Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 412974, 6 pages
http://dx.doi.org/10.1155/2014/412974

*Research Article*

# Almost Conservative Four-Dimensional Matrices through de la Vallée-Poussin Mean

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Received 1 November 2013; Accepted 17 December 2013; Published 2 January 2014

Academic Editor: Mohammad Mursaleen



The purpose of this paper is to generalize the concept of almost convergence for double sequence through the notion of de la Vallée-Poussin mean for double sequences. We also define and characterize the generalized regularly almost conservative and almost coercive four-dimensional matrices. Further, we characterize the infinite matrices which transform the sequence belonging to the space of absolutely convergent double series into the space of generalized almost convergence.

## 1. Introduction and Preliminaries

Let $l_\infty$ be the Banach space of real bounded sequences $x = x_n$ with the usual norm $\|x\| = \sup |x_n|$. There exist continuous linear functionals on $l_\infty$ called Banach limits [1]. It is well known that any Banach limit of $x$ lies between $\liminf x$ and $\limsup x$. The idea of almost convergence of Lorentz [2] is narrowly connected with the limits of S. Banach; that is, a sequence $x_n \in l_\infty$ is almost convergent to $l$ if all of its Banach limits are equal. As an application of almost convergence, Mohiuddine [3] obtained some approximation theorems for sequence of positive linear operator through this notion. For double sequence, the notion of almost convergence was first introduced by Móricz and Rhoades [4]. The authors of [5] introduced the notion of Banach limit for double sequence and characterized the spaces of almost and strong almost convergence for double sequences through some sublinear functionals. For more details on these concepts, one can refer to [6–12].

We say that a double sequence $x = (x_{j,k} : j,k = 0,1,2,\ldots)$ of real or complex numbers is bounded if

$$\|x\| = \sup_{j,k} \left|x_{j,k}\right| < \infty, \tag{1}$$

denoted by $\mathscr{L}_\infty$, the space of all bounded sequence $(x_{j,k})$.

A double sequence $x = (x_{j,k})$ of reals is called *convergent* to some number $L$ in *Pringsheim's sense* (briefly, *P-convergent* to $L$) [13] if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{j,k} - L| < \epsilon$ whenever $j,k \geq N$, where $\mathbb{N} := \{1,2,3,\ldots\}$.

If a double sequence $x = (x_{j,k})$ in $\mathscr{L}_\infty$ and $x$ is also $P$-convergent to $L$, then we say that it is *boundedly P-convergent* to $L$ (briefly, *BP-convergent* to $L$).

A double sequence $x = (x_{jk})$ is said to *converge regularly* to $L$ (briefly, *R-convergent* to $L$) if $x$ is converges in Pringsheim's sense, and the limits $x^j := \lim_k x_{jk} (j \in \mathbb{N})$ and $x^k := \lim_j x_{jk} (k \in \mathbb{N})$ exist. Note that in this case the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the $P$-limit of $x$.

Throughout this paper, by $\mathscr{C}_P$, $\mathscr{C}_{BP}$, and $\mathscr{C}_R$, we denote the space of all $P$-convergent, $BP$-convergent, and $R$-convergent double sequences, respectively. Also, the linear space of all continuous linear functionals on $\mathscr{C}_R$ is denoted by $\mathscr{C}'_R$.

Let $B = (b_{p,q,j,k} : j,k = 0,1,2,\ldots)$ be a four-dimensional infinite matrix of real numbers for all $p$, $q = 0,1,2,\ldots$ and $S_1$ a space of double sequences. Let $S_2$ be a double sequences space, converging with respect to a convergence rule $\nu \in \{P, BP, R\}$. Define

$$S_1^{B,\nu} = \left\{ x = \left(x_{j,k}\right) : Bx = \left(B_{p,q}(x)\right) = \nu - \sum_{j,k} b_{p,q,j,k} x_{j,k} \text{ exists and } Bx \in S_1 \right\}. \tag{2}$$

Then, we say that a four-dimensional matrix $B$ maps the space $S_2$ into the space $S_1$ if $S_2 \subset S_1^{B,\nu}$ and is denoted by $(S_2, S_1)$.

