))_B|^s w^\alpha dx\right)^{1/s} \leq C\left(\int_{\rho B} |u|^s w^\alpha dx\right)^{1/s}. \quad (31)$$

Also, using the procedure developed to extend the local inequalities into the John domains, we have the following global Poincaré-type inequality.

Theorem 16. *Let $u \in D'(\Omega, \wedge^1)$ be a solution of the nonhomogeneous A-harmonic* (7) *and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in* (2). *Assume that $w \in A_r(\Omega)$ for some $1 < r < \infty$ and s is a fixed exponent associated with the A-harmonic* (7). *Then, there exists a constant C, independent of u, such that*

$$\left(\int_\Omega |T(u) - (T(u))_{Q_0}|^s w dx\right)^{1/s} \leq C\left(\int_\Omega |u|^s w dx\right)^{1/s} \quad (32)$$

for any bounded δ-John domain $\Omega \subset \mathbb{R}^n$. Here $Q_0 \subset \Omega$ is a fixed cube.

By the same method used to prove the imbedding inequalities, we can prove the following local and global imbedding inequalities, Theorems 17 and 18, respectively.

Theorem 17. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a smooth differential form in a bounded domain Ω and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in* (2). *Assume that $\rho > 1$ and $w(x) \in A_r(\Omega)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|T(u) - (T(u))_B\|_{W^{1,s}(B),w^\alpha} \leq C|B| \operatorname{diam}(B)\|u\|_{s,\rho B,w^\alpha} \quad (33)$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha \leq 1$.

Theorem 18. *Let $u \in D'(\Omega, \wedge^1)$ be a solution of the nonhomogeneous A-harmonic (7) and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in (2). Assume that $w \in A_r(\Omega)$ for some $1 < r < \infty$ and s is a fixed exponent associated with the A-harmonic (7). Then, there exists a constant C, independent of u, such that*

$$\left\|T(u) - (T(u))_{Q_0}\right\|_{W^{1,s}(\Omega),w} \leq C\|u\|_{s,\Omega,w} \quad (34)$$

for any bounded δ-John domain $\Omega \subset \mathbb{R}^n$. Here $Q_0 \subset \Omega$ is a fixed cube.

So far, we have presented the $A_r(\Omega)$-weighted Poincaré-type estimates for the homotopy operator T. Now, we state other estimates with different weights, such as $A_r(\lambda, \Omega)$-weights and $A_r^\lambda(\Omega)$-weights.

Theorem 19. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying the nonhomogeneous A-harmonic (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Assume that $w \in A_r(\lambda, \Omega)$ for some $r > 1$ and $\lambda > 0$. Then, there exists a constant C, independent of u, such that*

$$\left\|T(u) - (T(u))_B\right\|_{s,B,w^{\alpha\lambda}} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w^\alpha}, \quad (35)$$

$$\left\|T(u) - (T(u))_B\right\|_{W^{1,s}(B),w^{\alpha\lambda}} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w^\alpha} \quad (36)$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$. Here $\rho > 1$ is some constant.

Note that inequality (35) can be written as

$$\left(\int_B \left|T(u) - (T(u))_B\right|^s w^{\alpha\lambda} dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\rho B} |u|^s w^\alpha dx\right)^{1/s}. \quad (37)$$

Theorem 20. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Assume that $\rho > 1$ and $w \in A_r^\lambda(\Omega)$ for some $r > 1$ and $\lambda > 0$. Then, there exists a constant C, independent of u, such that*

$$\left\|T(u) - (T(u))_B\right\|_{s,B,w^\alpha} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w^{\alpha\lambda}} \quad (38)$$

$$\left\|T(u) - (T(u))_B\right\|_{W^{1,s}(B),w^\alpha} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w^{\alpha\lambda}} \quad (39)$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

The above inequalities have integral representations; for example, inequality (38) can be written as

$$\left(\int_B \left|T(u) - (T(u))_B\right|^s w^\alpha dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\rho B} |u|^s w^{\alpha\lambda} dx\right)^{1/s}. \quad (40)$$

The above estimates can be extended into the following two-weight case.

Theorem 21. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Suppose that $\rho > 1$ and $(w_1, w_2) \in A_r(\lambda, \Omega)$ for some $\lambda > 0$ and $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B \left|T(u) - (T(u))_B\right|^s w_1^{\alpha\lambda} dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\rho B} |u|^s w_2^\alpha dx\right)^{1/s} \quad (41)$$

$$\left\|T(u) - (T(u))_B\right\|_{W^{1,s}(B),w_1^{\alpha\lambda}} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w_2^\alpha}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

In Theorem 21, we have assumed that $(w_1, w_2) \in A_r(\lambda, \Omega)$. If the weights w_1 and w_2 satisfy some other condition, say $(w_1, w_2) \in A_{r,\lambda}(\Omega)$, we have the following version of Poincaré-type inequality.

Theorem 22. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Suppose that $\rho > 1$ and $(w_1, w_2) \in A_{r,\lambda}(\Omega)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B \left|T(u) - (T(u))_B\right|^s w_1^\alpha dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\rho B} |u|^s w_2^\alpha dx\right)^{1/s}, \quad (*)$$

$$\left\|T(u) - (T(u))_B\right\|_{W^{1,s}(B),w_1^\alpha} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w_2^\alpha} \quad (42)$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < \lambda$.

Note that inequality $(*)$ can be written as

$$\left\|T(u) - (T(u))_B\right\|_{s,B,w_1^\alpha} \leq C|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w_2^\alpha}. \quad (*)'$$

Similarly, if $(w_1, w_2) \in A_r^\lambda(\Omega)$, we have the following version of two-weight Poincaré inequality for differential forms.

Theorem 23. *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Suppose that $(w_1, w_2) \in A^\lambda_r(\Omega)$ for some $r > 1$ and $\lambda > 0$. If $0 < \alpha < 1$ and $\sigma > 1$, then there exists a constant C, independent of u, such that*

$$\left(\int_B |T(u) - (T(u))_B|^s w_1^\alpha dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\sigma B} |u|^s w_2^{\alpha\lambda} dx\right)^{1/s}, \tag{43}$$

$$\|T(u) - (T(u))_B\|_{W^{1,s}(B), w_1^\alpha} \leq C|B| \operatorname{diam}(B) \|u\|_{s, \sigma B, w_2^{\alpha\lambda}}$$

for all balls B with $\sigma B \subset \Omega$.

If we choose $\lambda = 1/\alpha$ in Theorem 23, we have the following version of the Poincaré inequality with $(w_1, w_2) \in A^{1/\alpha}_r(\Omega)$.

Corollary 24. *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Suppose that $(w_1, w_2) \in A^{1/\alpha}_r(\Omega)$ for some $r > 1$. If $0 < \alpha < 1$ and $\sigma > 1$, then there exists a constant C, independent of u, such that*

$$\left(\int_B |T(u) - (T(u))_B|^s w_1^\alpha dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\sigma B} |u|^s w_2 dx\right)^{1/s} \tag{44}$$

for all balls B with $\sigma B \subset \Omega$.

Choosing $\alpha = 1/s$ in Theorem 23, we obtain the following two-weighted Poincaré inequality.

Corollary 25. *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Suppose that $(w_1, w_2) \in A^\lambda_r(\Omega)$ for some $r > 1$, $\lambda > 0$ and $\sigma > 1$, then there exists a constant C, independent of u, such that*

$$\left(\int_B |T(u) - (T(u))_B|^s w_1^{1/s} dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\sigma B} |u|^s w_2^{\lambda/s} dx\right)^{1/s} \tag{45}$$

for all balls B with $\sigma B \subset \Omega$.

Letting $\lambda = 1$ in Corollary 25, we find the following symmetric two-weighted inequality.

Corollary 26. *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Suppose that $(w_1, w_2) \in A_r(\Omega)$ for some $r > 1$ and $\sigma > 1$, then there exists a constant C, independent of u, such that*

$$\left(\int_B |T(u) - (T(u))_B|^s w_1^{1/s} dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\sigma B} |u|^s w_2^{1/s} dx\right)^{1/s} \tag{46}$$

for all balls B with $\sigma B \subset \Omega$.

Finally, when $\lambda = s$ in Theorem 23, we have the following two-weighted inequality.

Corollary 27. *Let $u \in L^s_{loc}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator defined in (2). Suppose that $(w_1, w_2) \in A^s_r(\Omega)$ for some $r > 1$. If $0 < \alpha < 1$ and $\sigma > 1$, then there exists a constant C, independent of u, such that*

$$\left(\int_B |T(u) - (T(u))_B|^s w_1^\alpha dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\sigma B} |u|^s w_2^{\alpha s} dx\right)^{1/s}, \tag{47}$$

$$\|T(u) - (T(u))_B\|_{W^{1,s}(B), w_1^\alpha} \leq C|B| \operatorname{diam}(B) \|u\|_{s, \sigma B, w_2^{\alpha s}}$$

for all balls B with $\sigma B \subset \Omega$.

3. L^φ-Norm Inequalities

The following local Poincaré-type inequality with the L^φ-norm was proved in [13], which can be used to establish the global inequality.

Theorem 28. *Let φ be a Young function in the class $G(p, q, C)$, $1 \leq p < q < \infty$, $C \geq 1$, Ω be a bounded and convex domain, and let $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in (2). Assume that $\varphi(|u|) \in L^1_{loc}(\Omega, m)$ and u is a solution of the nonhomogeneous A-harmonic (7) in Ω. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi(|T(u) - (T(u))_B|) \, dm \leq C \int_{\sigma B} \varphi(|u|) \, dm \tag{48}$$

for all balls B with $\sigma B \subset \Omega$.

Proof. From (18), we have

$$\|T(u) - (T(u))_B\|_{q,B} \leq C_1 |B|^{1+1/n} \|u\|_{q,B} \tag{49}$$

for all balls B with $B \subset \Omega$. From Lemma 1, for any positive numbers p and q, it follows that

$$\left(\int_B |u|^q dm\right)^{1/q} \leq C_2 |B|^{(p-q)/pq} \left(\int_{\sigma B} |u|^p dm\right)^{1/p}, \tag{50}$$

where σ is a constant $\sigma > 1$. Using Jensen's inequality for h^{-1}, (14), (49), (50), and (i) in Definition 2, and noticing the fact that φ and h are doubling and φ is an increasing function, we obtain

$$\begin{aligned}
&\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right) dm \\
&= h\left(h^{-1}\left(\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right) dm\right)\right) \\
&\leq h\left(\int_B h^{-1}\left(\varphi\left(\left|T(u)-(T(u))_B\right|\right)\right) dm\right) \\
&\leq h\left(C_3 \int_B \left|T(u)-(T(u))_B\right|^q dm\right) \\
&\leq C_4 \varphi\left(\left(C_3 \int_B \left|T(u)-(T(u))_B\right|^q dm\right)^{1/q}\right) \\
&\leq C_4 \varphi\left(C_5 |B|^{1+1/n}\left(\int_B |u|^q dm\right)^{1/q}\right) \\
&\leq C_4 \varphi\left(C_6 |B|^{1+1/n+(p-q)/pq}\left(\int_{\sigma B} |u|^p dm\right)^{1/p}\right) \\
&\leq C_4 \varphi\left(\left(C_6^p |B|^{p(1+1/n)+(p-q)/q} \int_{\sigma B} |u|^p dm\right)^{1/p}\right) \\
&\leq C_7 g\left(C_6^p |B|^{p(1+1/n)+(p-q)/q} \int_{\sigma B} |u|^p dm\right) \\
&= C_7 g\left(\int_{\sigma B} C_6^p |B|^{p(1+1/n)+(p-q)/q} |u|^p dm\right) \\
&\leq C_7 \int_{\sigma B} g\left(C_6^p |B|^{p(1+1/n)+(p-q)/q} |u|^p\right) dm \\
&\leq C_8 \int_{\sigma B} \varphi\left(C_6 |B|^{1+(1/n)+((p-q)/pq)} |u|\right) dm.
\end{aligned} \tag{51}$$

Since $p \geq 1$, then $1+(1/n)+((p-q)/pq) > 0$. Hence, we have $|B|^{1+(1/n)+((p-q)/pq)} \leq |\Omega|^{1+(1/n)+((p-q)/pq)} \leq C_5$. Note that φ is doubling, we obtain

$$\varphi\left(C_6 |B|^{1+(1/n)+((p-q)/pq)} |u|\right) \leq C_9 \varphi(|u|). \tag{52}$$

Combining (51) and (52) yields

$$\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right) dm \leq C_{10} \int_{\sigma B} \varphi(|u|) dm. \tag{53}$$

We have completed the proof of Theorem 28. □

Since each of φ, g, and h in Definition 2 is doubling, from the proof of Theorem 28 or directly from (48), we have

$$\int_B \varphi\left(\frac{\left|T(u)-(T(u))_B\right|}{\lambda}\right) dm \leq C \int_{\sigma B} \varphi\left(\frac{|u|}{\lambda}\right) dm \tag{54}$$

for all balls B with $\sigma B \subset \Omega$ and any constant $\lambda > 0$. From (13) and (54), the following Poincaré inequality with the Luxemburg norm

$$\left\|T(u)-(T(u))_B\right\|_{\varphi(B)} \leq C\|u\|_{\varphi(\sigma B)} \tag{55}$$

holds under the conditions described in Theorem 28.

