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Research Article

A New Iterative Method for the Set of Solutions of Equilibrium Problems and of Operator Equations with Inverse-Strongly Monotone Mappings

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The purpose of the paper is to present a new iteration method for finding a common element for the set of solutions of equilibrium problems and of operator equations with a finite family of λ_i-inverse-strongly monotone mappings in Hilbert spaces.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and let G be a bifunction from $C\times C$ into $(-\infty,+\infty)$. The equilibrium problem for G is to find $u^*\in C$ such that

$$G(u^*,v)\geq 0,\quad \forall v\in C. \tag{1}$$

The set of solutions of (1) is denoted by EP(G).

Equilibrium problem (1) includes the numerous problems in physics, optimization, economics, transportation, and engineering, as special cases.

Assume that the bifunction G satisfies the following standard properties.

Assumption A. Let $G: C\times C\to(-\infty,+\infty)$ be a bifunction satisfying the conditions (A1)–(A4):

(A1) $G(u,u)=0$, $\forall u\in C$;

(A2) $G(u,v)+G(v,u)\leq 0$, $\forall(u,v)\in C\times C$;

(A3) for each $u\in C$, $G(u,\cdot): C\to(-\infty,+\infty)$ is lower semicontinuous and convex;

(A4) $\overline{\lim}_{t\to+0}G((1-t)u+tz,v)\leq G(u,v)$, $\forall(u,z,v)\in C\times C\times C$.

Let $\{T_i\}$, $i=1,\ldots,N$, be a finite family of k_i-strictly pseudocontractive mappings from C into C with the set of fixed points $F(T_i)$; that is,

$$F(T_i)=\{x\in C: T_ix=x\}. \tag{2}$$

Assume that

$$\mathcal{S}:=\bigcap_{i=1}^{N}F(T_i)\cap\mathrm{EP}(G)\neq\emptyset. \tag{3}$$

The problem of finding an element

$$u^*\in\mathcal{S} \tag{4}$$

is studied intensively in [1–27].

Recall that a mapping T in H is said to be a k-strictly pseudocontractive mapping in the terminology of Browder and Petryshyn [28] if there exists a constant $0\leq k<1$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2, \tag{5}$$

for all $x,y\in D(T)$, the domain of T, where I is the identity operator in H. Clearly, if $k=0$, then T is nonexpansive; that is,

$$\|T(x)-T(y)\|\leq\|x-y\|. \tag{6}$$

We know that the class of k-strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings.

In the case that $T_i \equiv I$, (4) is reduced to the equilibrium problem (1) and shown in [5, 23] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, and certain fixed point problems (see also [29]). For finding approximative solutions of (1) there exist several methods: the regularization approach in [7, 9, 15, 24, 30, 31], the gap-function approach in [8, 15, 16, 18, 19], and the iterative procedure approach in [1–4, 6, 8, 11–14, 19–22, 32, 33].

In the case that $G \equiv 0$ and $N = 1$, (4) is a problem of finding a fixed point for a k-strictly pseudocontractive mapping in C and is given by Marino and Xu [17].

Theorem 1 (see [17]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping for some $0 \leq k < 1$, and assume that*

$$F(T) \neq \emptyset. \tag{7}$$

Let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned} &x_0 \in C, \\ &y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ &C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\qquad + (1-\alpha_n)(k-\alpha_n)\|x_n - Tx_n\|^2\}, \\ &Q_n = \{z \in C : \langle x_n - z,\ x_0 - x_n\rangle \geq 0\}, \\ &x_{n+1} = P_{C_n \cap Q_n} x_0. \end{aligned} \tag{8}$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n < 1$ for all n. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, the projection of x_0 onto $F(T)$.

For the case that $G \equiv 0$ and $N > 1$, (4) is a problem of finding a common fixed point for a finite family of k_i-strictly pseudocontractive mappings T_i in C and is studied in [27].

Let $x_0 \in C$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ three sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, and let $\{u_n\}$ be a sequence in C. Then the sequence $\{x_n\}$ generated by

$$\begin{aligned} &x_1 = \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\ &x_2 = \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\ &\vdots \\ &x_N = \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\ &x_{N+1} = \alpha_{N+1} x_N + \beta_{N+1} T_1 x_{N+1} + \gamma_{N+1} u_{N+1}, \\ &\vdots \end{aligned} \tag{9}$$

is called the implicit iteration process with mean errors for a finite family of strictly pseudocontractive mappings $\{T_i\}_{i=1}^N$.