Móricz and Rhoades [4] extended the notion of almost convergence from single to double sequence and characterized some matrix classes involving this concept. A double sequence $x = (x_{j,k})$ of real numbers is said to be *almost convergent* to a number $L$ if

$$\lim_{p,q\to\infty} \sup_{m,n>0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{j,k} - L \right| = 0. \tag{3}$$

For more details on double sequences and 4-dimensional matrices, one can refer to [14–20].

Using the notion of almost convergence for single sequence, King [21] introduced a slightly more general class of matrices than the conservative and regular matrices, that is, almost conservative and almost regular matrices, and presented its characterization. In [22], Schaefer presented some interesting characterization for almost convergence. The Steinhaus-type theorem for the concepts of almost regular and almost coercive matrices was proved by Başar and Solak [23]. In this paper, we generalize the concept of almost convergence for double sequences with the help of double generalized de la Vallée-Poussin mean and called it $(\Lambda)$almost convergence. Using this concept, we define the notions of regularly $(\Lambda)$almost conservative and $(\Lambda)$almost coercive four-dimensional matrices and obtain their necessity and sufficient conditions. Further, we introduce the space $\mathscr{L}_1$ of all absolutely convergent double series and characterize the matrix class $(\mathscr{L}_1, \mathscr{F}_\Lambda)$, where $\mathscr{F}_\Lambda$ denotes the space of $(\Lambda)$almost convergence for double sequences.

## 2. Main Results

*Definition 1.* Let $\lambda = (\lambda_m : m = 0, 1, 2, \ldots)$ and $\mu = (\mu_n : n = 0, 1, 2, \ldots)$ be two nondecreasing sequences of positive reals with each tending to $\infty$ such that $\lambda_{m+1} \le \lambda_m + 1$, $\lambda_1 = 0$, $\mu_{n+1} \le \mu_n + 1$, $\mu_1 = 0$, and

$$\mathfrak{T}_{m,n}(x) = \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j,k} \tag{4}$$

is called the *double generalized de la Vallée-Poussin mean*, where $J_m = [m-\lambda_m+1, m]$ and $I_n = [n-\mu_n+1, n]$. We denote the set of all $\lambda$ and $\mu$ type sequences by using the symbol $\Lambda$.

*Definition 2.* A double sequence $x = (x_{j,k})$ of reals is said to be $(\Lambda)$*almost convergent* (briefly, $\mathscr{F}_\Lambda$*-convergent*) to some number $L$ if $x \in \mathscr{F}_{\lambda,\mu}$, where

$$\mathscr{F}_\Lambda = \Big\{ x = (x_{jk}) : P\text{-}\lim_{m,n\to\infty} \Omega_{m,n,s,t}(x) = L \text{ exists, uniformly in } s, t; L = \mathscr{F}_\Lambda\text{-}\lim x \Big\}, \tag{5}$$

$$\Omega_{m,n,s,t}(x) = \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t},$$

denoted by $\mathscr{F}_\Lambda$, the space of all $(\Lambda)$almost convergent sequences $(x_{j,k})$. Note that $\mathscr{C}_{BP} \subset \mathscr{F}_\Lambda \subset \mathscr{L}_\infty$.

*Remark 3.* If we take $\lambda_m = m$ and $\mu_n = n$, then the notion of $(\Lambda)$almost convergence reduced to almost convergence due to Móricz and Rhoades [4].

*Definition 4.* A four-dimensional matrix $B = (b_{p,q,j,k})$ is said to be *regularly* $(\Lambda)$*almost conservative* if it maps every $R$-convergent double sequence into $\mathscr{F}_\Lambda$-convergent double sequence; that is, $B \in (\mathscr{C}_R, \mathscr{F}_\Lambda)$. In addition, if $\mathscr{F}_\Lambda\text{-}\lim Ax = R\text{-}\lim x$, then $B$ is *regularly* $(\Lambda)$*almost regular*.

