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Research Article

Algebraic L^2 Decay for Weak Solutions of the Nonlinear Heat Equations in Whole Space $\mathbf{R}^3$

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We obtained the algebraic L^2 time decay rate for weak solutions of the nonlinear heat equations with the nonlinear term $|\nabla u|^2 u$ in whole space $\mathbf{R}^3$. The methods are based on energy methods and Fourier analysis technique.

1. Introduction

In this study we consider the Cauchy problem of the following nonlinear heat equations in whole spaces:

$$u_t - \Delta u + |\nabla u|^2 u = 0, \quad x \in \mathbf{R}^3, \ t > 0, \tag{1}$$

with the initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbf{R}^3. \tag{2}$$

Here, $u = u(x, t)$ is the unknown function and

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^{\,2}}, \qquad \nabla u = \left(\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u\right). \tag{3}$$

This nonlinear model appears to be relevant in the theory of the viscous incompressible Newtonian and non-Newtonian fluid flows (refer to [1–4] and the references therein). As for the generalized nonlinear heat equations

$$u_t - \Delta u + f(u, \nabla u) = 0, \quad x \in \mathbf{R}^3, \ t > 0, \tag{4}$$

there is much study on the well-posedness and large time behavior for solutions to these nonlinear heat equations [5–7]. For example, Ponce [8] and Zheng and Chen [9] studied global weak solutions of the generalized nonlinear heat equations by the difference methods. Escobedo and Zuazua [10] and Zhang [11, 12] studied the L^2 time decay for weak solutions of the nonlinear heat equation with filtration-type nonlinear terms. Zhao [13] recently examined asymptotic behavior for weak solutions of the nonlinear heat equations with the nonlinear term satisfying some growth conditions. One may also refer to some interesting time decay results for weak solutions of the nonlinear partial differential equations models [1, 14, 15]. More precisely, the optimal decay estimates for solutions of those nonlinear models such as

$$\|u\|_{L^q(R^n)} \le C t^{-(n/2)(1/p-1/q)} \|u_0\|_{L^p(R^n)}, \quad t > 0. \tag{5}$$

Motivated by the results [16, 17], the main purpose of this study is to investigate the algebraic L^2 time decay for weak solutions of the nonlinear heat equation (1)-(2). More precisely, when the initial data u_0 satisfies some integrable condition, we will show the weak solution of the nonlinear heat equation (1)-(2) decays as that of the linear heat equation. The result is proved by developing some analysis techniques, such as Fourier transformation and energy inequality.

2. Statement of the Main Results

In this section we first introduce some notations. We denote by $L^q(\mathbf{R}^3)$ with $1 \le q \le \infty$ the usual Lebesgue space with the norm $\|\cdot\|_q$. In particular, $\|\cdot\| = \|\cdot\|_2$. $W^{m,p}(\mathbf{R}^3)$ is the usual Sobolev space with the norm $\|\cdot\|_{m,p}$ (refer to [18]). $C > 0$ is any constant which may only depend on the initial data u_0

but never depend on $t > 0$. The Fourier transformation of g is $\widehat{g}$ or $F[g]$,

$$F[g](\xi) = \widehat{g}(\xi) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} g(x) e^{-ix\cdot\xi} dx, \qquad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3. \tag{6}$$

For a weak solution of the nonlinear heat equation (1)-(2), we mean a function

$$u \in L^\infty\left((0,T); L^2\left(\mathbf{R}^3\right)\right), \quad \nabla u \in L^2\left((0,T); L^2\left(\mathbf{R}^3\right)\right), \quad \forall T > 0, \tag{7}$$

which satisfies

$$\begin{aligned}&\int_{\mathbf{R}^3} u(t)\cdot\varphi(t)\,dx + 2\int_{\mathbf{R}^3} \nabla u\cdot\nabla\varphi\,dx + 2\int_{\mathbf{R}^3} |\nabla u|^2 u\varphi\,dx \\ &= \int_{\mathbf{R}^3} u\cdot\frac{\partial\varphi}{\partial t}\,dx + \int_{\mathbf{R}^3} u_0\cdot\varphi(0)\,dx, \quad \forall\varphi \in C_0^\infty\left([0,T)\times\mathbf{R}^3\right),\end{aligned} \tag{8}$$

and energy inequality

$$\frac{d}{dt}\int_{\mathbf{R}^3} |u|^2 dx + 2\int_{\mathbf{R}^3} |\nabla u|^2 dx + 2\int_{\mathbf{R}^3} |u|^2|\nabla u|^2 dx \le 0. \tag{9}$$

Our main result reads as follows.