The scheme (9) can be expressed in the compact form as

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n, \tag{10}$$

where $T_n = T_{n \bmod N}$.

Theorem 2 (see [27]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of strictly pseudocontractive mappings of C into itself such that*

$$\bigcap_{i=1}^{N} F(T_i) \neq \emptyset. \tag{11}$$

Let $x_0 \in C$ and let $\{u_n\}$ be a bounded sequence in C; let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$;

(ii) *there exist constants* σ_1, σ_2 *such that* $0 < \sigma_1 \leq \beta_n \leq \sigma_2 < 1$, $\forall n \geq 1$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the implicit iterative sequence $\{x_n\}$ defined by (9) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$. Moreover, if there exists $i_0 \in \{1, 2, \ldots, N\}$ such that T_{i_0} is demicompact, then $\{x_n\}$ converges strongly.

If G is an arbitrary bifunction satisfying Assumption A and $N = 1$, then (4) is a problem of finding a common element of the fixed point set for a k-strictly pseudocontractive mapping in C and of the solution set of equilibrium problem for G (see [26]).

Theorem 3 (see [26]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying Assumption A, and let T be a nonexpansive mapping of C into H such that*

$$F(T) \cap EP(G) \neq \emptyset. \tag{12}$$

Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{gathered} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T u_n, \end{gathered} \tag{13}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\begin{gathered} \lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{gathered} \tag{14}$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap EP(G)$, where

$$z = P_{F(T)\cap EP(G)} f(z). \tag{15}$$

Set $A_i = I - T_i$. Obviously, A_i are λ_i-inverse-strongly monotone; that is,

$$\langle A_i(x) - A_i(y), x - y\rangle \geq \lambda_i \|A_i(x) - A_i(y)\|^2, \quad \forall x, y \in D(A_i), \quad \lambda_i = \frac{1-k_i}{2}. \tag{16}$$

From now on, let $\{A_i\}_{i=1}^N$ be a finite family of λ_i-inverse-strongly monotone mappings in H with $C \subset \bigcap_{i=1}^N D(A_i)$ and $\lambda_i > 0$, $i = 1, \ldots, N$. On the other hand, if there exists $i_0 \in \{1, 2, \ldots, N\}$ such that $\lambda_{i_0} > 1$, then A_{i_0} is a contraction; that is, $\|A_{i_0}(x) - A_{i_0}(y)\| \leq (1/\lambda_{i_0})\|x - y\|$ with $1/\lambda_{i_0} < 1$. And hence, A_{i_0} has only one solution and, consequently, the stated problem does not have sense. So, without loss of generality, assume that $0 < \lambda_i \leq 1$, $i = 1, \ldots, N$.

Set

$$S = \bigcap_{i=1}^N S_i, \tag{17}$$

where $S_i = \{x \in C : A_i(x) = 0\}$ is the solution set of A_i in C.

Assume that $\mathrm{EP}(G) \cap S \neq \emptyset$.

Our problem is to find an element

$$u^* \in \mathrm{EP}(G) \cap S. \tag{18}$$

Since the mapping $A = I - T$ is $(1/2)$-inverse-strongly monotone for each nonexpansive mapping T, the problem of finding an element $u^* \in C$, which is not only a solution of a variational inequality involving an inverse-strongly monotone mapping but also a fixed point of a nonexpansive mapping, is a particular case of (18).

For instance, the case that $G(u, v) \equiv \langle A(u), v - u\rangle$, where A is some inverse-strongly monotone mapping and $N = 1$, is studied in [25].