*Definition 5.* A matrix $B = (b_{p,q,j,k})$ is said to be $(\Lambda)$*almost coercive* if it maps every $BP$-convergent double sequence $(x_{j,k})$ into $\mathscr{F}_\Lambda$-convergent double sequence, briefly, a matrix $B$ in $(\mathscr{C}_{BP}, \mathscr{F}_\Lambda)$.

**Theorem 6.** *A matrix $B = (b_{p,q,j,k})$ is regularly $(\Lambda)$almost conservative if and only if*

- $(\mathrm{CR}_1)$ $\|B\| = \sup_{p,q} \sum_{j,k} |b_{p,q,j,k}| < \infty$,
- $(\mathrm{CR}_2)$ $\lim_{m,n\to\infty} \alpha(m,n,s,t,j,k) = u_{jk}$, *for each $j$, $k$ (uniformly in $s$, $t$),*
- $(\mathrm{CR}_3)$ $\lim_{m,n\to\infty} \sum_k |\alpha(m,n,s,t,j,k)| = u_{j0}$, *for each $j$ (uniformly in $s$, $t$),*
- $(\mathrm{CR}_4)$ $\lim_{m,n\to\infty} \sum_j |\alpha(m,n,s,t,j,k)| = u_{0k}$, *for each $k$ (uniformly in $s$, $t$),*
- $(\mathrm{CR}_5)$ $\lim_{m,n\to\infty} \sum_{j,k} \alpha(m,n,s,t,j,k) = u$, *(uniformly in $s$, $t$),*

*where*

$$\beta(m,n,s,t,j,k) = \frac{1}{\lambda_m \mu_n} \sum_{p\in J_m} \sum_{q\in I_n} b_{p+s,q+t,j,k}. \tag{6}$$

*In this case, the $\mathscr{F}_\Lambda$-limit of $Bx$ is*

$$\ell u + \sum_{j=0}^{\infty} (\ell_j - \ell) u_{j0} + \sum_{k=0}^{\infty} (h_k - \ell) u_{0k} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (x_{j,k} - \ell_j - h_k - \ell) u_{jk}, \tag{7}$$

*where $\ell = R\text{-}\lim x$.*

*Proof. Necessity.* Suppose that $B$ is regularly $(\Lambda)$almost conservative matrix. Fix $s, t \in \mathbb{Z}$, the set of integers. Let

$$\Omega_{m,n,s,t}(x) = \frac{1}{\lambda_m \mu_n} \sum_{p\in J_m} \sum_{q\in I_m} \rho_{p+s,q+t}(x), \tag{8}$$

where

$$\rho_{p,q}(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{p,q,j,k} x_{j,k}. \tag{9}$$

It is clear that

$$\rho_{p,q} \in \mathscr{C}_R', p, \quad q = 0, 1, 2, \ldots. \tag{10}$$

Hence $\Omega_{m,n,s,t} \in \mathscr{C}'_R$ for $m, n \in \mathbb{N}$. Since $B$ is regularly $(\Lambda)$almost conservative, we have

$$P\text{-}\lim_{m,n\to\infty} \Omega_{m,n,s,t}(x) = \Omega(x) \quad (\text{say}), \tag{11}$$

uniformly in $s, t$. It follows that $(\Omega_{m,n,s,t}(x))$ is bounded for $x \in \mathscr{C}_R$ and fixed $s, t$. Hence, $\|\Omega_{m,n,s,t}(x)\|$ is bounded by the uniform boundedness principle.