Theorem 29. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, $q(n-p) < np$, Ω be a bounded domain, and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in (2). Assume that $u \in D'(\Omega, \wedge^l)$ is any differential l-form, $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega, m)$. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right) dm \leq C \int_B \varphi(|u|) dm \tag{56}$$

for all balls B with $B \subset \Omega$.

Proof. From (53), we have

$$\begin{aligned}
&\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right) dm \\
&\qquad \leq C_1 \varphi\left(\left(\int_B \left|T(u)-(T(u))_B\right|^q dm\right)^{1/q}\right).
\end{aligned} \tag{57}$$

If $1 < p < n$, by assumption, we have $q < np/(n-p)$. Using the Poincaré-type inequality for differential forms $T(u)$

$$\begin{aligned}
&\left(\int_B \left|T(u)-(T(u))_B\right|^{np/(n-p)} dm\right)^{(n-p)/np} \\
&\qquad \leq C_2 \left(\int_B |d(T(u))|^p dm\right)^{1/p},
\end{aligned} \tag{58}$$

we find that

$$\left(\int_B \left|T(u)-(T(u))_B\right|^q dm\right)^{1/q} \leq C_3 \left(\int_B |d(T(u))|^p dm\right)^{1/p}. \tag{59}$$

We all know that for any differential form u, $d(T(u)) = u_B$, and $\|u_B\|_{p,B} \leq C_4 \|u\|_{p,B}$. Hence,

$$\left(\int_B |d(T(u))|^p dm\right)^{1/p} \leq C_5 \left(\int_B |u|^p dm\right)^{1/p}. \tag{60}$$

Combining (57), (59), and (60), we obtain

$$\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right) dm \leq C_1 \varphi\left(C_6 \left(\int_B |u|^p dm\right)^{1/p}\right) \tag{61}$$

for $1 < p < n$. Note that the L^p-norm of $|T(u)-(T(u))_B|$ increases with p and $np/(n-p) \to \infty$ as $p \to n$, it follows that (59) still holds when $p \geq n$. Since φ is increasing, from (57) and (59), we obtain

$$\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right) dm \leq C_1 \varphi\left(C_6 \left(\int_B |u|^p dm\right)^{1/p}\right). \tag{62}$$

Applying (62), (i) in Definition 2, Jensen's inequality, and noticing that φ and g are doubling, we have

$$\begin{aligned}\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right)dm &\le C_1\varphi\left(C_6\left(\int_B |u|^p dm\right)^{1/p}\right)\\ &\le C_1 g\left(C_7\left(\int_B |u|^p dm\right)\right)\\ &\le C_8\int_B g\left(|u|^p\right)dm.\end{aligned} \tag{63}$$

Using (i) in Definition 2 again yields

$$\int_B g\left(|u|^p\right)dm \le C_9\int_B \varphi(|u|)\,dm. \tag{64}$$

Combining (63) and (64), we obtain

$$\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right)dm \le C_{10}\int_B \varphi(|u|)\,dm. \tag{65}$$

The proof of Theorem 29 has been completed. □

Similar to (55), from (18) and (56), the following Orlicz norm inequality

$$\left\|T(u)-(T(u))_B\right\|_{\varphi(B)} \le C\|u\|_{\varphi(B)} \tag{66}$$

holds if all conditions of Theorem 29 are satisfied.

4. Lipschitz and BMO Norm Inequalities

In this section, we will present Lipschitz and BMO norm inequalities for the homotopy operator. All results presented in this section and next section can be found in [14]. Let us recall the definitions of Lipschitz and BMO norms first.

Let $\omega \in L^1_{\mathrm{loc}}(M,\wedge^l)$, $l = 0,1,\dots,n$. We write $\omega \in \mathrm{locLip}_k(M,\wedge^l)$, $0\le k\le 1$, if

$$\|\omega\|_{\mathrm{locLip}_k,M} = \sup_{\sigma Q\subset M}|Q|^{-(n+k)/n}\left\|\omega-\omega_Q\right\|_{1,Q} < \infty \tag{67}$$

for some $\sigma \ge 1$. Further, we write $\mathrm{lip}_k(M,\wedge^l)$ for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\|\omega\|_{\mathrm{lip}_k,M}$ for this norm. Similarly, for $\omega\in L^1_{\mathrm{loc}}(M,\wedge^l)$, $l=0,1,\dots,n$, we write $\omega\in \mathrm{BMO}(M,\wedge^l)$ if

$$\|\omega\|_{\star,M} = \sup_{\sigma Q\subset M}|Q|^{-1}\left\|\omega-\omega_Q\right\|_{1,Q} < \infty \tag{68}$$

for some $\sigma \ge 1$. When ω is a 0-form, (68) reduces to the classical definition of $\mathrm{BMO}(M)$. The definitions of the above Lipschitz and BMO norms can be found in [1].

The following Theorem 30 indicates that we can use the L^s-norm of u to estimate the Lipschitz norm of $T(u)$.

Theorem 30. *Let $u\in L^s(M,\wedge^l)$, $l=1,2,\dots,n$, $1<s<\infty$, be a solution of the A-harmonic* (1) *in a bounded, convex domain M and let $T: C^\infty(M,\wedge^l)\to C^\infty(M,\wedge^{l-1})$ be the homotopy operator defined in* (7). *Then, there exists a constant C, independent of u, such that*

$$\|T(u)\|_{\mathrm{locLip}_k,M} \le C\|u\|_{s,M}, \tag{69}$$

where k is a constant with $0\le k\le 1$.

Proof. From Theorem 6, we have

$$\left\|T(u)-(T(u))_B\right\|_{s,B} \le C_1|B|\,\mathrm{diam}(B)\,\|u\|_{s,\sigma B} \tag{70}$$

for all balls B with $\sigma B\subset M$, where $\sigma>1$ is a constant. Using the Hölder inequality with $1=1/s+(s-1)/s$, we find that

$$\begin{aligned}\left\|T(u)-(T(u))_B\right\|_{1,B} &= \int_B\left|T(u)-(T(u))_B\right|dx\\ &\le\left(\int_B\left|T(u)-(T(u))_B\right|^s dx\right)^{1/s}\\ &\quad\times\left(\int_B 1^{s/(s-1)}dx\right)^{(s-1)/s}\\ &= |B|^{(s-1)/s}\left\|T(u)-(T(u))_B\right\|_{s,B}\\ &= |B|^{1-1/s}\left\|T(u)-(T(u))_B\right\|_{s,B}\\ &\le |B|^{1-1/s}\left(C_1|B|\,\mathrm{diam}(B)\,\|u\|_{s,\sigma B}\right)\\ &\le C_2|B|^{2-1/s+1/n}\|u\|_{s,\sigma B}.\end{aligned} \tag{71}$$

Using the definition of the Lipschitz norm, (71), and $2-1/s+1/n-1-k/n = 1-1/s+1/n-k/n>0$, we obtain

$$\begin{aligned}\|T(u)\|_{\mathrm{locLip}_k,M} &= \sup_{\sigma B\subset M}|B|^{-(n+k)/n}\left\|T(u)-(T(u))_B\right\|_{1,B}\\ &= \sup_{\sigma B\subset M}|B|^{-1-k/n}\left\|T(u)-(T(u))_B\right\|_{1,B}\\ &\le \sup_{\sigma B\subset M}|B|^{-1-k/n}C_2|B|^{2-1/s+1/n}\|u\|_{s,\sigma B}\\ &= \sup_{\sigma B\subset M}C_2|B|^{1-1/s+1/n-k/n}\|u\|_{s,\sigma B}\\ &\le \sup_{\sigma B\subset M}C_2|M|^{1-1/s+1/n-k/n}\|u\|_{s,\sigma B}\\ &\le C_3\sup_{\sigma B\subset M}\|u\|_{s,\sigma B}\\ &\le C_3\|u\|_{s,M}.\end{aligned} \tag{72}$$

The proof of Theorem 30 has been completed. □

Using the similar method involved in the proof of Theorem 30, we have the following Lipschitz norm inequalities for Green's operator G and the projection operator H;

see [1] for more properties about Green's operator G and the projection operator H.

Theorem 31. *Let $u \in L^s(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n-1$, $1 < s < \infty$, be a solution of the A-harmonic (7) in a bounded domain Ω, and let G be Green's operator and let H be the projection operator. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\mathrm{locLip}_k,\Omega} \le C\|du\|_{s,\Omega},$$
$$\|H(u)\|_{\mathrm{locLip}_k,\Omega} \le C\|du\|_{s,\Omega}, \tag{73}$$

where k is a constant with $0 \le k \le 1$.

We have discussed some estimates for the Lipschitz norm $\|\cdot\|_{\mathrm{locLip}_k,\Omega}$ above. Next, we will focus on the estimates for the BMO norm $\|\cdot\|_{\star,\Omega}$. For this, let $u \in \mathrm{locLip}_k(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, $0 \le k \le 1$, and let Ω be a bounded domain. Then, from the definitions of the Lipschitz and BMO norms, we have

$$\begin{aligned}
&\|u\|_{\star,\Omega}\\
&\quad = \sup_{\sigma B\subset\Omega} |B|^{-1}\|u-u_B\|_{1,B}\\
&\quad = \sup_{\sigma B\subset\Omega} |B|^{k/n}|B|^{-(n+k)/n}\|u-u_B\|_{1,B}\\
&\quad \le \sup_{\sigma B\subset\Omega} |\Omega|^{k/n}|B|^{-(n+k)/n}\|u-u_B\|_{1,B}\\
&\quad \le |\Omega|^{k/n}\sup_{\sigma B\subset\Omega} |B|^{-(n+k)/n}\|u-u_B\|_{1,B}\\
&\quad \le C_1\sup_{\sigma B\subset\Omega} |B|^{-(n+k)/n}\|u-u_B\|_{1,B}\\
&\quad \le C_1\|u\|_{\mathrm{locLip}_k,\Omega},
\end{aligned} \tag{74}$$

where C_1 is a positive constant. Hence, we have proved the following inequality between the Lipschitz norm and the BMO norm.

Theorem 32. *If a differential form $u \in \mathrm{locLip}_k(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, $0 \le k \le 1$, in a bounded domain Ω, then $u \in \mathrm{BMO}(\Omega, \wedge^l)$ and*

$$\|u\|_{\star,\Omega} \le C\|u\|_{\mathrm{locLip}_k,\Omega}, \tag{75}$$

where C is a constant.