Theorem 4 (see [25]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\lambda > 0$. Let A be a λ-inverse-strongly monotone mapping of C into H, and let T be a nonexpansive mapping of C into itself such that*

$$F(T) \cap VI(C, A) \neq \emptyset, \tag{19}$$

where $VI(C, A)$ denotes the solution set of the following variational inequality: find $x^* \in C$ such that

$$\langle A(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{20}$$

Let $\{x_n\}$ be a sequence defined by

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C(x_n - \lambda_n A(x_n)), \tag{21}$$

for every $n = 0, 1, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\lambda)$ and $\{\alpha_n\} \subset (c, d)$ for some $c, d \in (0, 1)$. Then, $\{x_n\}$ converges weakly to $z \in F(T) \cap VI(C, A)$, where

$$z = \lim_{n\to\infty} P_{F(T)\cap VI(C,A)} x_n. \tag{22}$$

The following theorem is an improvement of Theorem 4 for the case of nonself-mapping.

Theorem 5 (see [34]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a λ-inverse-strongly monotone mapping of C into H, and let T be a nonexpansive nonself-mapping of C into H such that*

$$F(T) \cap VI(C, A) \neq \emptyset. \tag{23}$$

Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = P_C(\alpha_n x + (1 - \alpha_n) T P_C(x_n - \lambda_n A(x_n))) \tag{24}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \tag{25}$$

then $\{x_n\}$ converges strongly to $P_{F(T)\cap VI(C,A)} x$.

We know that λ-inverse-strongly monotone mapping is $(1/\lambda)$-Lipschitz continuous and monotone. Therefore, for the case that $G(u, v) \equiv \langle A(u), v - u\rangle$, where A is not inverse-strongly monotone, but Lipschitz continuous and monotone, Nadezhkina and Takahashi [35] prove the following theorem.

Theorem 6 (see [35]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H, and let T be a nonexpansive mapping of C into itself such that*

$$F(T) \cap VI(C, A) \neq \emptyset. \tag{26}$$

Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n A(x_n)), \\ z_n &= P_C(x_n - \lambda_n A(y_n)), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{27}$$

for every $n = 0, 1, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\alpha_n \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $P_{F(T)\cap VI(C,A)} x$.

Some similar results are also considered in [36, 37].

Buong [38] introduced two new implicit iteration methods for solving problem (18).

We construct a regularization solution u_n of the following single equilibrium problem: find $u_n \in C$ such that

$$\mathcal{F}(u_n, v) \geq 0, \quad \forall v \in C, \tag{28}$$

where

$$\mathcal{F}(u,v) := G(u,v) + \sum_{i=1}^{N} \alpha_n^{\mu_i} G_i(u,v) + \alpha_n \langle u, v-u \rangle, \quad \alpha_n > 0, \tag{29}$$

$$G_i(u,v) = \langle A_i(u), v-u \rangle, \quad i = 1,\dots,N,$$

$$0 < \mu_i < \mu_{i+1} < 1, \quad i = 2,\dots,N-1,$$

and $\{\alpha_n\}$ is the positive sequence of regularization parameters that converges to 0, as $n \to +\infty$.

The first one is the following theorem.

Theorem 7 (see [38]). *For each $\alpha_n > 0$, problem* (28) *has a unique solution u_n such that*

(i) $\lim_{n\to+\infty} u_n = u^*$, $u^* \in EP(G) \cap S$, $\|u^*\| \le \|y\|$, $\forall y \in EP(G) \cap S$;

(ii)

$$\|u_n - u_m\| \le (\|u^*\| + dN) \frac{|\alpha_n - \alpha_m|}{\alpha_n}, \tag{30}$$

where d is a positive constant.

Next, we introduce the second result. Let $\{\tilde{c}_n\}$ and $\{\gamma_n\}$ be some sequences of positive numbers, and let z_0 and z_1 be two arbitrary elements in C. Then, the sequence $\{z_n\}$ of iterations is defined by the following equilibrium problem: find $z_{n+1} \in C$ such that

$$\tilde{c}_n \left(G(z_{n+1}, v) + \sum_{i=1}^{N} \alpha_n^{\mu_i} G_i(z_{n+1}, v) + \alpha_n \langle z_{n+1}, v - z_{n+1} \rangle \right) + \langle z_{n+1} - z_n, v - z_{n+1} \rangle - \gamma_n \langle z_n - z_{n-1}, v - z_{n+1} \rangle \ge 0, \quad \forall v \in C. \tag{31}$$

Theorem 8 (see [38]). *Assume that the parameters $\tilde{c}_n$, γ_n, and α_n are chosen such that*

(i) $0 < c_0 < \tilde{c}_n$, $0 \le \gamma_n < \gamma_0$,

(ii) $\sum_{n=1}^{\infty} b_n = +\infty$, $b_n = \tilde{c}_n\alpha_n/(1+\tilde{c}_n\alpha_n)$,

(iii) $\sum_{n=1}^{\infty} \gamma_n b_n^{-1} \|z_n - z_{n-1}\| < +\infty$,

(iv) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+1}|/\alpha_n b_n) = 0$.

