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Research Article

A Fixed Point Theorem for Multivalued Mappings with δ-Distance

Özlem Acar and Ishak Altun

Department of Mathematics, Faculty of Science and Arts, Kirikkale University, Yahsihan, 71450 Kirikkale, Turkey

Correspondence should be addressed to Özlem Acar; acarozlem@ymail.com

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We mainly study fixed point theorem for multivalued mappings with δ-distance using Wardowski's technique on complete metric space. Let (X,d) be a metric space and let $B(X)$ be a family of all nonempty bounded subsets of X. Define $\delta : B(X) \times B(X) \to \mathbb{R}$ by $\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$. Considering δ-distance, it is proved that if (X,d) is a complete metric space and $T : X \to B(X)$ is a multivalued certain contraction, then T has a fixed point.

1. Introduction

Fixed point theory concern itself with a very basic mathematical setting. It is also well known that one of the fundamental and most useful results in fixed point theory is Banach fixed point theorem. This result has been extended in many directions for single and multivalued cases on a metric space X (see [1–9]). Fixed point theory for multivalued mappings is studied by both Pompeiu-Hausdorff metric H [10, 11], which is defined on $CB(X)$ (the family of all nonempty, closed, and bounded subsets of X), and δ-distance, which is defined on $B(X)$ (the family of all nonempty and bounded subsets of X). Using Pompeiu-Hausdorff metric, Nadler [12] introduced the concept of multivalued contraction mapping and show that such mapping has a fixed point on complete metric space. Then many authors focused on this direction [13–18]. On the other hand, Fisher [19] obtained different type of multivalued fixed point theorems defining δ-distance between two bounded subsets of a metric space X. We can find some results about this way in [20–23].

In this paper, we give some new multivalued fixed point results by considering the δ-distance. For this we use the recent technique, which was given by Wardowski [24]. For the sake of completeness,we will discuss its basic lines. Let $\mathcal{F}$ be the set of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing; that is, for all $\alpha, \beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(F2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1 (see [24]). Let (X,d) be a metric space and let $T : X \to X$ be a mapping. Given $F \in \mathcal{F}$, we say that T is F-contraction, if there exists $\tau > 0$ such that

$$x, y \in X, \quad d(Tx,Ty) > 0 \Longrightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y)). \tag{1}$$

Taking different functions $F \in \mathcal{F}$ in (1), one gets a variety of F-contractions, some of them being already known in the literature. The following examples will certify this assertion.

Example 2 (see [24]). Let $F_1 : (0,\infty) \to \mathbb{R}$ be given by the formulae $F_1(\alpha) = \ln\alpha$. It is clear that $F_1 \in \mathcal{F}$. Then each self-mapping T on a metric space (X,d) satisfying (1) is an F_1-contraction such that

$$d(Tx,Ty) \le e^{-\tau} d(x,y), \quad \forall x,y \in X,\ Tx \ne Ty. \tag{2}$$

It is clear that for x, $y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \le e^{-\tau} d(x,y)$ also holds. Therefore

T satisfies Banach contraction with $L = e^{-\tau}$; thus T is a contraction.

Example 3 (see [24]). Let $F_2 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_2(\alpha) = \alpha + \ln \alpha$. It is clear that $F_2 \in \mathcal{F}$. Then each self-mapping T on a metric space (X, d) satisfying (1) is an F_2-contraction such that

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \le e^{-\tau}, \quad \forall x, y \in X, \ Tx \ne Ty. \tag{3}$$

We can find some different examples for the function F belonging to $\mathcal{F}$ in [24]. In addition, Wardowski concluded that every F-contraction T is a contractive mapping, that is,

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in X, \ Tx \ne Ty. \tag{4}$$

Thus, every F-contraction is a continuous mapping.

Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \le F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He noted that, for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, it was obtained that every Banach contraction satisfies the contractive condition (3). On the other side, [24, Example 2.5] shows that the mapping T is not an F_1-contraction (Banach contraction) but still is an F_2-contraction. Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem 4 (see [24]). *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

Following Wardowski, Mınak et al. [25] introduced the concept of Ćirić type generalized F-contraction. Let (X, d) be a metric space and let $T : X \to X$ be a mapping. Given $F \in \mathcal{F}$, we say that T is a Ćirić type generalized F-contraction if there exists $\tau > 0$ such that

$$\begin{gathered} x, y \in X, \\ d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \le F(m(x, y)), \end{gathered} \tag{5}$$

where

$$\begin{aligned} m(x, y) = \max \Big\{ & d(x, y), d(x, Tx), d(y, Ty), \\ & \frac{1}{2}[d(x, Ty) + d(y, Tx)] \Big\}. \end{aligned} \tag{6}$$

Then the following theorem was given.