Using Theorems 32 and 30, we obtain the following inequality between the BMO norm and the L^s norm.

Theorem 33. *Let $u \in L^s(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the A-harmonic (7) in a bounded, convex domain M and let $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be the homotopy operator defined in (2). Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{\star,M} \le C\|u\|_{s,M}. \tag{76}$$

Proof. Since inequality (75) holds for any differential form, we may replace u by Tu in inequality (75). Thus, it follows that

$$\|Tu\|_{\star,M} \le C_1\|Tu\|_{\mathrm{locLip}_k,M}, \tag{77}$$

where k is a constant with $0 \le k \le 1$. On the other hand, from Theorem 30 we have

$$\|T(u)\|_{\mathrm{locLip}_k,M} \le C_2\|u\|_{s,M}. \tag{78}$$

Combination of (77) and (78) yields $\|Tu\|_{\star,M} \le C_3\|u\|_{s,M}$. The proof of Theorem 33 has been completed. □

As in the proof of Theorem 33, using inequality (75) and Theorem 31, we obtain the following result immediately.

Theorem 34. *Let $u \in L^s(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n-1$, $1 < s < \infty$, be a solution of the A-harmonic (7) in a bounded domain Ω, and let G be Green's operator and let H be the projection operator. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\star,\Omega} \le C\|du\|_{s,\Omega},$$
$$\|H(u)\|_{\star,\Omega} \le C\|du\|_{s,\Omega}. \tag{79}$$

5. Weighted Lipschitz and BMO Norm Inequalities

In this section, we present the weighted Lipschitz and BMO norms inequalities. For $\omega \in L^1_{\mathrm{loc}}(\Omega, \wedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $\omega \in \mathrm{locLip}_k(\Omega, \wedge^l, w^\alpha)$, $0 \le k \le 1$, if

$$\|\omega\|_{\mathrm{locLip}_k,\Omega,w^\alpha} = \sup_{\sigma Q\subset\Omega} (\mu(Q))^{-(n+k)/n}\|\omega-\omega_Q\|_{1,Q,w^\alpha} < \infty \tag{80}$$

for some $\sigma > 1$, where Ω is a bounded domain, the measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight, and α is a real number. For convenience, we will write the following simple notation $\mathrm{locLip}_k(\Omega, \wedge^l)$ for $\mathrm{locLip}_k(\Omega, \wedge^l, w^\alpha)$. Similarly, for $\omega \in L^1_{\mathrm{loc}}(\Omega, \wedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we will write $\omega \in \mathrm{BMO}(\Omega, \wedge^l, w^\alpha)$ if

$$\|\omega\|_{\star,\Omega,w^\alpha} = \sup_{\sigma Q\subset\Omega} (\mu(Q))^{-1}\|\omega-\omega_Q\|_{1,Q,w^\alpha} < \infty \tag{81}$$

for some $\sigma > 1$, where the measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight, and α is a real number. Again, we will write $\mathrm{BMO}(\Omega, \wedge^l)$ to replace $\mathrm{BMO}(\Omega, \wedge^l, w^\alpha)$ when it is clear that the integral is weighted.

Theorem 35. *Let $u \in L^s(M, \wedge^l, \mu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic (7) in a bounded, convex domain M and let T be the homotopy operator defined in (2), where the measure μ is defined by $d\mu = w^\alpha dx$ and $w \in A_r(M)$ for some $r > 1$ with $w(x) \ge \varepsilon > 0$ for any $x \in M$. Then, there exists a constant C, independent of u, such that*

$$\|T(u)\|_{\mathrm{locLip}_k,M,w^\alpha} \le C\|u\|_{s,M,w^\alpha}, \tag{82}$$

where k and α are constants with $0 \le k \le 1$ and $0 < \alpha \le 1$.

Proof. First, we note that $\mu(B) = \int_B w^\alpha dx \geq \int_B \varepsilon^\alpha dx = C_1|B|$, which implies that

$$\frac{1}{\mu(B)} \leq \frac{C_2}{|B|} \tag{83}$$

for any ball B. Using (30) and the Hölder inequality with $1 = 1/s + (s-1)/s$, we find that

$$\begin{aligned}
&\|T(u) - (T(u))_B\|_{1,B,w^\alpha} \\
&= \int_B |T(u) - (T(u))_B| \, d\mu \\
&\leq \left(\int_B |T(u) - (T(u))_B|^s d\mu\right)^{1/s} \left(\int_B 1^{s/(s-1)} d\mu\right)^{(s-1)/s} \\
&= (\mu(B))^{(s-1)/s} \|T(u) - (T(u))_B\|_{s,B,w^\alpha} \\
&= (\mu(B))^{1-1/s} \|T(u) - (T(u))_B\|_{s,B,w^\alpha} \\
&\leq (\mu(B))^{1-1/s} \left(C_3 |B| \operatorname{diam}(B) \|u\|_{s,\sigma B,w^\alpha}\right) \\
&\leq C_4 (\mu(B))^{1-1/s} |B|^{1+1/n} \|u\|_{s,\sigma B,w^\alpha}.
\end{aligned} \tag{84}$$

Next, from the definition of the weighted Lipschitz norm, (80), and (84), we obtain

$$\begin{aligned}
&\|T(u)\|_{\mathrm{locLip}_k,M,w^\alpha} \\
&\quad = \sup_{\sigma B \subset M} (\mu(B))^{-(n+k)/n} \|T(u) - (T(u))_B\|_{1,B,w^\alpha} \\
&\quad = \sup_{\sigma B \subset M} (\mu(B))^{-1-k/n} \|T(u) - (T(u))_B\|_{1,B,w^\alpha} \\
&\quad \leq C_5 \sup_{\sigma B \subset M} (\mu(B))^{-1/s-k/n} |B|^{1+1/n} \|u\|_{s,\sigma B,w^\alpha} \\
&\quad \leq C_6 \sup_{\sigma B \subset M} |B|^{-1/s-k/n+1+1/n} \|u\|_{s,\sigma B,w^\alpha} \\
&\quad \leq C_6 \sup_{\sigma B \subset M} |M|^{-1/s-k/n+1+1/n} \|u\|_{s,\sigma B,w^\alpha} \\
&\quad \leq C_6 |M|^{-1/s-k/n+1+1/n} \sup_{\sigma B \subset M} \|u\|_{s,\sigma B,w^\alpha} \\
&\quad \leq C_7 \|u\|_{s,M,w^\alpha}
\end{aligned} \tag{85}$$

since $-1/s - k/n + 1 + 1/n > 0$ and $|M| < \infty$. We have completed the proof of Theorem 35. □

Next, we present the $\|\cdot\|_{\star,\Omega,w^\alpha}$ norm estimate. Let $u \in \mathrm{locLip}_k(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, $0 \leq k \leq 1$, in a bounded domain Ω. From the definitions of the weighted Lipschitz and the weighted BMO norms, we have

$$\begin{aligned}
&\|u\|_{\star,\Omega,w^\alpha} \\
&\quad = \sup_{\sigma B \subset \Omega} (\mu(B))^{-1} \|u - u_B\|_{1,B,w^\alpha} \\
&\quad = \sup_{\sigma B \subset \Omega} (\mu(B))^{k/n} (\mu(B))^{-(n+k)/n} \|u - u_B\|_{1,B,w^\alpha} \\
&\quad \leq \sup_{\sigma B \subset \Omega} (\mu(\Omega))^{k/n} (\mu(B))^{-(n+k)/n} \|u - u_B\|_{1,B,w^\alpha} \\
&\quad \leq (\mu(\Omega))^{k/n} \sup_{\sigma B \subset \Omega} (\mu(B))^{-(n+k)/n} \|u - u_B\|_{1,B,w^\alpha} \\
&\quad \leq C_1 \sup_{\sigma B \subset \Omega} (\mu(B))^{-(n+k)/n} \|u - u_B\|_{1,B,w^\alpha} \\
&\quad \leq C_1 \|u\|_{\mathrm{locLip}_k,\Omega,w^\alpha},
\end{aligned} \tag{86}$$

where C_1 is a positive constant. Hence, we have obtained the following theorem.

Theorem 36. *Let $u \in \mathrm{locLip}_k(\Omega, \wedge^l, \mu)$, $l = 0, 1, \ldots, n$, $0 \leq k \leq 1$, be any differential form in a bounded domain Ω, where $w \in A_r(\Omega)$ is a weight for some $r > 1$. Then, $u \in \mathrm{BMO}(\Omega, \wedge^l, w^\alpha)$ and*

$$\|u\|_{\star,\Omega,w^\alpha} \leq C \|u\|_{\mathrm{locLip}_k,\Omega,w^\alpha}, \tag{87}$$

where C and α are constants with $0 < \alpha \leq 1$.

Theorem 37. *Let $u \in L^s(M, \wedge^l, \mu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic (7) in a bounded, convex domain M and let T be the homotopy operator defined in (2), where the measure μ is defined by $d\mu = w^\alpha dx$ and $w \in A_r(M)$ for some $r > 1$ with $w(x) \geq \varepsilon > 0$ for any $x \in M$. Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{\star,M,w^\alpha} \leq C \|u\|_{s,M,w^\alpha}, \tag{88}$$

where α is a constant with $0 < \alpha \leq 1$.

Proof. Replacing u by Tu in Theorem 36, we have

$$\|Tu\|_{\star,M,w^\alpha} \leq C_1 \|Tu\|_{\mathrm{locLip}_k,M,w^\alpha}, \tag{89}$$

where k is a constant with $0 \leq k \leq 1$. Now, from Theorem 35, we find that

$$\|T(u)\|_{\mathrm{locLip}_k,M,w^\alpha} \leq C_2 \|u\|_{s,M,w^\alpha}. \tag{90}$$

Substituting (90) into (89), we obtain $\|Tu\|_{\star,M,w^\alpha} \leq C_3 \|u\|_{s,M,w^\alpha}$. The proof of Theorem 37 has been completed. □

6. Global L^φ-Inequalities

In this section, we discuss the global inequalities in the following $L^\varphi(m)$-averaging domains. See [13] for detailed proofs.

Definition 38 (see [20]). Let φ be an increasing convex function on $[0,\infty)$ with $\varphi(0) = 0$. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L^\varphi(m)$-averaging domain, if $m(\Omega) < \infty$ and there exists a constant C such that

$$\int_\Omega \varphi\left(\tau\left|u-u_{B_0}\right|\right)dm \le C\sup_{B\subset\Omega}\int_B \varphi\left(\sigma\left|u-u_B\right|\right)dm \quad (91)$$

for some ball $B_0 \subset \Omega$ and all u such that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega, m)$, where τ, σ are constants with $0 < \tau < \infty$, $0 < \sigma < \infty$ and the supremum is over all balls $B \subset \Omega$.

From the above definition, we see that L^s-averaging domains and $L^s(m)$-averaging domains are special $L^\varphi(m)$-averaging domains when $\varphi(t) = t^s$ in Definition 38. Also, uniform domains and John domains are very special $L^\varphi(m)$-averaging domains; see [20, 21] for more results about domains.

Theorem 39. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, and let Ω be any bounded $L^\varphi(m)$-averaging domain and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in* (2)*. Assume that $\varphi(|u|) \in L^1(\Omega, m)$ and $u \in D'(\Omega, \wedge^1)$ is a solution of the nonhomogeneous A-harmonic* (7) *in Ω. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi\left(\left|T(u)-(T(u))_{B_0}\right|\right)dm \le C\int_\Omega \varphi(|u|)\,dm, \quad (92)$$

where $B_0 \subset \Omega$ is some fixed ball.