For each $i, \nu \in \mathbb{Z}^+$, define the sequence $y = y(m, n, s, t)$ by

$$y_{j,k} = \begin{cases} \operatorname{sgn}\left(\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right), & \text{if } 0 \le j \le i,\ 0 \le k \le \nu, \\ 0, & \text{if } \nu < k,\ i < j. \end{cases} \tag{12}$$

Then, a double sequence $y \in \mathscr{C}_R$, $\|y\| = 1$, and

$$\left|\Omega_{m,n,s,t}(y)\right| = \frac{1}{\lambda_m\mu_n}\sum_{j=0}^{i}\sum_{k=0}^{\nu}\left|\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right|. \tag{13}$$

Hence

$$\left|\Omega_{m,n,s,t}(y)\right| \le \|\Omega_{m,n,s,t}\| \, \|y\| = \|\Omega_{m,n,s,t}\|. \tag{14}$$

Therefore

$$\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right| \le \|\Omega_{m,n,s,t}\|, \tag{15}$$

so that condition $(CR_1)$ follows.

The sequences $F^{(b,c)} = (f^{(b,c)}_{j,k})$, $F^{(b)} = (f^{(b)}_{j,k})$, $G^{(c)} = (g^{(c)}_{j,k})$, and $G = (g_{j,k})$ are defined by

$$\begin{gathered} f^{(q,r)}_{j,k} = \begin{cases} 1, & \text{if } (j,k) = (b,c), \\ 0, & \text{otherwise}, \end{cases} \\ f^{(b)}_{j,k} = \begin{cases} 1, & \text{if } j = b, \\ 0, & \text{otherwise}, \end{cases} \\ g^{(c)}_{j,k} = \begin{cases} 1, & \text{if } k = c, \\ 0, & \text{otherwise}, \end{cases} \\ g_{j,k} = 1, \quad \forall j,k. \end{gathered} \tag{16}$$

Since $F^{(j,k)}, F^{(j)}, G^{(k)}, G \in \mathscr{C}_R$, the $P$-limit of $\Omega_{m,n,s,t}(F^{(j,k)})$, $\Omega_{m,n,s,t}(F^{(j)})$, $\Omega_{m,n,s,t}(G^{(k)})$, and $\Omega_{m,n,s,t}(G)$ must exist, uniformly in $s, t$. Hence, the conditions $(CR_2)$–$(CR_5)$ must hold, respectively.

*Sufficiency.* Suppose that the conditions $(CR_1)$–$(CR_5)$ hold and a double sequence $x = (x_{jk}) \in \mathscr{C}_R$. Fix $s, t \in \mathbb{Z}$. Then

$$\begin{gathered} \Omega_{m,n,s,t}(x) = \frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}x_{j,k}, \\ \left|\Omega_{m,n,s,t}(x)\right| \le \frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right| \left\|x_{j,k}\right\|. \end{gathered} \tag{17}$$

Therefore, by $(CR_1)$, we have $|\Omega_{m,n,s,t}(x)| \le C_{s,t}\|x\|$, where $C_{s,t}$ is a constant independent of $m, n$. Hence $\Omega_{m,n,s,t} \in \mathscr{C}'_R$ and the sequence $(\|\Omega_{m,n,s,t}\|)$ is bounded for each $s, t \in \mathbb{Z}^+$. It follow from the conditions $(CR_2)$, $(CR_3)$, $(CR_4)$, and $(CR_5)$ that the $P$-limit of $\Omega_{m,n,s,t}(F^{(j,k)})$, $\Omega_{m,n,s,t}(F^{(j)})$, $\Omega_{m,n,s,t}(G^{(k)})$, and $\Omega_{m,n,s,t}(G)$ exist for all $j, k, s$, and $t$. Since $G, F^{(j)}, G^{(k)}$ and $F^{(j,k)}$ is a fundamental set in $\mathscr{C}_R$ (see [24]), it follows that

$$\lim_{m,n\to\infty} \Omega_{m,n,s,t}(x) = \Omega_{s,t}(x) \tag{18}$$

exists and $\Omega_{s,t} \in \mathscr{C}'_R$. Therefore, $\Omega_{s,t}$ has the form