Theorem 5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Ćirić type generalized F-contraction. If T or F is continuous, then T has a unique fixed point in X.*

Considering the Pompeiu-Hausdorff metric H, both Theorems 4 and 5 were extended to multivalued cases in [26] and [27], respectively (see also [28, 29]). In this work, we give a fixed point result for multivalued mappings using the δ-distance. First recall some definitions and notations which are used in this paper.

Let (X, d) be a metric space. For $A, B \in B(X)$ we define

$$\begin{gathered} \delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}, \\ D(a, B) = \inf\{d(a, b) : b \in B\}. \end{gathered} \tag{7}$$

If $A = \{a\}$ we write $\delta(A, B) = \delta(a, B)$ and also if $B = \{b\}$, then $\delta(a, B) = d(a, b)$. It is easy to prove that for $A, B, C \in B(X)$

$$\begin{gathered} \delta(A, B) = \delta(B, A) \ge 0, \\ \delta(A, B) \le \delta(A, C) + \delta(C, B), \\ \delta(A, A) = \sup\{d(a, b) : a, b \in A\} = \operatorname{diam} A, \\ \delta(A, B) = 0, \quad \text{implies that } A = B = \{a\}. \end{gathered} \tag{8}$$

If $\{A_n\}$ is a sequence in $B(X)$, we say that $\{A_n\}$ converges to $A \subseteq X$ and write $A_n \to A$ if and only if

(i) $a \in A$ implies that $a_n \to a$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in \mathbb{N}$,

(ii) for any $\varepsilon > 0$, $\exists m \in \mathbb{N}$ such that $A_n \subseteq A_\varepsilon$ for $n > m$, where

$$A_\varepsilon = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}. \tag{9}$$

Lemma 6 (see [20]). *Suppose $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ and (X, d) is a complete metric space. If $A_n \to A \in B(X)$ and $B_n \to B \in B(X)$ then $\delta(A_n, B_n) \to \delta(A, B)$.*

Lemma 7 (see [20]). *If $\{A_n\}$ is a sequence of nonempty bounded subsets in the complete metric space (X, d) and if $\delta(A_n, y) \to 0$ for some $y \in X$, then $A_n \to \{y\}$.*

2. Main Result

In this section, we prove a fixed point theorem for multivalued mappings with δ-distance and give an illustrative example.

Definition 8. Let (X, d) be a metric space and let $T : X \to B(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\tau + F(\delta(Tx, Ty)) \le F(M(x, y)), \tag{10}$$

for all $x, y \in X$ with $\min\{\delta(Tx, Ty), d(x, y)\} > 0$, where

$$\begin{aligned} M(x, y) = \max \Big\{ & d(x, y), D(x, Tx), D(y, Ty), \\ & \frac{1}{2}[D(x, Ty) + D(y, Tx)] \Big\}. \end{aligned} \tag{11}$$

Theorem 9. *Let (X, d) be a complete metric space and let $T : X \to B(X)$ be a multivalued F-contraction. If F is continuous and Tx is closed for all $x \in X$, then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and so the proof is completed. Thus, suppose that, for every $n \in \mathbb{N} \cup \{0\}$, $x_n \neq x_{n+1}$. So $d(x_n, x_{n+1}) > 0$ and $\delta(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$. Then, we have from (10)

$$\begin{aligned}
&\tau + F(d(x_n, x_{n+1})) \\
&\leq \tau + F(\delta(Tx_{n-1}, Tx_n)) \\
&\leq F(M(x_{n-1}, x_n)) \\
&= F\left(\max\left\{\begin{matrix} d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \\ \frac{1}{2}[D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})] \end{matrix}\right\}\right) \\
&\leq F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\
&= F(d(x_{n-1}, x_n)),
\end{aligned} \tag{12}$$

and so

$$\begin{aligned}
F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\
&\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\
&\vdots \\
&\leq F(d(x_0, x_1)) - n\tau.
\end{aligned} \tag{13}$$

Denote $a_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \ldots$. Then, $a_n > 0$ for all n and, using (10), the following holds:

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{14}$$

From (14), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0. \tag{15}$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0. \tag{16}$$

By (14), the following holds for all $n \in \mathbb{N}$:

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0. \tag{17}$$

Letting $n \to \infty$ in (17), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{18}$$