Proof. From Definition 38, (48), and noticing that φ is doubling, we have

$$\begin{aligned}\int_\Omega \varphi\left(\left|T(u)-(T(u))_{B_0}\right|\right)dm &\le C_1\sup_{B\subset\Omega}\int_B \varphi\left(\left|T(u)-(T(u))_B\right|\right)dm\\ &\le C_1\sup_{B\subset\Omega}\left(C_2\int_{\sigma B}\varphi(|u|)\,dm\right)\\ &\le C_1\sup_{B\subset\Omega}\left(C_2\int_\Omega\varphi(|u|)\,dm\right)\\ &\le C_3\int_\Omega\varphi(|u|)\,dm.\end{aligned} \quad (93)$$

We have completed the proof of Theorem 39. □

Similar to the local case, the following global inequality with the Orlicz norm

$$\left\|u-u_{B_0}\right\|_{\varphi(\Omega)} \le C\|du\|_{\varphi(\Omega)} \quad (94)$$

holds if all conditions in Theorem 39 are satisfied. Also, by the same way, we can extend Theorem 28 into the following global result in $L^\varphi(m)$-averaging domains.

Theorem 40. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, Ω be a bounded $L^\varphi(m)$-averaging domain and $q(n-p) < np$, and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in* (2)*. Assume that $u \in D'(\Omega, \wedge^1)$ and $\varphi(|u|) \in L^1(\Omega, m)$. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi\left(\left|T(u)-(T(u))_{B_0}\right|\right)dm \le C\int_\Omega \varphi(|u|)\,dm, \quad (95)$$

where $B_0 \subset \Omega$ is some fixed ball.

Note that (95) can be written as

$$\left\|T(u)-(T(u))_{B_0}\right\|_{\varphi(\Omega)} \le C\|u\|_{\varphi(\Omega)}. \quad (96)$$

It has been proved that any John domain is a special $L^\varphi(m)$-averaging domain. Hence, we have the following results.

Corollary 41. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, and let Ω be a bounded John domain and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in* (2)*. Assume that $\varphi(|u|) \in L^1(\Omega, m)$ and $u \in D'(\Omega, \wedge^1)$ is a solution of the nonhomogeneous A-harmonic* (7) *in Ω. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi\left(\left|T(u)-(T(u))_{B_0}\right|\right)dm \le C\int_\Omega \varphi(|u|)\,dm, \quad (97)$$

where $B_0 \subset \Omega$ is some fixed ball.

Choosing $\varphi(t) = t^p\log_+^\alpha t$ in Theorems 39 and 40, respectively, we obtain the following Poincaré inequalities with the $L^p(\log_+^\alpha L)$-norms.

Corollary 42. *Let $\varphi(t) = t^p\log_+^\alpha t$, $p \ge 1$, $\alpha \in \mathbb{R}$, and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the homotopy operator defined in* (2)*. Assume that $\varphi(|u|) \in L^1(\Omega, m)$ and $u \in D'(\Omega, \wedge^1)$ is a solution of the nonhomogeneous A-harmonic* (7)*. Then, there exists a constant C, independent of u, such that*

$$\begin{aligned}\int_\Omega &\left|T(u)-(T(u))_{B_0}\right|^p\log_+^\alpha\left(\left|T(u)-(T(u))_{B_0}\right|\right)dm\\ &\le C\int_\Omega |u|^p\log_+^\alpha(|u|)\,dm\end{aligned} \quad (98)$$

for any bounded $L^\varphi(m)$-averaging domain Ω and $B_0 \subset \Omega$ is some fixed ball.

Note that (98) can be written as the following version with the Luxemburg norm

$$\left\|T(u)-(T(u))_{B_0}\right\|_{L^p(\log_+^\alpha L)(\Omega)} \le C\|u\|_{L^p(\log_+^\alpha L)(\Omega)} \quad (99)$$

provided the conditions in Corollary 42 are satisfied.

Corollary 43. *Let $\varphi(t) = t^p\log_+^\alpha t$, $1 \le p_1 < p < p_2$, $\alpha \in \mathbb{R}$, Ω be a bounded $L^\varphi(m)$-averaging domain and $p_2(n-p_1) < np_1$, and $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, be the* *homotopy operator defined in* (2). *Assume that* $u \in D'(\Omega, \wedge^1)$, $\varphi(|u|) \in L^1(\Omega, m)$. *Then, there exists a constant* C, *independent of* u, *such that*

$$\int_\Omega \left|T(u) - (T(u))_{B_0}\right|^p \log_+^\alpha \left(\left|T(u) - (T(u))_{B_0}\right|\right) dm \leq C \int_\Omega |u|^p \log_+^\alpha (|u|)\, dm, \tag{100}$$

where $B_0 \subset \Omega$ *is some fixed ball.*

7. Composition of Homotopy and Projection Operators

In this section, we present the norm estimates for the composition of the homotopy operator and projection operator. The results presented in this section can be found in [15, 16]. We assume that M is a domain in an oriented, compact, C^∞ smooth Riemannian manifold of dimension $n \geq 2$. Let $\wedge^l M$ be the lth exterior power of the cotangent bundle, and let $C^\infty(\wedge^l M)$ be the space of smooth l-forms on M and $\mathscr{W}(\wedge^l M) = \{u \in L^1_{\text{loc}}(\wedge^l M) : u \text{ has generalized gradient}\}$.The harmonic l-fields are defined by $\mathscr{H}(\wedge^l M) = \{u \in \mathscr{W}(\wedge^l M) : du = d^\star u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$. The orthogonal complement of $\mathscr{H}$ in L^1 is defined by $\mathscr{H}^\perp = \{u \in L^1 :< u, h >= 0 \text{ for all } h \in \mathscr{H}\}$. Then, Green's operator G is defined as $G : C^\infty(\wedge^l M) \to \mathscr{H}^\perp \cap C^\infty(\wedge^l M)$ by assigning $G(u)$ be the unique element of $\mathscr{H}^\perp \cap C^\infty(\wedge^l M)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\wedge^l M)$ onto $\mathscr{H}$ so that $H(u)$ is the harmonic part of u. See [1, 22, 23] for more properties of these operators.

Lemma 44 (see [20]). *Let* ϕ *be a strictly increasing convex function on* $[0, \infty)$ *with* $\phi(0) = 0$, *and let* D *be a domain in* $\mathbb{R}^n$. *Assume that* u *is a function in* D *such that* $\phi(|u|) \in L^1(D, \mu)$ *and* $\mu(\{x \in D : |u - c| > 0\}) > 0$ *for any constant* c, *where* μ *is a Radon measure defined by* $d\mu(x) = w(x)dx$ *for a weight* $w(x)$. *Then, we have*

$$\int_D \phi\left(\frac{a}{2}\left|u - u_{D,\mu}\right|\right) d\mu \leq \int_D \phi(a|u|)\, d\mu \tag{101}$$

for any positive constant a, *where* $u_{D,\mu} = (1/\mu(D)) \int_D u\,d\mu$.

Lemma 45 (see [24]). *Let* $u \in C^\infty(\wedge^l M)$ *and* $l = 1, 2, \ldots, n$, $1 < s < \infty$. *Then, there exists a positive constant* $C = C(s)$, *independent of* u, *such that*

$$\left\|dd^* G(u)\right\|_{s,M} + \left\|d^* dG(u)\right\|_{s,M} + \|dG(u)\|_{s,M} + \left\|d^* G(u)\right\|_{s,M} + \|G(u)\|_{s,M} \leq C(s)\|u\|_{s,M}. \tag{102}$$

Lemma 46 (see [12]). *Each* Ω *has a modified Whitney cover of cubes* $\mathscr{V} = \{Q_i\}$ *such that* $\cup_i Q_i = \Omega$, $\sum_{Q_i \in \mathscr{V}} \chi_{\sqrt{(5/4)}Q} \leq N\chi_\Omega$ *and some* $N > 1$, *and if* $Q_i \cap Q_j \neq \emptyset$, *then there exists a cube* R *(this cube need not be a member of* $\mathscr{V}$*) in* $Q_i \cap Q_j$ *such that* $Q_i \cup Q_j \subset NR$. *Moreover, if* Ω *is* δ*-John, then there is a distinguished cube* $Q_0 \in \mathscr{V}$ *which can be connected with every cube* $Q \in \mathscr{V}$ *by a chain of cubes* $Q_0, Q_1, \ldots, Q_k = Q$ *from* $\mathscr{V}$ *and such that* $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, *for some* $\rho = \rho(n, \delta)$.

Lemma 47. *Let* $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, $H : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^l)$ *be the projection operator, and* $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ *be the homotopy operator. Then, there exists a constant* $C = C(n, s, \Omega)$, *independent of* u, *such that*

$$\|T(H(u))\|_{s,B} \leq C(n, s, \Omega)|B| \operatorname{diam}(B)\|u\|_{s,B} \tag{103}$$

for all balls $B \subset \Omega$.

Proof. Let T be the homotopy operator and let u be locally L^s integrable l form. Then, there exists a constant $C_1(n, s, \Omega)$, independent of u, such that

$$\|Tu\|_{s,B} \leq C_1(n, s, \Omega)|B| \operatorname{diam}(B)\|u\|_{s,B}. \tag{104}$$

By using Lemma 45, we have

$$\begin{aligned}\|\Delta G(u)\|_{s,B} &= \left\|(dd^* + d^* d)G(u)\right\|_{s,B} \\ &\leq \left\|dd^* G(u)\right\|_{s,B} + \left\|d^* dG(u)\right\|_{s,B} \\ &\leq C_2(s)\|u\|_{s,B}.\end{aligned} \tag{105}$$

Thus, by (104) and (105), we have

$$\begin{aligned}\|TH(u)\|_{s,B} &\leq C_1(n, s, \Omega)|B| \operatorname{diam}(B)\|H(u)\|_{s,B} \\ &= C_1(n, s, \Omega)|B| \operatorname{diam}(B)\|u - \Delta G(u)\|_{s,B} \\ &\leq C_1(n, s, \Omega)|B| \operatorname{diam}(B)\left(\|u\|_{s,B} + \|\Delta G(u)\|_{s,B}\right) \\ &\leq C_1(n, s, \Omega)|B| \operatorname{diam}(B)\left(\|u\|_{s,B} + C_2(s)\|u\|_{s,B}\right) \\ &\leq C_3(n, s, \Omega)|B| \operatorname{diam}(B)\|u\|_{s,B}\end{aligned} \tag{106}$$

which ends the proof of Lemma 47. □

Lemma 48. *Let* $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be a solution of the nonhomogeneous A-harmonic* (7) *in a bounded and convex domain* Ω, *let* H *be the projection operator, and let* T *be the homotopy operator. Then, there exists a constant* $C(n, s, \alpha, \lambda, \Omega)$, *independent of* u, *such that*

$$\left(\int_B |T(H(u))|^s \frac{1}{d^\alpha(x, \partial\Omega)} dx\right)^{1/s} \leq C(n, s, \alpha, \lambda, \Omega)|B|^\gamma \left(\int_{\rho B} |u|^s \frac{1}{\left|x - x_B\right|^\lambda} dx\right)^{1/s} \tag{107}$$

for all balls B *with* $\rho B \subset \Omega$, $\rho > 1$, *and any real number* α *and* λ *with* $\alpha > \lambda \geq 0$ *and* $\gamma = 1 + (1/n) - ((\alpha - \lambda)/ns)$. *Here* x_B *is the center of the ball.*

Theorem 49. *Let* $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be a solution of the nonhomogeneous A-harmonic* (7) *in a bounded domain Ω, let $H : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^l)$ be the projection operator, and let $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\|T(H(u)) - (T(H(u)))_B\|_{s,B} \le C|B| \operatorname{diam}(B) \|u\|_{s,\sigma B} \tag{108}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

Theorem 50. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a smooth differential form in a bounded domain Ω, let H be the projection operator, and let T be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\begin{aligned} &\|T(H(u)) - (T(H(u)))_B\|_{s,B} \\ &\quad \le C|B| \operatorname{diam}(B) \left(\|u\|_{s,B} + \|du\|_{s,B}\right) \end{aligned} \tag{109}$$

for all balls $B \subset \Omega$.