Then, the sequence $\{z_n\}$ defined by (31) *converges strongly to the element u^*, as $n \to +\infty$.*

In this paper, we consider the new another iteration method: for an arbitrary element x_0 in H, the sequence $\{x_n\}$ of iterations is defined by finding $u_n \in C$ such that

$$G(u_n, y) + \langle u_n - x_n, y - u_n \rangle \ge 0, \quad \forall y \in C,$$

$$\begin{aligned} x_{n+1} &= P_C\left(x_n - \beta_n \left[x_n - u_n + \sum_{i=1}^{N} \alpha_n^{\mu_i} A_i(x_n) + \alpha_n x_n \right] \right) \\ &= P_C\left(x_n - \beta_n \left[\sum_{i=1}^{N} \alpha_n^{\mu_i} A_i(x_n) + (1+\alpha_n) x_n - u_n \right] \right), \end{aligned} \tag{32}$$

where P_C is the metric projection of H onto C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers.

The strong convergence of the sequence $\{x_n\}$ defined by (32) is proved under some suitable conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ in the next section.

2. Main Results

We formulate the following lemmas for the proof of our main theorems.

Lemma 9 (see [9]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let G be a bifunction of $C \times C$ into $(-\infty, +\infty)$ satisfying Assumption A. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z,y) + \frac{1}{r}\langle z - x, y - z \rangle \ge 0, \quad \forall y \in C. \tag{33}$$

Lemma 10 (see [9]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $G : C \times C \to (-\infty, +\infty)$ satisfies Assumption A. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle z - x, y - z \rangle \ge 0 \right\}, \quad \forall y \in C. \tag{34}$$

Then, the following statements hold:

(i) *T_r is single valued;*

(ii) *T_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y \rangle; \tag{35}$$

(iii) *$F(T_r) = EP(G)$;*

(iv) *$EP(G)$ is closed and convex.*

Lemma 11 (see [36]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be the sequences of positive numbers satisfying the following conditions:*

(i) $a_{n+1} \le (1 - b_n)a_n + c_n$,

(ii) $\sum_{n=0}^{\infty} b_n = +\infty$, $b_n < 1$, $\lim_{n\to+\infty}(c_n/b_n) = 0$.

Then, $\lim_{n\to+\infty} a_n = 0$.

Lemma 12 (see [38]). *Let A be any inverse-strongly monotone mapping from C into H with the solution set $S_A := \{x \in C : A(x) = 0\}$, and let C_0 be a closed convex subset of C such that*

$$S_A \cap C_0 \neq \emptyset. \tag{36}$$

Then, the solution set of the following variational inequality

$$\langle A(\tilde{y}), x - \tilde{y}\rangle \geq 0, \quad \forall x \in C_0,\ \tilde{y} \in C_0, \tag{37}$$

is coincided with $S_A \cap C_0$.

From Lemma 9, we can consider the firmly nonexpansive mapping T_0 defined by

$$T_0(x) = \{z \in C : G(z, y) + \langle z - x, y - z\rangle \geq 0,\ \forall y \in C\}, \quad \forall x \in H. \tag{38}$$

From Lemma 10, we know that T_0 is nonexpansive. Consequently, $A_0 := I - T_0$ is $(1/2)$-inverse-strongly monotone. Let

$$S_0 := \{x \in C : A_0(x) = 0\}. \tag{39}$$

Then, $S_0 = \mathrm{EP}(G)$ and problem (18) are equivalent to finding

$$u^* \in S_0 \cap S. \tag{40}$$

Now, we construct a regularization solution y_n for (40) by solving the following variational inequality problem: find $y_n \in C$ such that

$$\left\langle \sum_{i=0}^{N} \alpha_n^{\mu_i} A_i(y_n) + \alpha_n y_n, v - y_n \right\rangle \geq 0, \quad \forall v \in C, \\ \mu_0 = 0 < \mu_1 < \cdots < \mu_N < 1, \tag{41}$$

where the positive regularization parameter $\alpha_n \to 0$, as $n \to +\infty$.