$$\begin{aligned} \Omega_{s,t}(x) = {} & \ell\Omega_{s,t}(G) + \sum_{j=0}^{\infty}\left(\ell_j - \ell\right)\Omega_{s,t}\left(F^{(j)}\right) \\ & + \sum_{k=0}^{\infty}\left(h_k - \ell\right)\Omega_{s,t}\left(G^{(k)}\right) \\ & + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(x_{j,k} - \ell_j - h_k + \ell\right)\Omega_{s,t}\left(F^{(j,k)}\right). \end{aligned} \tag{19}$$

But $\Omega_{s,t}(F^{(j,k)}) = u_{jk}$, $\Omega_{s,t}(F^{(j)}) = u_{j0}$, $\Omega_{s,t}(G^{(k)}) = u_{0k}$, and $\Omega_{s,t}(G) = u$ by the conditions $(CR_2)$–$(CR_5)$, respectively. Hence

$$\lim_{m,n\to\infty} \Omega_{m,n,s,t}(x) = \Omega(x) \tag{20}$$

exists for each $x \in \mathscr{C}_R$ and $s, t = 0, 1, 2, \ldots$ with

$$\begin{aligned} \Omega(x) = {} & \ell u + \sum_{j=0}^{\infty}\left(\ell_j - \ell\right)u_{j0} + \sum_{k=0}^{\infty}\left(h_k - \ell\right)u_{0k} \\ & + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(x_{j,k} - \ell_j - h_k - \ell\right)u_{jk}. \end{aligned} \tag{21}$$

Since $\Omega_{m,n,s,t}(x) \in \mathscr{C}'_R$ for each $m, n, s$, and $t$, it has the form

$$\begin{aligned} \Omega_{m,n,s,t}(x) = {} & \ell\Omega_{m,n,s,t}(G) + \sum_{j=0}^{\infty}\left(\ell_j - \ell\right)\Omega_{m,n,s,t}\left(F^{(j)}\right) \\ & + \sum_{k=0}^{\infty}\left(h_k - \ell\right)\Omega_{m,n,s,t}\left(G^{(k)}\right) \\ & + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(x_{j,k} - \ell_j - h_k - \ell\right)\Omega_{m,n,s,t}\left(F^{(j,k)}\right). \end{aligned} \tag{22}$$

It is easy to see from (21) and (22) that the convergence of $(\Omega_{m,n,s,t}(x))$ to $\Omega(x)$ is uniform in $s, t$, since $\Omega_{m,n,s,t}(G) \to u$, $\Omega_{m,n,s,t}(F^{(j)}) \to u_{j0}$, $\Omega_{m,n,s,t}(G^{(k)}) \to u_{0k}$, and $\Omega_{m,n,s,t}(F^{(j,k)}) \to u_{jk}$ $(m, n \to \infty)$ uniformly in $s, t$. Therefore, $B$ is regularly $(\Lambda)$almost conservative. □

Let us recall the following lemma, which is proved by Mursaleen and Mohiuddine [25].

**Lemma 7.** *Let $A(s,t) = (a_{m,n,j,k}(s,t))$, $s, t = 0, 1, 2, \ldots$, be a sequence of infinite matrices such that*

(i) $\|A(s,t)\| < H < +\infty$ *for all s, t; and*

(ii) *for each j, k* $\lim_{m,n} a_{m,n,j,k}(s,t) = 0$ *uniformly in s, t.*

*Then*

$$\lim_{m,n} \sum_j \sum_k a_{m,n,j,k}(s,t)\, x_{j,k} = 0 \quad \textit{uniformly in } s, t \textit{ for each } x \in \mathscr{L}^{\infty} \tag{23}$$

*if and only if*

$$\lim_{m,n} \sum_j \sum_k \left| a_{m,n,j,k}(s,t) \right| = 0 \quad \textit{uniformly in } s, t. \tag{24}$$