In applications, such as in calculating electric or magnetic fields, we often face the fact that the integrand contains a singular factor. So, the above result was extended into the following singular weighted case.

Theorem 51. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation in a bounded domain Ω, let H be the projection operator, and let T be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\begin{aligned} &\left(\int_B |T(H(u)) - (T(H(u)))_B|^s \frac{1}{|x - x_B|^\alpha} dx\right)^{1/s} \\ &\quad \le C|B|^\gamma \left(\int_{\sigma B} |u|^s \frac{1}{|x - x_B|^\lambda} dx\right)^{1/s} \end{aligned} \tag{110}$$

for all balls B with $\sigma B \subset \Omega$ and any real numbers α and λ with $\alpha > \lambda \ge 0$, where $\gamma = 1 + (1/n) - ((\alpha - \lambda)/ns)$ and x_B is the center of ball B and $\sigma > 1$ is a constant.

Proof. Let $\varepsilon \in (0, 1)$ be small enough such that $\varepsilon n < \alpha - \lambda$ and $B \subset \Omega$ be any ball with center x_B and radius r_B. Choose $t = s/(1 - \varepsilon)$; then, $t > s$. Write $\beta = t/(t - s)$, and using the Hölder inequality and Theorem 49, we have

$$\begin{aligned} &\left(\int_B \left(|TH(u) - (TH(u))_B|\right)^s \frac{1}{|x - x_B|^\alpha} dx\right)^{1/s} \\ &= \left(\int_B \left(|TH(u) - (TH(u))_B| \frac{1}{|x - x_B|^{\alpha/s}}\right)^s dx\right)^{1/s} \\ &\le \|TH(u) - (TH(u))_B\|_{t,B} \left(\int_B \left(\frac{1}{|x - x_B|}\right)^{t\alpha/(t-s)} dx\right)^{(t-s)/st} \\ &= \|TH(u) - (TH(u))_B\|_{t,B} \left(\int_B |x - x_B|^{-\alpha\beta} dx\right)^{1/\beta s} \\ &\le C_1 |B| \operatorname{diam}(B) \|u\|_{t,\nu B} \left\||x - x_B|^{-\alpha}\right\|^{1/s}_{\beta,B}, \end{aligned} \tag{111}$$

where $\nu > 1$ is a constant. We may assume that $x_B = 0$. Otherwise, we can move the center to the origin by a simple transformation. Then, for any $x \in B$, $|x - x_B| \ge |x| - |x_B| = |x|$. By using the polar coordinate substitution, we have

$$\int_B |x - x_B|^{-\alpha\beta} dx \le C \int_0^{r_B} \rho^{-\alpha\beta} \rho^{n-1} d\rho \le \frac{C}{n - \alpha\beta} (r_B)^{n-\alpha\beta}. \tag{112}$$

Choose $m = nst/(ns + \alpha t - \lambda t)$, then $0 < m < s$. By the reverse Hölder inequality, we find that

$$\|u\|_{t,\nu B} \le C_2 |B|^{(m-t)/mt} \|u\|_{m,\sigma B}, \tag{113}$$

where $\sigma > \nu > 1$ is a constant. By the Hölder inequality again, we obtain

$$\begin{aligned} &\|u\|_{m,\sigma B} \\ &= \left(\int_{\sigma B} \left(|u| |x - x_B|^{-\lambda/s} |x - x_B|^{\lambda/s}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\sigma B} \left(|u| |x - x_B|^{-\lambda/s}\right)^s dx\right)^{1/s} \\ &\quad \times \left(\int_{\sigma B} \left(|x - x_B|^{\lambda/s}\right)^{ms/(s-m)} dx\right)^{(s-m)/ms} \\ &\le \left(\int_{\sigma B} |u|^s |x - x_B|^{-\lambda} dx\right)^{1/s} C_3 (\sigma r_B)^{\lambda/s + n(s-m)/ms} \\ &\le C_4 \left(\int_{\sigma B} |u|^s |x - x_B|^{-\lambda} dx\right)^{1/s} (r_B)^{\lambda/s + n(s-m)/ms}. \end{aligned} \tag{114}$$

Note that

$$\begin{aligned} \operatorname{diam}(B) \cdot |B|^{1 + (1/t) - (1/m)} &= |B|^{1 + (1/n) + (1/t) - ((ns + \alpha t - \lambda t)/nst)} \\ &= |B|^{1 + (1/n) - ((\alpha - \lambda)/ns)}. \end{aligned} \tag{115}$$

Substituting (112), (113), and (114) in (111) and using (115), we have

$$\begin{aligned} &\left(\int_B \left(|TH(u) - (TH(u))_B|\right)^s \frac{1}{|x - x_B|^\alpha} dx\right)^{1/s} \\ &\quad \le C_5 |B|^\gamma \left(\int_{\sigma B} |u|^s |x - x_B|^{-\lambda} dx\right)^{1/s}. \end{aligned} \tag{116}$$

We have completed the proof of Theorem 51. □

Remark 52. (1) Replacing α by 2α and λ by α in Theorem 51, we have

$$\begin{aligned}\left(\int_B \left|T(H(u)) - (T(H(u)))_B\right|^s \frac{1}{|x-x_B|^{2\alpha}} dx\right)^{1/s} \\ \le C|B|^{1+(1/n)-(\alpha/ns)} \left(\int_{\sigma B} |u|^s \frac{1}{|x-x_B|^{\alpha}} dx\right)^{1/s}.\end{aligned} \tag{117}$$

(2) If $\lambda = 0$, inequality (110) reduces to

$$\begin{aligned}\left(\int_B \left|T(H(u)) - (T(H(u)))_B\right|^s \frac{1}{|x-x_B|^{\alpha}} dx\right)^{1/s} \\ \le C|B|^{1+(1/n)-(\alpha/ns)} \left(\int_{\sigma B} |u|^s dx\right)^{1/s},\end{aligned} \tag{118}$$

which does not contain a singular factor in the integral on the right side of the inequality.

The following definition of $L^s(\mu)$-averaging domains can be found in [1]. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L^s(\mu)$-averaging domain, $s \ge 1$, if $\mu(\Omega) < \infty$, and there exists a constant C such that

$$\begin{aligned}\left(\frac{1}{\mu(\Omega)} \int_\Omega \left|u - u_{B_0,\mu}\right|^s d\mu\right)^{1/s} \\ \le C \sup_{4B\subset\Omega} \left(\frac{1}{\mu(B)} \int_B \left|u - u_{B,\mu}\right|^s d\mu\right)^{1/s}\end{aligned} \tag{119}$$

for some ball $B_0 \subset \Omega$ and all $u \in L^s_{\mathrm{loc}}(\Omega;\mu)$. Here the supremum is over all balls $B \subset \Omega$ with $4B \subset \Omega$ and μ is a measure defined by $d\mu = w(x)dx$ for a weight $w(x)$ and $u_{B,\mu} = (1/\mu(B)) \int_B u(x)dx$.

Theorem 53. *Let $u \in D'(\Omega, \wedge^1)$ be a solution of the nonhomogeneous A-harmonic equation, let H be the projection operator, and let T be the homotopy operator. Assume that s is a fixed exponent associated with the nonhomogeneous A-harmonic equation. Then, there exists a constant C, independent of u, such that*

$$\begin{aligned}\left(\int_\Omega \left|T(H(u)) - (T(H(u)))_{B_0}\right|^s \frac{1}{d(x,\partial\Omega)^{\alpha}} dx\right)^{1/s} \\ \le C\left(\int_\Omega |u|^s \frac{1}{d(x,\partial\Omega)^{\lambda}} dx\right)^{1/s}\end{aligned} \tag{120}$$

for any bounded and convex $L^s(\mu)$-averaging domain $\Omega \subset \mathbb{R}^n$. Here $B_0 \subset \Omega$ is a fixed ball and α and λ are constants with $0 \le \lambda < \alpha < \min\{n, s+\lambda+n(s-1)\}$.

Proof. Let r_B be the radius of a ball $B \subset \Omega$. We may assume the center of B is 0. Then, $d(x,\partial\Omega) \ge r_B - |x|$ for any $x \in B$. Therefore, $d^{-1}(x,\partial\Omega) \le 1/(r_B - |x|)$ for any $x \in B$. Similar to the proof of Theorem 51, we have

$$\begin{aligned}\left(\int_B \left|TH(u) - (TH(u))_B\right|^s \frac{1}{d(x,\partial\Omega)^{\alpha}} dx\right)^{1/s} \\ \le C_1|B|^{\gamma}\left(\int_{\sigma B} |u|^s \frac{1}{d(x,\partial\Omega)^{\lambda}} dx\right)^{1/s}\end{aligned} \tag{121}$$

for all balls B with $\sigma B \subset \Omega$, $\sigma > 1$, and any real numbers α and λ with $\alpha > \lambda \ge 0$, where $\gamma = 1 + (1/n) - ((\alpha-\lambda)/ns)$. Write $d\mu = (1/d(x,\partial\Omega)^{\alpha})dx$. Then, $\mu(B) = \int_B d\mu = \int_B (1/d(x,\partial\Omega)^{\alpha})dx \ge \int_B (1/(\mathrm{diam}(\Omega))^{\alpha})dx = C_1|B|$, and hence $1/\mu(B) \le C_2/|B|$. Since Ω is an $L^s(\mu)$-averaging domain, using (121) and noticing that $\gamma - 1/s = (1-1/s) + (s+\lambda-\alpha)/ns > 0$, we have

$$\begin{aligned}&\left(\frac{1}{\mu(\Omega)} \int_\Omega \left|T(H(u)) - (T(H(u)))_{B_0}\right|^s \frac{1}{d(x,\partial\Omega)^{\alpha}} dx\right)^{1/s} \\ &= \left(\frac{1}{\mu(\Omega)} \int_\Omega \left|T(H(u)) - (T(H(u)))_{B_0}\right|^s d\mu\right)^{1/s} \\ &\le C_3 \sup_{4B\subset\Omega}\left(\frac{1}{\mu(B)} \int_B \left|T(H(u)) - (T(H(u)))_B\right|^s d\mu\right)^{1/s} \\ &\le C_4 \sup_{4B\subset\Omega}\left(\frac{1}{|B|} \int_B \left|T(H(u)) - (T(H(u)))_B\right|^s d\mu\right)^{1/s} \\ &\le C_5 \sup_{4B\subset\Omega} |B|^{\gamma-1/s}\left(\int_{\sigma B} |u|^s \frac{1}{d(x,\partial\Omega)^{\lambda}} dx\right)^{1/s} \\ &\le C_5|\Omega|^{\gamma-1/s}\left(\int_\Omega |u|^s \frac{1}{d(x,\partial\Omega)^{\lambda}} dx\right)^{1/s} \\ &\le C_6\left(\int_\Omega |u|^s \frac{1}{d(x,\partial\Omega)^{\lambda}} dx\right)^{1/s},\end{aligned} \tag{122}$$

which is equivalent to

$$\begin{aligned}\left(\int_\Omega \left|T(H(u)) - (T(H(u)))_{B_0}\right|^s \frac{1}{d(x,\partial\Omega)^{\alpha}} dx\right)^{1/s} \\ \le C\left(\int_\Omega |u|^s \frac{1}{d(x,\partial\Omega)^{\lambda}} dx\right)^{1/s}.\end{aligned} \tag{123}$$

We have completed the proof of Theorem 53. □

We recall the following definition of δ-John domains with $\delta > 0$.