Theorem 1. *Let $u_0 \in L^2(\mathbf{R}^3) \cap L^1(\mathbf{R}^3)$; then the weak solution $u(x,t)$ of the nonlinear heat equation (1)-(2) decays as*

$$\|u(t)\| \le C(1+t)^{-3/4}, \quad \forall t > 0. \tag{10}$$

Remark 2. It is well known that the solution $v(t)$ of the linear heat equation

$$v_t - \Delta v = 0 \tag{11}$$

has the same time decay properties as

$$\|v(t)\| \le C(1+t)^{-3/4}, \quad \forall t > 0, \tag{12}$$

with the initial data $u_0 \in L^2(\mathbf{R}^3) \cap L^1(\mathbf{R}^3)$. Therefore our result cannot be improved under the same condition on initial data. Moreover, it seems as an interesting and important problem to further consider the explicit error estimates between the above nonlinear heat equation and the linear heat equation; we will study this issue in the forthcoming paper.

Remark 3. Compared with the previous results on the time decay issue of the nonlinear heat equations, the main difficulty here lies in the uniform estimates of the nonlinear term $|\nabla u|^2 u$. In order to overcome this difficulty, we further develop the classic so-called Fourier splitting methods which are first introduced by Schonbek [16] when she studied the time decay of Navier-Stokes equations. And then we further need to apply Hölder inequality, Young inequality, and energy inequality, together with the rigorous computation.

3. Proof of Theorem 1

In order to prove Theorem 1, the existence of weak solution of the nonlinear heat equation can be derived by the standard methods (refer also to [19], e.g.); we first deduce some auxiliary lemmas which will be employed in the proof of the main theorems.

Lemma 4. *Under the same condition in Theorem 1, one has*

$$\left|\widehat{u}(\xi,t)\right| \le C. \tag{13}$$

Proof of Lemma 4. We first take Fourier transformation to both sides of the nonlinear heat equation (1)

$$\widehat{u}_t + |\xi|^2\widehat{u} + F\left[|\nabla u|^2 u\right] = 0. \tag{14}$$

We can look at the above equation (14) as a linear ordinary differential equation with respect to $\widehat{u}(t)$, and according to the fundamental theory of the ordinary differential equation, it is easy to check that the solution of (14) can be expressed as

$$\widehat{u}(t) = e^{-|\xi|^2 t}\widehat{u}_0 - \int_0^t F\left[|\nabla u|^2 u\right] e^{-|\xi|^2(t-s)} ds. \tag{15}$$

Now we need to estimate the right-hand side of (15). Since

$$\begin{aligned}\left|F\left[|\nabla u|^2 u\right]\right| &= \left|\left(\frac{1}{2\pi}\right)^{3/2}\int_{\mathbf{R}^3} e^{-ix\cdot\xi}|\nabla u|^2 u\,dx\right| \\ &\le C\int_{\mathbf{R}^3} |\nabla u|^2\,|u|\,dx \\ &\le C\left(\int_{\mathbf{R}^3} |\nabla u|^2 dx\right)^{1/2}\left(\int_{\mathbf{R}^3} |\nabla u|^2|u|^2 dx\right)^{1/2} \\ &\le C\int_{\mathbf{R}^3} |\nabla u|^2 dx + C\int_{\mathbf{R}^3} |\nabla u|^2|u|^2 dx,\end{aligned} \tag{16}$$

we have used Hölder inequality and Young inequality.

Moreover, integrating the energy inequality (9) in time from 0 to t gives

$$\begin{aligned}&\int_{\mathbf{R}^3} |u(t)|^2 dx + 2\int_0^t\int_{\mathbf{R}^3} |\nabla u(x,s)|^2 dx\,ds \\ &\quad + 2\int_0^t\int_{\mathbf{R}^3} |u|^2|\nabla u|^2 dx\,ds \le \int_{\mathbf{R}^3} \left|u_0\right|^2 dx\end{aligned} \tag{17}$$

which, together with $u_0 \in L^2$, yields

$$\begin{aligned}&\int_0^\infty \left|F\left[|\nabla u|^2 u\right]\right| dt \\ &\le C\int_0^\infty\int_{\mathbf{R}^3} |\nabla u|^2 dx\,dt + C\int_0^\infty\int_{\mathbf{R}^3} |\nabla u|^2|u|^2 dx\,dt \\ &\le C\int_{\mathbf{R}^3} \left|u_0\right|^2 dx \le C.\end{aligned} \tag{18}$$