**Theorem 8.** *A matrix $B = (b_{p,q,j,k})$ is $(\Lambda)$almost coercive if and only if*

$(AC_1)$ $\|B\| = \sup_{p,q} \sum_{j,k} |b_{p,q,j,k}| < \infty$,

$(AC_2)$ $\lim_{m,n\to\infty} \alpha(m,n,s,t,j,k) = u_{jk}$, *for each j, k (uniformly in s, t),*

$(AC_3)$ $\lim_{m,n\to\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (1/\lambda_m \mu_n) | \sum_{p\in J_m} \sum_{q\in I_n} b_{p+s,q+t,j,k} - u_{jk}| = 0$, *uniformly in s, t.*

*In this case, the $\mathscr{F}_{\Lambda}$-limit of $Bx$ is $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} x_{j,k}$ for every $(x_{j,k}) \in \mathscr{L}_{\infty}$.*

*Proof. Sufficiency.* Assume that conditions $(AC_1)$–$(AC_3)$ hold. For any positive integers $J$, $K$

$$\begin{aligned}
\sum_{j=1}^{J} \sum_{k=1}^{K} |u_{jk}| &= \sum_{j=1}^{J} \sum_{k=1}^{K} \lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} \left| \sum_{p\in J_m} \sum_{q\in I_n} b_{p+s,q+t,j,k} \right| \\
&= \lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} \sum_{j=1}^{J} \sum_{k=1}^{K} \left| \sum_{p\in J_m} \sum_{q\in I_n} b_{p+s,q+t,j,k} \right| \\
&\le \limsup_{m,n} \frac{1}{\lambda_m \mu_n} \sum_{p\in J_m} \sum_{q\in I_n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| b_{p+s,q+t,j,k} \right| \\
&\le \|B\|.
\end{aligned} \tag{25}$$

This shows that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |u_{jk}|$ converges and that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} x_{j,k}$ is defined for every double sequence $x = (x_{j,k}) \in \mathscr{L}_{\infty}$.

Let $(x_{j,k})$ be any arbitrary bounded double sequence. For every positive integers $m$, $n$

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_m \mu_n} \sum_{p\in J_m} \sum_{q\in I_n} b_{p+s,q+t,j,k} - u_{jk} \right) x_{j,k} \right\| \\
&= \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ \frac{1}{\lambda_m \mu_n} \sum_{p\in J_m} \sum_{q\in I_n} \left[ b_{p+s,q+t,j,k} - u_{jk} \right] \right] x_{j,k} \right\| \\
&\le \sup_{s,t} \left[ \left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ \frac{1}{\lambda_m \mu_n} \sum_{p\in J_m} \sum_{q\in I_n} \left[ b_{p+s,q+t,j,k} - u_{jk} \right] \right] x_{j,k} \right| \right] \\
&\le \|x\| \sup_{s,t} \left[ \frac{1}{\lambda_m \mu_n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| \sum_{p\in J_m} \sum_{q\in I_n} \left[ b_{p+s,q+t,j,k} - u_{jk} \right] \right| \right].
\end{aligned} \tag{26}$$

Letting $p, q \to \infty$ and using condition $(AC_3)$, we get

$$\frac{1}{\lambda_m \mu_n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{p\in J_m} \sum_{q\in I_n} b_{p+s,q+t,j,k} x_{j,k} \longrightarrow \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} x_{j,k}. \tag{27}$$

Hence, $Bx \in \mathscr{F}_{\Lambda}$ with $\mathscr{F}_{\Lambda}\text{-}\lim Bx = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} x_{j,k}$.

*Necessity.* Let $B$ be $(\Lambda)$almost coercive matrix. This implies that a four-dimensional matrix $B$ is $(\Lambda)$almost conservative; then we have conditions $(AC_1)$ and $(AC_2)$ from Theorem 6. Now we have to show that $(AC_3)$ holds.