Definition 54. A proper subdomain $\Omega \subset \mathbb{R}^n$ is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi,\partial\Omega) \ge \delta\left|x - \xi\right| \tag{124}$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

Theorem 55. *Let $u \in D'(\Omega, \wedge^1)$ be a solution of the nonhomogeneous A-harmonic (7), let H be the projection operator, and let T be the homotopy operator. Assume that s is a fixed exponent associated with the nonhomogeneous A-harmonic equation. Then, there exists a constant $C(n, N, s, \alpha, \lambda, Q_0, \Omega)$, independent of u, such that*

$$\left(\int_\Omega \left|T(H(u)) - (T(H(u)))_{Q_0}\right|^s \frac{1}{d^\alpha(x,\partial\Omega)} dx\right)^{1/s} \le C(n, N, s, \alpha, \lambda, Q_0, \Omega)\left(\int_\Omega |u|^s g(x)\,dx\right)^{1/s} \tag{125}$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$, where $g(x) = \sum_i \chi_{Q_i}(1/|x - x_{Q_i}|^\lambda)$. Here α and λ are constants with $0 \le \lambda < \alpha < \min\{n, s + \lambda + n(s-1)\}$, and the fixed cube $Q_0 \subset \Omega$, the cubes $Q_i \subset \Omega$, and the constant $N > 1$ appeared in Lemma 46.

Proof. We use the notation appearing in Lemma 46. There is a modified Whitney cover of cubes $\mathscr{V} = \{Q_i\}$ for Ω such that $\Omega = \cup Q_i$, and $\sum_{Q_i \in \mathscr{V}} \chi_{\sqrt{(5/4)}Q_i} \le N\chi_\Omega$ for some $N > 1$. Since $\Omega = \cup Q_i$, for any $x \in \Omega$, it follows that $x \in Q_i$ for some i. Applying Lemma 48 to Q_i, we have

$$\left(\int_{Q_i} |TH(u)|^s \frac{1}{d^\alpha(x,\partial\Omega)} dx\right)^{1/s} \le C_1(n, s, \alpha, \lambda, \Omega)\,|Q_i|^\gamma \left(\int_{\sigma Q_i} |u|^s \frac{1}{d^\lambda(x,\partial\Omega)} dx\right)^{1/s}, \tag{126}$$

where $\sigma > 1$ is a constant. Let $\mu(x)$ and $\mu_1(x)$ be the Radon measures defined by $d\mu = (1/d^\alpha(x,\partial\Omega))dx$ and $d\mu_1(x) = g(x)dx$, respectively. Then,

$$\mu(Q) = \int_Q \frac{1}{d^\alpha}(x,\partial\Omega)\,dx \ge \int_Q \frac{1}{(\operatorname{diam}(\Omega))^\alpha} dx = M(n,\alpha,\Omega)\,|Q|, \tag{127}$$

where $M(n,\alpha,\Omega)$ is a positive constant. Then, by the elementary inequality $(a+b)^s \le 2^s(|a|^s + |b|^s)$, $s \ge 0$, we have

$$\begin{aligned}
&\left(\int_\Omega \left|T(H(u)) - (T(H(u)))_{Q_0}\right|^s \frac{1}{d^\alpha(x,\partial\Omega)} dx\right)^{1/s} \\
&= \left(\int_{\cup Q} \left|T(H(u)) - (T(H(u)))_{Q_0}\right|^s d\mu\right)^{1/s} \\
&\le \left(\sum_{Q\in\mathscr{V}} \left(2^s \int_Q \left|T(H(u)) - (T(H(u)))_Q\right|^s d\mu + 2^s \right.\right. \\
&\quad \left.\left. \times \int_Q \left|(T(H(u)))_Q - (T(H(u)))_{Q_0}\right|^s d\mu\right)\right)^{1/s} \\
&\le C_1(s)\left(\left(\sum_{Q\in\mathscr{V}} \int_Q \left|T(H(u)) - (T(H(u)))_Q\right|^s d\mu\right)^{1/s}\right. \\
&\quad \left. + \left(\sum_{Q\in\mathscr{V}} \int_Q \left|(T(H(u)))_Q - (T(H(u)))_{Q_0}\right|^s d\mu\right)^{1/s}\right)
\end{aligned} \tag{128}$$

for a fixed $Q_0 \subset \Omega$. The first sum in (128) can be estimated by using Lemma 44 with $\varphi = t^s$, $a = 2$, and Lemma 48:

$$\begin{aligned}
&\sum_{Q\in\mathscr{V}} \int_Q \left|T(H(u)) - (T(H(u)))_Q\right|^s d\mu \\
&\le \sum_{Q\in\mathscr{V}} \int_Q 2^s |T(H(u))|^s d\mu \\
&\le C_2(n, s, \alpha, \lambda, \Omega) \sum_{Q\in\mathscr{V}} |Q|^{\gamma s} \int_{\rho Q} |u|^s d\mu_1 \\
&\le C_3(n, s, \alpha, \lambda, \Omega)\,|\Omega|^{\gamma s} \sum_{Q\in\mathscr{V}} \int_\Omega \left(|u|^s d\mu_1\right)\chi_{\rho Q} \\
&\le C_4(n, N, s, \alpha, \lambda, \Omega)\,|\Omega|^{\gamma s} \int_\Omega |u|^s d\mu_1 \\
&\le C_5(n, N, s, \alpha, \lambda, \Omega) \int_\Omega |u|^s g(x)\,dx.
\end{aligned} \tag{129}$$

To estimate the second sum in (128), we need to use the property of δ-John domain. Fix a cube $Q \in \mathscr{V}$ and let $Q_0, Q_1, \ldots, Q_k = Q$ be the chain in Lemma 46.

$$\left|(T(H(u)))_Q - (T(H(u)))_{Q_0}\right| \le \sum_{i=0}^{k-1} \left|(T(H(u)))_{Q_i} - (T(H(u)))_{Q_{i+1}}\right|. \tag{130}$$

The chain $\{Q_i\}$ also has property that, for each i, $i = 0, 1, \ldots, k-1$, with $Q_i \cap Q_{i+1} \neq \emptyset$; there exists a cube D_i such that $D_i \subset Q_i \cap Q_{i+1}$ and $Q_i \cup Q_{i+1} \subset ND_i$, $N > 1$:

$$\frac{\max\{|Q_i|, |Q_{i+1}|\}}{|Q_i \cap Q_{i+1}|} \le \frac{\max\{|Q_i|, |Q_{i+1}|\}}{|D_i|} \le C_6(N). \tag{131}$$

For such D_j, $j = 0, 1, \ldots, k-1$, let $|D^\star| = \min\{|D_0|, |D_1|, \ldots, |D_{k-1}|\}$; then

$$\frac{\max\{|Q_i|, |Q_{i+1}|\}}{|Q_i \cap Q_{i+1}|} \le \frac{\max\{|Q_i|, |Q_{i+1}|\}}{|D^\star|} \le C_7(N). \tag{132}$$

By (127), (132), and Lemma 48, we have

$$\begin{aligned}
&\left|(T(H(u)))_{Q_i}-(T(H(u)))_{Q_{i+1}}\right|^s\\
&=\frac{1}{\mu(Q_i\cap Q_{i+1})}\\
&\quad\times\int_{Q_i\cap Q_{i+1}}\left|(T(H(u)))_{Q_i}-(T(H(u)))_{Q_{i+1}}\right|^s\frac{dx}{d^{\alpha}(x,\partial\Omega)}\\
&\leq C_8(n,\alpha,\Omega)\frac{1}{|Q_i\cap Q_{i+1}|}\\
&\quad\times\int_{Q_i\cap Q_{i+1}}\left|(T(H(u)))_{Q_i}-(T(H(u)))_{Q_{i+1}}\right|^s\frac{dx}{d^{\alpha}(x,\partial\Omega)}\\
&\leq C_8(n,\alpha,\Omega)\frac{C_7(N)}{\max\{|Q_i|,|Q_{i+1}|\}}\\
&\quad\times\int_{Q_i\cap Q_{i+1}}\left|(T(H(u)))_{Q_i}-(T(H(u)))_{Q_{i+1}}\right|^s d\mu\\
&\leq C_9(n,N,s,\alpha,\Omega)\\
&\quad\times\sum_{j=i}^{i+1}\frac{1}{|Q_j|}\int_{Q_j}\left|T(H(u))-(T(H(u)))_{Q_j}\right|^s d\mu\\
&\leq C_{10}(n,N,s,\alpha,\lambda,\Omega)\\
&\quad\times\sum_{j=i}^{i+1}\frac{|Q_j|^{\gamma s}}{|Q_j|}\int_{\rho Q_j}|u|^s d\mu_1\\
&=C_{10}(n,N,s,\alpha,\lambda,\Omega)\\
&\quad\times\sum_{j=i}^{i+1}|Q_j|^{\gamma s-1}\int_{\rho Q_j}|u|^s g(x)\,dx.
\end{aligned}\tag{133}$$

Since $Q\subset NQ_j$ for $j=i,i+1$, $0\leq i\leq k-1$, from (133)

$$\begin{aligned}
&\left|(T(H(u)))_{Q_i}-(T(H(u)))_{Q_{i+1}}\right|^s\chi_Q(x)\\
&\leq C_{11}(n,N,s,\alpha,\lambda,\Omega)\sum_{j=i}^{i+1}\chi_{NQ_j}(x)\left|Q_j\right|^{\gamma s-1}\int_{\rho Q_j}|u|^s g(x)\,dx\\
&\leq C_{12}(n,N,s,\alpha,\lambda,\Omega)\sum_{j=i}^{i+1}\chi_{NQ_j}(x)\,|\Omega|^{\gamma s-1}\int_{\rho Q_j}|u|^s d\mu_1.
\end{aligned}\tag{134}$$

We know that $|\Omega|^{\gamma-1/s}<\infty$ since Ω is bounded and $\gamma-(1/s)=1+(1/n)+(\lambda/ns)-(1/s)-(\alpha/ns)>0$ when $\alpha<s+\lambda+n(s-1)$. Thus, from $(a+b)^{1/s}\leq 2^{1/s}(|a|^{1/s}+|b|^{1/s})$, (130), and (134),

$$\begin{aligned}
&\left|(T(H(u)))_Q-(T(H(u)))_{Q_0}\right|\chi_Q(x)\\
&\quad\leq C_{13}(n,N,s,\alpha,\lambda,\Omega)\sum_{D\in\mathcal{V}}\left(\int_{\rho D}|u|^s d\mu_1\right)^{1/s}\cdot\chi_{ND}(x)
\end{aligned}\tag{135}$$

for every $x\in\mathbb{R}^n$. Then,

$$\begin{aligned}
&\sum_{Q\in\mathcal{V}}\int_Q\left|(T(H(u)))_Q-(T(H(u)))_{Q_0}\right|^s d\mu\\
&\quad\leq C_{13}(n,N,s,\alpha,\lambda,\Omega)\\
&\qquad\times\int_{\mathbb{R}^n}\left|\sum_{D\in\mathcal{V}}\left(\int_{\rho D}|u|^s d\mu_1\right)^{1/s}\chi_{ND}(x)\right|^s d\mu.
\end{aligned}\tag{136}$$

Notice that

$$\sum_{D\in\mathcal{V}}\chi_{ND}(x)\leq\sum_{D\in\mathcal{V}}\chi_{\rho ND}(x)\leq N\chi_{\Omega}(x).\tag{137}$$

Using elementary inequality $|\sum_{i=1}^{M}t_i|^s\leq M^{s-1}\sum_{i=1}^{M}|t_i|^s$, we finally have

$$\begin{aligned}
&\sum_{Q\in\mathcal{V}}\int_Q\left|(T(H(u)))_Q-(T(H(u)))_{Q_0}\right|^s d\mu\\
&\leq C_{14}(n,N,s,\alpha,\lambda,\Omega)\int_{\mathbb{R}^n}\left(\sum_{D\in\mathcal{V}}\left(\int_{\rho D}|u|^s d\mu_1\right)\chi_D(x)\right)d\mu\\
&=C_{14}(n,N,s,\alpha,\lambda,\Omega)\sum_{D\in\mathcal{V}}\left(\int_{\rho D}|u|^s d\mu_1\right)\\
&\leq C_{15}(n,N,s,\alpha,\lambda,\Omega)\int_{\Omega}|u|^s g(x)\,dx.
\end{aligned}\tag{138}$$

Substituting (129) and (138) in (128), we have proved Theorem 55. □

The following L^s-imbedding inequality with a singular factor in the John domain was also proved in [12].