Now we are in a position to introduce and prove the main results.

Theorem 13. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying Assumption A and let $\{A_i\}_{i=1}^N$ be a finite family of λ_i-inverse-strongly monotone mappings in H with $C \subset \bigcap_{i=1}^N D(A_i)$ and $\lambda_i > 0$, $i = 1, \ldots, N$, such that*

$$EP(G) \cap S \neq \emptyset, \tag{42}$$

where $EP(G)$ denotes the set of solutions for (1) and

$$S = \bigcap_{i=1}^{N} S_i, \quad S_i = \{x \in C : A_i(x) = 0\}. \tag{43}$$

Then, for each $\alpha_n > 0$, problem (41) has a unique solution y_n such that

(i) $\lim_{n\to+\infty} y_n = u^*$, $u^* \in EP(G) \cap S$,

(ii) $\|u^*\| \leq \|y\|$, $\forall y \in EP(G) \cap S$,

(iii)

$$\|y_n - y_m\| \leq \frac{|\alpha_n - \alpha_m|}{\alpha_n}\left(\|u^*\| + dN\right), \tag{44}$$

where d is some positive constant.

Proof. From Lemma 12, we know that S_0 is the set of solutions for the following variational inequality problem: find $u^* \in C$ such that

$$\langle A_0(u^*), v - u^*\rangle \geq 0, \quad \forall v \in C. \tag{45}$$

If we define the new bifunction $G_0(u, v)$ by

$$G_0(u, v) = \langle A_0(u^*), v - u^*\rangle, \tag{46}$$

then problem (41) is the same as (28) with a new $G(u, v)$, and the proof for the theorem is a complete repetition of the proof for Theorem 2.1 in [38].

Set

$$L = \max\left\{2, \frac{1}{\lambda_i},\ i = 1, \ldots, N\right\}. \tag{47}$$

□

Theorem 14. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying Assumption A and let $\{A_i\}_{i=1}^N$ be a finite family of λ_i-inverse-strongly monotone mappings in H with $C \subset \bigcap_{i=1}^N D(A_i)$ and $\lambda_i > 0$, $i = 1, \ldots, N$, such that*

$$EP(G) \cap S \neq \emptyset, \tag{48}$$

where $EP(G)$ denotes the set of solutions for (1) and

$$S = \bigcap_{i=1}^{N} S_i, \quad S_i = \{x \in C : A_i(x) = 0\}. \tag{49}$$

Suppose that α_n, β_n satisfy the following conditions:

$$\alpha_n, \beta_n > 0\ (\alpha_n \leq 1), \qquad \lim_{n\to\infty} \alpha_n = 0, \\ \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2 \beta_n} = 0, \qquad \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty, \\ \overline{\lim_{n\to\infty}}\, \beta_n \frac{(L(N+1) + \alpha_n)^2}{\alpha_n} < 1. \tag{50}$$

Then, the sequence $\{x_n\}$ defined by (32) converges strongly to $u^* \in EP(G) \cap S$; that is,

$$\lim_{n\to\infty} x_n = u^* \in EP(G) \cap S. \tag{51}$$

Proof. Let y_n be the solution of (41). Then,

$$y_n = P_C\left(y_n - \beta_n\left[\sum_{i=0}^{N} \alpha_n^{\mu_i} A_i(y_n) + \alpha_n y_n\right]\right). \tag{52}$$

Set $\Delta_n = \|x_n - y_n\|$. Obviously,

$$\Delta_{n+1} = \|x_{n+1} - y_{n+1}\| \le \|x_{n+1} - y_n\| + \|y_{n+1} - y_n\|. \quad (53)$$