Suppose that, for some $s$, $t$, we have

$$\limsup_{m,n} \frac{1}{\lambda_m \mu_n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| \sum_{p\in J_m} \sum_{q\in I_n} \left[ b_{p+s,q+t,j,k} - u_{jk} \right] \right| = N > 0. \tag{28}$$

Since $\|B\|$ is finite, therefore $N$ is also finite. We observe that since $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |u_{jk}| < +\infty$ and $B$ is $(\Lambda)$almost coercive, the matrix $A = (a_{p,q,j,k})$, where $a_{p,q,j,k} = b_{p,q,j,k} - u_{jk}$ is also $(\Lambda)$almost coercive matrix. By an argument similar to that of Theorem 2.1 in [26] for single sequences, one can find $x \in \mathscr{L}_{\infty}$ for which $Ax \notin \mathscr{F}_{\Lambda}$. This contradiction implies the necessity of $(AC_3)$.

Now, we use Lemma 7 to show that this convergence is uniform in $s$, $t$. Let

$$h_{m,n,j,k}(s,t) = \frac{1}{\lambda_m \mu_n} \sum_{p\in J_m} \sum_{q\in I_n} \left[ b_{p,q,j,k} - u_{jk} \right] \tag{29}$$

and let $H(s,t)$ be the matrix $(h_{m,n,j,k}(s,t))$. It is easy to see that $\|H(s,t)\| \le 2\|B\|$ for every $s$, $t$; and from condition $(AC_2)$

$$\lim_{m,n} h_{m,n,j,k}(s,t) = 0 \quad \text{for each } j, k, \text{uniformly in } s, t. \tag{30}$$

For any $x \in \mathscr{L}_{\infty}$,

$$\lim_{m,n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} h_{m,n,j,k}(s,t)\, x_{j,k} = \mathscr{F}_{\Lambda}\text{-}\lim Bx - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u_{jk} x_{j,k} \tag{31}$$

and the limit exists uniformly in $s$, $t$, since $Bx \in \mathscr{F}_\Lambda$. Moreover, this limit is zero since

$$\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} h_{m,n,j,k}(s,t)\, x_{j,k}\right| \leq \frac{\|x\| \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left|\sum_{p\in J_m}\sum_{q\in I_n}\left[b_{p,q,j,k}-u_{jk}\right]\right|}{\lambda_m\mu_n}. \tag{32}$$

Hence

$$\lim_{m,n}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left|h_{m,m,j,k}(s,t)\right| = 0 \quad \text{uniformly in } s,t. \tag{33}$$

This shows that matrix $B = (b_{p,q,j,k})$ satisfies condition $(\mathrm{AC}_3)$. □

In the following theorem, we characterize the four-dimensional matrices of type $(\mathscr{L}_1, \mathscr{F}_\Lambda)$, where

$$\mathscr{L}_1 = \left\{ x = (x_{j,k}) : \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|x_{j,k}\right| < \infty \right\}, \tag{34}$$

the space of all absolutely convergent double series.

**Theorem 9.** *A matrix $B \in (\mathscr{L}_1, \mathscr{F}_\Lambda)$ if and only if it satisfies the following conditions:*

(i) $\sup_{m,n,s,t,j,k}\left|(1/\lambda_m\mu_n)\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right| < \infty$,

*and the condition $(CR_2)$ of Theorem 6 holds.*

*Proof. Sufficiency.* Suppose that conditions (i) and $(\mathrm{CR}_2)$ hold. For any double sequence $x = (x_{j,k}) \in \mathscr{L}_1$, we see that

$$\lim_{m,n,\to\infty}\frac{1}{\lambda_m\mu_n}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}x_{j,k} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} u_{jk}x_{j,k}, \tag{35}$$

uniformly in $s$, $t$ and it also converges absolutely. Furthermore,

$$\frac{1}{\lambda_m\mu_n}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}x_{j,k} \tag{36}$$