Theorem 56. *Let $u\in D'(\Omega,\wedge^1)$ be a solution of the nonhomogeneous A-harmonic (7), let H be the projection operator, and let T be the homotopy operator. Assume that s is a fixed exponent associated with the nonhomogeneous A-harmonic equation. Then, there exists a constant $C(n,s,\alpha,\lambda,\Omega)$, independent of u, such that*

$$\|\nabla(T(H(u)))\|_{s,\Omega,w_1}\leq C(n,s,\alpha,\lambda,\Omega)\|u\|_{s,\Omega,w_2},\tag{139}$$

$$\|T(H(u))\|_{W^{1,s}(\Omega),w_1}\leq C(n,s,\alpha,\lambda,\Omega)\|u\|_{s,\Omega,w_2}\tag{140}$$

for any bounded and convex δ-John domain $\Omega\subset\mathbb{R}^n$. Here the weights are defined by $w_1(x)=1/d^{\alpha}(x,\partial\Omega)$ and $w_2(x)=\sum_i\chi_{Q_i}(1/|x-x_{Q_i}|^{\lambda})$, respectively. α and λ are constants with $0\leq\lambda<\alpha<\lambda+(n+1)s$.

Theorem 57. *Let $u\in D'(\Omega,\wedge^1)$ be a solution of the nonhomogeneous A-harmonic (7), let H be the projection operator, and let T be the homotopy operator. Assume that s is a fixed exponent associated with the nonhomogeneous A-harmonic equation. Then, there exists a constant $C(n,N,s,\alpha,\lambda,Q_0,\Omega)$, independent of u, such that*

$$\begin{aligned}
&\left\|T(H(u))-(T(H(u)))_{Q_0}\right\|_{W^{1,s}(\Omega),w_1}\\
&\quad\leq C(n,N,s,\alpha,\lambda,Q_0,\Omega)\|u\|_{s,\Omega,w_2}
\end{aligned}\tag{141}$$

for any bounded, convex δ-John domain $\Omega \subset \mathbb{R}^n$*. Here the weights are defined by* $w_1(x) = 1/d^{\alpha}(x, \partial\Omega)$ *and* $w_2(x) = \sum_i \chi_{Q_i}(1/|x - x_{Q_i}|^{\lambda})$*,* α *and* λ *are constants with* $0 \le \lambda < \alpha < \min\{n, \lambda + n(s-1)\}$*, and the fixed cube* $Q_0 \subset \Omega$ *and the constant* $N > 1$ *appeared in Lemma 46.*

Proof. Since $(T(H(u)))_{Q_0}$ is a closed form, $\nabla((T(H(u)))_{B_0}) = d((T(H(u)))_{Q_0}) = 0$. Thus, by using Theorem 55 and (139), we have

$$\begin{aligned}
&\left\| T(H(u)) - (T(H(u)))_{Q_0} \right\|_{W^{1,s}(\Omega), w_1} \\
&= \operatorname{diam}(\Omega)^{-1} \left\| T(H(u)) - (T(H(u)))_{Q_0} \right\|_{s,\Omega,w_1} \\
&\quad + \left\| \nabla \left(T(H(u)) - (T(H(u)))_{Q_0} \right) \right\|_{s,\Omega,w_1} \\
&= \operatorname{diam}(\Omega)^{-1} \left\| T(H(u)) - (T(H(u)))_{Q_0} \right\|_{s,\Omega,w_1} \\
&\quad + \left\| \nabla (T(H(u))) \right\|_{s,\Omega,w_1} \\
&\le C_1(n, N, s, \alpha, \lambda, Q_0, \Omega) \|u\|_{s,\Omega,w_2} \\
&\quad + C_2(n, s, \alpha, \lambda, \Omega) \|u\|_{s,\Omega,w_2} \\
&\le C_3(n, N, s, \alpha, \lambda, Q_0, \Omega) \|u\|_{s,\Omega,w_2}.
\end{aligned} \tag{142}$$

Thus, (141) holds. We have completed the proof of Theorem 57. □

Remark 58. Since the usual p-harmonic equation $\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions is the special case of the nonhomogeneous A-harmonic equation for differential forms, all results proved in Theorems 55, 56, and 57 are still true for p-harmonic functions.

8. Composition of Homotopy and Potential Operators

Recently, Bi extended the definition of the potential operator to the case of differential forms; see [3]. For any differential l-form $u(x)$, the potential operator P is defined by

$$Pu(x) = \sum_I \int_E K(x, y) u_I(y) \, dy dx_I, \tag{143}$$

where the kernel $K(x, y)$ is a nonnegative measurable function defined for $x \ne y$ and the summation is over all ordered l-tuples I. The $l = 0$ case reduces to the usual potential operator:

$$Pf(x) = \int_E \mathrm{K}(x, y) f(y) \, dy, \tag{144}$$

where $f(x)$ is a function defined on $E \subset \mathbb{R}^n$. See [3, 25] for more results about the potential operator. We say a kernel K on $\mathbb{R}^n \times \mathbb{R}^n$ satisfies the standard estimates if there exist δ, $0 < \delta \le 1$, and constant C such that for all distinct points x and y in $\mathbb{R}^n$, and all z with $|x - z| < (1/2)|x - y|$, the kernel K satisfies (i) $K(x, y) \le C|x - y|^{-n}$; (ii) $|K(x, y) - K(z, y)| \le C|x - z|^{\delta}|x - y|^{-n-\delta}$; and (iii) $|K(y, x) - K(y, z)| \le C|x - z|^{\delta}|x - y|^{-n-\delta}$.

In this paper, we always assume that P is the potential operator defined in (143) with the kernel $K(x, y)$ satisfying condition (i) of the standard estimates. Recently, Bi in [3] proved the following inequality for the potential operator:

$$\|P(u)\|_{p,E} \le C\|u\|_{p,E}, \tag{145}$$

where $u \in D'(E, \wedge^l)$, $l = 0, 1, \ldots, n-1$, is a differential form defined in a bounded and convex domain E and $p > 1$ is a constant.

In this section, we prove the local L^{φ} imbedding inequalities for $T \circ P$ applied to solutions of the nonhomogeneous A-harmonic equation in a bounded domain. For any subset $E \subset \mathbb{R}^n$, we use $W^{1,\varphi}(E, \wedge^l)$ to denote the Orlicz-Sobolev space of l-forms which equals $L^{\varphi}(E, \wedge^l) \cap L_1^{\varphi}(E, \wedge^l)$ with norm

$$\|u\|_{W^{1,\varphi}(E)} = \|u\|_{W^{1,\varphi}(E,\wedge^l)} = \operatorname{diam}(E)^{-1} \|u\|_{L^{\varphi}(E)} + \|\nabla u\|_{L^{\varphi}(E)}. \tag{$**$}$$

If we choose $\varphi(t) = t^p$, $p > 1$ in $(**)$, we obtain the usual L^p norm for $W^{1,p}(E, \wedge^l)$

$$\|u\|_{W^{1,p}(E)} = \|u\|_{W^{1,p}(E,\wedge^l)} = \operatorname{diam}(E)^{-1} \|u\|_{p,E} + \|\nabla u\|_{p,E}. \tag{$**$}'$$

In 2013, the following Theorems 59 to 61 were recently proved in [18].

Theorem 59. *Let* φ *be a Young function in the class* $G(p, q, C)$*,* $1 \le p < q < \infty$*,* $C \ge 1$*,* Ω *be a bounded domain,* $T : C^{\infty}(\Omega, \wedge^l) \to C^{\infty}(\Omega, \wedge^{l-1})$*,* $l = 1, 2, \ldots, n$*, be the homotopy operator defined in* (2)*, and let* P *be the potential operator defined in* (143) *with the kernel* $K(x, y)$ *satisfying condition (i) of the standard estimates. Assume that* $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ *and* u *is a solution of the nonhomogeneous* A*-harmonic* (7) *in* Ω*. Then, there exists a constant* C*, independent of* u*, such that*

$$\left\| T(P(u)) - (T(P(u)))_B \right\|_{L^{\varphi}(B)} \le C \operatorname{diam}(B) \|u\|_{L^{\varphi}(\sigma B)} \tag{146}$$

for all balls B *with* $\sigma B \subset \Omega$ *for some* $\sigma > 1$*.*

Theorem 60. *Let* φ *be a Young function in the class* $G(p, q, C)$*,* $1 \le p < q < \infty$*,* $C \ge 1$*,* Ω *be a bounded domain,* T *be the homotopy operator defined in* (2)*, and let* P *be the potential operator defined in* (143) *with the kernel* $K(x, y)$ *satisfying condition (i) of the standard estimates. Assume that* $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ *and* u *is a solution of the nonhomogeneous* A*-harmonic* (7) *in* Ω*. Then, there exists a constant* C*, independent of* u*, such that*

$$\|Td(T(P(u)))\|_{L^{\varphi}(B)} \le C|B| \operatorname{diam}(B) \|u\|_{L^{\varphi}(\sigma B)} \tag{147}$$

for all balls B *with* $\sigma B \subset \Omega$ *for some* $\sigma > 1$*.*

Theorem 61. *Let* φ *be a Young function in the class* $G(p, q, C)$*,* $1 \le p < q < \infty$*,* $C \ge 1$*,* Ω *be a bounded domain,* T *be the homotopy operator defined in* (2)*, and let* P *be the potential operator defined in* (143) *with the kernel* $K(x, y)$ *satisfying condition (i) of the standard estimates. Assume that*

$\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the nonhomogeneous A-harmonic (7) in Ω. Then, there exists a constant C, independent of u, such that

$$\|\nabla T d(T(P(u)))\|_{L^\varphi(B)} \leq C|B|\,\|u\|_{L^\varphi(\sigma B)} \tag{148}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

The following local L^φ-imbedding theorem was also obtained in [18].

Theorem 62. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, Ω be a bounded domain, T be the homotopy operator defined in (2), and let P be the potential operator defined in (143) with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the nonhomogeneous A-harmonic (7) in Ω. Then, there exists a constant C, independent of u, such that*

$$\left\|T(P(u)) - (T(P(u)))_B\right\|_{W^{1,\varphi}(B,\wedge^l)} \leq C|B|\,\|u\|_{L^\varphi(\sigma B)} \tag{149}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

Proof. From $(**)$, (147), and (148), we have

$$\begin{aligned}
&\left\|T(P(u)) - (T(P(u)))_B\right\|_{W^{1,\varphi}(B,\wedge^l)} \\
&= \|Td(T(P(u)))\|_{W^{1,\varphi}(B,\wedge^l)} \\
&= (\operatorname{diam}(B))^{-1}\|Td(T(P(u)))\|_{L^\varphi(B)} \\
&\quad + \|\nabla \mathrm{T}d(T(P(u)))\|_{L^\varphi(B)} \\
&\leq (\operatorname{diam}(B))^{-1}\left(C_1|B|\operatorname{diam}(B)\|u\|_{L^\varphi(\sigma_1 B)}\right) \\
&\quad + C_2|B|\,\|u\|_{L^\varphi(\sigma_2 B)} \\
&\leq C_1|B|\,\|u\|_{L^\varphi(\sigma_1 B)} + C_2|B|\,\|u\|_{L^\varphi(\sigma_2 B)} \\
&\leq C_3|B|\,\|u\|_{L^\varphi(\sigma B)}
\end{aligned} \tag{150}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma = \max\{\sigma_1, \sigma_2\}$. The proof of Theorem 62 has been completed. □

The following version of local imbedding will be used to establish a global imbedding theorem which indicates that the operator $T \circ P$ is bounded.