From the nonexpansivity of P_C, the monotone and Lipschitz continuous properties of A_i, $i = 0, \ldots, N$, (41), (52), and $y_n = T_0(x_n)$, we have

$$\begin{aligned}\|x_{n+1} - y_n\| &\le \left\| x_n - y_n - \beta_n \left[\sum_{i=0}^{N} \alpha_n^{\mu_i} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n) \right] \right\|,\end{aligned}$$

$$\begin{aligned}&\left\| x_n - y_n - \beta_n \left[\sum_{i=0}^{N} \alpha_n^{\mu_i} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n) \right] \right\|^2 \\ &= \|x_n - y_n\|^2 + \beta_n^2 \left\| \left[\sum_{i=0}^{N} \alpha_n^{\mu_i} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n) \right] \right\|^2 \\ &\quad - 2\beta_n \left\langle \sum_{i=0}^{N} \alpha_n^{\mu_i} (A_i(x_n) - A_i(y_n)) + \alpha_n (x_n - y_n), x_n - y_n \right\rangle \\ &\le \|x_n - y_n\|^2 \left[1 - 2\beta_n\alpha_n + \beta_n^2 \left(2 + \sum_{i=1}^{N} \alpha_n^{\mu_i} \frac{1}{\lambda_i} + \alpha_n \right)^2 \right].\end{aligned} \quad (54)$$

Thus,

$$\|x_{n+1} - y_n\| \le \Delta_n \left(1 - 2\beta_n\alpha_n + \beta_n^2 (L(N+1) + \alpha_n)^2\right)^{1/2}. \quad (55)$$

Therefore,

$$\begin{aligned}\Delta_{n+1} &\le \Delta_n \left(1 - 2\beta_n\alpha_n + \beta_n^2 (L(N+1) + \alpha_n)^2\right)^{1/2} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} (\|u^*\| + dN) \\ &\le \Delta_n (1 - \alpha_n\beta_n)^{1/2} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} (\|u^*\| + dN).\end{aligned} \quad (56)$$

We note that, for $\varepsilon > 0$, $a > 0$, $b > 0$, the inequality

$$(a+b)^2 \le (1+\varepsilon)\left(a^2 + \frac{b^2}{\varepsilon}\right) \quad (57)$$

holds. Thus, applying inequality (57) for $\varepsilon = \alpha_n\beta_n/2$, we obtain

$$\begin{aligned}0 &\le \Delta_{n+1}^2 \\ &\le \Delta_n^2 (1 - \alpha_n\beta_n)\left(1 + \frac{1}{2}\alpha_n\beta_n\right) + \left(\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} (\|u^*\| + dN)\right)^2 \frac{2}{\alpha_n\beta_n}\left(1 + \frac{1}{2}\alpha_n\beta_n\right) \\ &= \Delta_n^2 \left(1 - \frac{1}{2}\alpha_n\beta_n - \frac{1}{2}(\alpha_n\beta_n)^2\right) + \left(\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2\beta_n} (\|u^*\| + dN)\right)^2 2\alpha_n\beta_n \left(1 + \frac{1}{2}\alpha_n\beta_n\right).\end{aligned} \quad (58)$$

Set

$$\begin{aligned}b_n &= \alpha_n\beta_n \left(\frac{1}{2} + \frac{1}{2}\alpha_n\beta_n\right) \\ c_n &= \left(\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2\beta_n} (\|u^*\| + dN)\right)^2 2\alpha_n\beta_n \left(1 + \frac{1}{2}\alpha_n\beta_n\right).\end{aligned} \quad (59)$$

Then, it is not difficult to check that b_n and c_n satisfy the conditions in Lemma 11 for sufficiently large n. Hence, $\lim_{n\to+\infty} \Delta_n^2 = 0$. Since $\lim_{n\to\infty} y_n = u^*$, we have

$$\lim_{n\to\infty} x_n = u^* \in \mathrm{EP}(G) \cap S. \quad (60)$$

This completes the proof. □

Remark 15. The sequences $\alpha_n = (1+n)^{-p}$, $0 < p < 1/2$, and $\beta_n = \gamma_0\alpha_n$ with

$$0 < \gamma_0 < \frac{1}{(L(N+1) + \alpha_0)^2} \quad (61)$$

satisfy all the necessary conditions in Theorem 14.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

The main idea of this paper was proposed by Jong Kyu Kim. Jong Kyu Kim and Nguyen Buong prepared the paper initially and performed all the steps of proof in this research. All authors read and approved the final paper.

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