converges absolutely for each $m$, $n$, $s$, and $t$. Given $\epsilon > 0$, there exist $j_\circ = j_\circ(\epsilon)$ and $k_\circ = k_\circ(\epsilon)$ such that

$$\sum_{j>j_\circ}\sum_{k>k_\circ}\left|x_{j,k}\right| < \epsilon. \tag{37}$$

By the condition $(\mathrm{CR}_2)$, we can find $m_\circ, n_\circ \in \mathbb{N}$ such that

$$\left|\sum_{j\leq j_\circ}\sum_{k\leq k_\circ}\left[\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k} - u_{jk}\right]x_{j,k}\right| < \infty, \tag{38}$$

for all $m > m_\circ$ and $n > n_\circ$, uniformly in $s$, $t$. Now, by using the conditions (37), (38), and $(\mathrm{CR}_2)$, we get

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k} - u_{jk}\right]x_{j,k}\right| \\
&\quad\leq \left|\sum_{j\leq j_\circ}\sum_{k\leq k_\circ}\left[\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k} - u_{jk}\right]x_{j,k}\right| \\
&\qquad+ \sum_{j>j_\circ}\sum_{k>k_\circ}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k} - u_{jk}\right|\left|x_{j,k}\right|,
\end{aligned} \tag{39}$$

for all $m > m_\circ$, $n > n_\circ$ and uniformly in $s$, $t$. Hence (37) holds.

*Necessity.* Suppose that $B \in (\mathscr{L}_1, \mathscr{F}_\Lambda)$. The condition $(\mathrm{CR}_2)$ follows from the fact that $E \in \mathscr{L}_1$, where $E = (e^{(j,k)})$ with $e^{(j,k)} = 1$ for all $j$, $k$. To verify the condition (i), we define a continuous linear functional $L_{m,n,s,t}(x)$ on $\mathscr{L}_1$ by

$$L_{m,n,s,t}(x) = \frac{1}{\lambda_m\mu_n}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}x_{j,k}. \tag{40}$$

Now

$$\left|L_{m,n,s,t}(x)\right| \leq \sup_{j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right|\|x\|_1 \tag{41}$$

and hence

$$\left\|L_{m,n,s,t}\right\| \leq \sup_{j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right|. \tag{42}$$

For any fixed $j, k \in \mathbb{N}$, we define a double sequence $x = (x_{i,\ell})$ by

$$x_{i,\ell} = \begin{cases} \operatorname{sgn}\left(\dfrac{1}{\lambda_m\mu_n}\displaystyle\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right), & \text{for } (i,\ell) = (j,k), \\ 0, & \text{for } (i,\ell) \neq (j,k). \end{cases} \tag{43}$$

Then $\|x\|_1 = 1$, and

$$\begin{aligned}
\left|L_{m,n,s,t}(x)\right| &= \left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}x_{j,k}\right| \\
&= \left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right|\|x\|_1,
\end{aligned} \tag{44}$$

so that

$$\left\|L_{m,n,s,t}\right\| \geq \sup_{j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right|. \tag{45}$$

It follows from (42) and (45) that

$$\left\|L_{m,n,s,t}\right\| = \sup_{j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right|. \tag{46}$$

Since $B \in (\mathscr{L}_1, \mathscr{F}_\Lambda)$, we have

$$\begin{aligned}\sup_{m,n,s,t}&\left|L_{m,n,s,t}(x)\right| \\ &= \sup_{m,n,s,t}\left|\frac{1}{\lambda_m\mu_n}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}x_{j,k}\right| < \infty.\end{aligned} \tag{47}$$

Hence, by the uniform boundedness principle, we obtain

$$\sup_{m,n,s,t}\left\|L_{m,n,s,t}(x)\right\| = \sup_{m,n,s,t,j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} b_{p+s,q+t,j,k}\right| < \infty. \tag{48}$$

□

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. (130-100-D1434). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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