Theorem 63. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, Ω be a bounded domain, T be the homotopy operator defined in (2), and let P be the potential operator defined in (143) with the kernel $K(x,y)$ satisfying condition (i) of the standard estimates. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the nonhomogeneous A-harmonic (7) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|TP(u)\|_{W^{1,\varphi}(B,\wedge^l)} \leq C|B|\,\|u\|_{L^\varphi(\sigma B)} \tag{151}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

Proof. Applying (6) to $P(u)$, then using (145), we find that

$$\begin{aligned}
\|TP(u)\|_{q,B} &\leq C_1|B|\operatorname{diam}(B)\|P(u)\|_{q,B} \\
&\leq C_2|B|\operatorname{diam}(B)\|u\|_{q,B}, \\
\|\nabla TP(u)\|_{q,B} &\leq C_3|B|\operatorname{diam}(B)\|P(u)\|_{q,B} \\
&\leq C_4|B|\operatorname{diam}(B)\|u\|_{q,B}
\end{aligned} \tag{152}$$

for any differential form u and all balls B with $B \subset \Omega$, where $q > 1$ is a constant. Starting with (152) and using the similar method developed in the proof of Theorem 61, we obtain

$$\begin{aligned}
\|TP(u)\|_{L^\varphi(B)} &\leq C_5|B|\operatorname{diam}(B)\|u\|_{L^\varphi(\sigma_1 B)}, \\
\|\nabla TP(u)\|_{L^\varphi(B)} &\leq C_6|B|\,\|u\|_{L^\varphi(\sigma_2 B)},
\end{aligned} \tag{153}$$

respectively, where σ_1 and σ_2 are constants. From $(**)$, (153), we have

$$\begin{aligned}
&\|TP(u)\|_{W^{1,\varphi}(B,\wedge^l)} \\
&= (\operatorname{diam}(B))^{-1}\|TP(u)\|_{L^\varphi(B)} + \|\nabla TP(u)\|_{L^\varphi(B)} \\
&= (\operatorname{diam}(B))^{-1}\left(C_5|B|\operatorname{diam}(B)\|u\|_{L^\varphi(\sigma_1 B)}\right) \\
&\quad + C_6|B|\,\|u\|_{L^\varphi(\sigma_2 B)} \\
&\leq C_7|B|\,\|u\|_{L^\varphi(\sigma B)},
\end{aligned} \tag{154}$$

where $\sigma = \max\{\sigma_1, \sigma_2\}$. The proof of Theorem 63 has been completed. □

Note that if we choose $\varphi(t) = t^p \log_+^\alpha t$ or $\varphi(t) = t^p$ in Theorems 59–63, we will obtain some $L^p(\log_+^\alpha L)$-norm or L^p-norm inequalities, respectively. For example, let $\varphi(t) = t^p\log_+^\alpha t$ in Theorem 62; we have the following imbedding inequalities for $T \circ P$ with the $L^p(\log_+^\alpha L)$-norms.

Corollary 64. *Let $\varphi(t) = t^p\log_+^\alpha t$, $p \geq 1$, and $\alpha \in \mathbb{R}$, and Ω be a bounded domain. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the nonhomogeneous A-harmonic (7). Then, there exists a constant C, independent of u, such that*

$$\left\|T(P(u)) - (T(P(u)))_B\right\|_{W^{1,t^p\log_+^\alpha t}(B,\wedge^l)} \leq C|B|\,\|u\|_{L^p(\log_+^\alpha L)(\sigma B)} \tag{155}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

Selecting $\varphi(t) = t^p$ in Theorem 62, we obtain the usual imbedding inequalities $T \circ P$ with the L^p-norms.

$$\left\|T(P(u)) - (T(P(u)))_B\right\|_{W^{1,p}(B,\wedge^l)} \leq C|B|\,\|u\|_{p,\sigma B} \tag{156}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant. Now, we present the global imbedding theorem with the L^φ-norm as follows.

Theorem 65. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, Ω be any bounded L^φ-averaging domain, T be the homotopy operator defined in (2), and let P be* *the potential operator defined in* (143) *with the kernel* $K(x, y)$ *satisfying condition (i) of the standard estimates. Assume that* $\varphi(|v|) \in L^1(\Omega)$ *and* $v \in D'(\Omega, \wedge^1)$ *is a solution of the nonhomogeneous A-harmonic* (7) *in* Ω*. Then, there exists a constant C, independent of v, such that*

$$\left\|T(P(v)) - (T(P(v)))_{B_0}\right\|_{W^{1,\varphi}(\Omega)} \le C\|v\|_{L^\varphi(\Omega)} \tag{157}$$

where $B_0 \subset \Omega$ *is some fixed ball.*

It is well known that any John domain is a special L^φ-averaging domain; see [1]. Hence, we have the following global L^φ-imbedding theorem for John domains.

Theorem 66. *Let* φ *be a Young function in the class* $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$, Ω *be any bounded John domain, T be the homotopy operator defined in* (2)*, and let P be the potential operator defined in* (143) *with the kernel* $K(x, y)$ *satisfying condition (i) of the standard estimates. Assume that* $\varphi(|v|) \in L^1(\Omega)$ *and* $v \in D'(\Omega, \wedge^1)$ *is a solution of the nonhomogeneous A-harmonic* (7) *in* Ω*. Then, there exists a constant C, independent of v, such that*

$$\left\|T(P(v)) - (T(P(v)))_{B_0}\right\|_{W^{1,\varphi}(\Omega)} \le C\|v\|_{L^\varphi(\Omega)} \tag{158}$$

where $B_0 \subset \Omega$ *is some fixed ball.*

Next, let S be the set of all solutions of the nonhomogeneous A-harmonic equation in Ω. We have the following version of imbedding theorem with L^φ norm for any bounded domain, which says that the composite operator $T \circ P$ maps $W^{1,\varphi}(\Omega, \wedge^1) \cap S$ continuously into $L^\varphi(\Omega)$. See [18] for the proof of Theorem 67.

Theorem 67. *Let* φ *be a Young function in the class* $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$, *T be the homotopy operator defined in* (2)*, and let P be the potential operator defined in* (143) *with the kernel* $K(x, y)$ *satisfying condition (i) of the standard estimates. Assume that* $\varphi(|v|) \in L^1(\Omega)$ *and* $v \in D'(\Omega, \wedge^1) \cap S$ *in* Ω*. Then, the composite operator* $T \circ P$ *maps* $W^{1,\varphi}(\Omega, \wedge^1) \cap S$ *continuously into* $L^\varphi(\Omega)$*. Furthermore, there exists a constant C, independent of v, such that*

$$\|TP(v)\|_{W^{1,\varphi}(\Omega)} \le C\|v\|_{L^\varphi(\Omega)} \tag{159}$$

holds for any bounded domain Ω*.*

Selecting $\varphi(t) = t^p$ in Theorems 65, we have the following version of the imbedding inequality with L^p-norms.

Corollary 68. *Let* $\varphi(t) = t^p$, $p \ge 1$, *T be the homotopy operator defined in* (2)*, and let P be the potential operator defined in* (143)*. Assume that* $\varphi(|v|) \in L^1(\Omega)$ *and* $v \in D'(\Omega, \wedge^1)$ *is a solution of the nonhomogeneous A-harmonic* (7) *in* Ω*. Then, there exists a constant C, independent of v, such that*

$$\left\|TP(v) - (T(P(v)))_{B_0}\right\|_{W^{1,p}(\Omega)} \le C\|v\|_{p,\Omega} \tag{160}$$

holds for any bounded domain Ω*.*

Remark 69. (i) We know that the L^s-averaging domains are the special L^φ-averaging domains. Thus, Theorem 65 also holds for the L^s-averaging domain; (ii) Theorem 67 holds for any bounded domain in $\mathbb{R}^n$.

9. Composition of Homotopy and Green's Operators

In this section, we estimate the Lipschitz norm $\|\cdot\|_{\mathrm{locLip}_k, M}$ or BMO norm $\|\cdot\|_{\star,M}$ of composition $T \circ G$ in terms of the L^s norm. First, we present the following L^s norm inequality for the composition $T \circ G$ of the homotopy operator T and Green's operator G.

Theorem 70. *Let* $u \in L^s_{\mathrm{loc}}(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be a smooth differential form in a bounded, convex domain M and let* $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ *be the homotopy operator defined in* (2)*. Then, there exists a constant C, independent of u, such that*

$$\left\|T(G(u)) - (T(G(u)))_B\right\|_{s,B} \le C\,|B|\operatorname{diam}(B)\,\|u\|_{s,B} \tag{161}$$

for all balls $B \subset M$*.*

Using Theorem 70, we obtain the following inequality with Lipschitz norm.

Theorem 71. *Let* $u \in L^s(M, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, *be a smooth differential form in a bounded, convex domain M, let G be Green's operator, and let* $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ *be the homotopy operator defined in* (2)*. Then, there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{\mathrm{locLip}_k, M} \le C\|u\|_{s,M}, \tag{162}$$

where k is a constant with $0 \le k \le 1$*.*

The following Theorem 72 tells us the relationship between the Lipschitz norm $\|\cdot\|_{\mathrm{locLip}_k, M}$ and BMO norm $\|\cdot\|_{\star,M}$ of composition $T \circ G$.

Theorem 72. *Let* $u \in L^s_{\mathrm{loc}}(M, \wedge^1)$, $1 < s < \infty$, *be a solution of the nonhomogeneous A-harmonic* (7) *in a bounded, convex domain M. Let G be Green's operator and let T be the homotopy operator defined in* (2)*. Then, there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{\mathrm{locLip}_k, M} \le C\|u\|_{\star,M}, \tag{163}$$

where k is a constant with $0 \le k \le 1$*.*

The following theorem gives an estimate for BMO norm $\|\cdot\|_{\star,M}$ of composition $T \circ G$ in terms of L^s norm.

Theorem 73. *Let* $u \in L^s(M, \wedge^1)$, $1 < s < \infty$, *be a solution of the nonhomogeneous A-harmonic* (7) *in a bounded, convex domain M. Let G be Green's operator and let T be the homotopy operator defined in* (2)*. Then, there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{\star,M} \le C\|u\|_{s,M}. \tag{164}$$

Theorem 74. *Let $u \in L^s(M, \wedge^l, \nu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation in a bounded, convex domain M. Let G be Green's operator and let T be the homotopy operator defined in* (2). *The measures μ and ν are defined by $d\mu = w_1^\alpha dx$, $d\nu = w_2^\alpha dx$, and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $w_1(x) \geq \varepsilon > 0$ for any $x \in M$. Then, there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{\mathrm{locLip}_k, M, w_1^\alpha} \leq C\|u\|_{s,M,w_2^\alpha}, \tag{165}$$

where k and α are constants with $0 \leq k \leq 1$ *and* $0 < \alpha \leq 1$.

Finally, we can estimate the weighted $\|\cdot\|_{\star,M,w_1^\alpha}$ norm in terms of the L^s norm.

Theorem 75. *Let $u \in L^s(M, \wedge^l, \nu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation in a bounded, convex domain M. Let G be Green's operator and let T be the homotopy operator defined in* (2). *The measures μ and ν are defined by $d\mu = w_1^\alpha dx$, $d\nu = w_2^\alpha dx$, and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $w_1(x) \geq \varepsilon > 0$ for any $x \in M$. Then, there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{\star, M, w_1^\alpha} \leq C\|u\|_{s,M,w_2^\alpha}, \tag{166}$$

where α is a constant with $0 < \alpha \leq 1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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