From (18), there exits $n_1 \in \mathbb{N}$ such that $n a_n^k \leq 1$ for all $n \geq n_1$. So we have

$$a_n \leq \frac{1}{n^{1/k}}, \tag{19}$$

for all $n \geq n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (19), we have

$$\begin{aligned}
&d(x_n, x_m) \\
&\quad\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\quad= a_n + a_{n+1} + \cdots + a_{m-1} \\
&\quad= \sum_{i=n}^{m-1} a_i \\
&\quad\leq \sum_{i=n}^{\infty} a_i \\
&\quad\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned} \tag{20}$$

By the convergence of the series $\sum_{i=1}^{\infty}(1/i^{1/k})$, we get $d(x_n, x_m) \to 0$ as $n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$; that is, $\lim_{n\to\infty} x_n = z$. Now, suppose F is continuous. In this case, we claim that $z \in Tz$. Assume the contrary; that is, $z \notin Tz$. In this case, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$. (Otherwise, there exists $n_1 \in \mathbb{N}$ such that $x_n \in Tz$ for all $n \geq n_1$, which implies that $z \in Tz$. This is a contradiction, since $z \notin Tz$.) Since $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$, then we have

$$\begin{aligned}
&\tau + F\left(D\left(x_{n_k+1}, Tz\right)\right) \\
&\quad\leq \tau + F\left(\delta\left(Tx_{n_k}, Tz\right)\right) \\
&\quad\leq F\left(M\left(x_{n_k}, z\right)\right) \\
&\quad\leq F\left(\max\left\{d\left(x_{n_k}, z\right), d\left(x_{n_k}, x_{n_k+1}\right), D(z, Tz),\right.\right. \\
&\qquad\qquad\left.\left.\frac{1}{2}\left[D\left(x_{n_k}, Tz\right) + d\left(z, x_{n_k+1}\right)\right]\right\}\right).
\end{aligned} \tag{21}$$

Taking the limit $k \to \infty$ and using the continuity of F, we have $\tau + F(D(z, \overline{Tz})) \leq F(D(z, Tz))$, which is a contradiction. Thus, we get $z \in \overline{Tz} = Tz$. This completes the proof. ☐

Example 10. Let $X = \{0, 1, 2, 3, \ldots\}$ and $d(x, y) = \begin{cases} 0; & x = y \\ x + y; & x \neq y \end{cases}$. Then (X, d) is a complete metric space. Define $T : X \to B(X)$ by

$$Tx = \begin{cases} \{0\}; & x = 0 \\ \{0, 1, 2, 3, \ldots, x - 1\}; & x \neq 0. \end{cases} \tag{22}$$

We claim that T is multivalued F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$. Because of the $\min\{\delta(Tx, Ty),$

$d(x, y)\} > 0$, we can consider the following cases while $x \neq y$ and $\{x, y\} \cap \{0, 1\}$ is empty or singleton.

Case 1. For $y = 0$ and $x > 1$, we have

$$\begin{aligned} \frac{\delta(Tx, Ty)}{M(x, y)} e^{\delta(Tx,Ty)-M(x,y)} &= \frac{x-1}{x} e^{x-1-x} \\ &= \frac{x-1}{x} e^{-1} < e^{-1}. \end{aligned} \tag{23}$$

Case 2. For $y = 1$ and $x > 1$, we have

$$\begin{aligned} \frac{\delta(Tx, Ty)}{M(x, y)} e^{\delta(Tx,Ty)-M(x,y)} &= \frac{x-1}{x} e^{x-1-x} \\ &= \frac{x-1}{x} e^{-1} < e^{-1}. \end{aligned} \tag{24}$$

Case 3. For $x > y > 1$, we have

$$\begin{aligned} \frac{\delta(Tx, Ty)}{M(x, y)} e^{\delta(Tx,Ty)-M(x,y)} &= \frac{x+y-2}{x+y} e^{x+y-2-x-y} \\ &= \frac{x+y-2}{x+y} e^{-2} < e^{-1}. \end{aligned} \tag{25}$$

This shows that T is multivalued F-contraction; therefore, all conditions of theorem are satisfied and so T has a fixed point in X.

On the other hand, for $y = 0$ and $x \neq 0$, since $\delta(Tx, Ty) = x - 1$ and $d(x, y) = x$, we get

$$\lim_{n\to\infty} \frac{\delta(Tx, Ty)}{M(x, y)} = \lim_{n\to\infty} \frac{x-1}{x} = 1; \tag{26}$$

then T does not satisfy

$$\delta(Tx, Ty) \leq \lambda M(x, y), \tag{27}$$

for $\lambda \in [0, 1)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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