Plugging (18) into (15) gives that

$$\begin{aligned}|\widehat{u}(\xi,t)| &= \left|e^{-|\xi|^2 t}\widehat{u}_0 - \int_0^t F\left[|\nabla u|^2 u\right] e^{-|\xi|^2(t-s)} ds\right| \\ &\le \left|e^{-|\xi|^2 t}\widehat{u}_0\right| + \left|\int_0^t F\left[|\nabla u|^2 u\right] e^{-|\xi|^2(t-s)} ds\right| \\ &\le |\widehat{u}_0| + \int_0^t \left|F\left[|\nabla u|^2 u\right]\right| ds \\ &\le \left|\left(\frac{1}{2\pi}\right)^{3/2}\int_{\mathbf{R}^3} e^{-ix\cdot\xi} u_0 dx\right| + \int_0^\infty \left|F\left[|\nabla u|^2 u\right]\right| ds \\ &\le C\int_{\mathbf{R}^3} |u_0|\, dx + C \le C;\end{aligned} \tag{19}$$

here we have used the fact $u_0 \in L^1(\mathbf{R}^3)$.

Hence, we complete the proof of Lemma 4. □

Lemma 5 (Parseval equality [7]). *If $f(x) \in L^2(\mathbf{R}^3)$, then the Fourier transformation of $f(x)$ satisfies $\widehat{f}(x) \in L^2(\mathbf{R}^3)$ and*

$$\int_{\mathbf{R}^3} |f(x)|^2 dx = \int_{\mathbf{R}^3} \left|\widehat{f}(\xi)\right|^2 d\xi. \tag{20}$$

Now we begin to prove Theorem 1.

From the energy inequality (9), we have

$$\frac{d}{dt}\int_{\mathbf{R}^3} |u|^2 dx + 2\int_{\mathbf{R}^3} |\nabla u|^2 dx \le 0. \tag{21}$$

With the aid of Parseval equality, we rewrite (21) as

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3} |\widehat{u}|^2 d\xi + 2\int_{\mathbf{R}^3} |\xi|^2|\widehat{u}|^2 d\xi \le 0. \tag{22}$$

Choosing a smooth function $\varrho(t)$, such that

$$\varrho(0) = 1, \qquad \varrho(t) > 0, \qquad \varrho'(t) > 0, \tag{23}$$

and then multiplying $\varrho(t)$ with both sides of the inequality (22), one shows that

$$\begin{aligned}&\frac{d}{dt}\left(\varrho(t)\int_{\mathbf{R}^3} |\widehat{u}|^2 d\xi\right) \\ &\quad + 2\varrho(t)\int_{\mathbf{R}^3} |\xi|^2|\widehat{u}|^2 d\xi \le \varrho'(t)\int_{\mathbf{R}^3} |\widehat{u}|^2 d\xi.\end{aligned} \tag{24}$$

Let

$$\sigma(t) = \left\{\xi \in \mathbf{R}^3 : 2\varrho(t)|\xi|^2 \le \varrho'(t)\right\}; \tag{25}$$

then we have

$$\begin{aligned}&2\varrho(t)\int_{\mathbf{R}^3} |\xi|^2|\widehat{u}|^2 d\xi \\ &\quad \ge 2\varrho(t)\int_{\sigma(t)^c} |\xi|^2|\widehat{u}|^2 d\xi \\ &\quad \ge \varrho'(t)\int_{\mathbf{R}^3} |\widehat{u}|^2 d\xi - \varrho'(t)\int_{\sigma(t)} |\widehat{u}|^2 d\xi.\end{aligned} \tag{26}$$

Inserting the above inequality into (24), it follows that

$$\frac{d}{dt}\left(\varrho(t)\int_{\mathbf{R}^3} |\widehat{u}(\xi,t)|^2 d\xi\right) \le \varrho'(t)\int_{\sigma(t)} |\widehat{u}(\xi,t)|^2 d\xi \tag{27}$$

and integrating in time yields

$$\begin{aligned}\varrho(t)\int_{\mathbf{R}^3} |\widehat{u}(\xi,t)|^2 d\xi &\le \int_{\mathbf{R}^3} |\widehat{u}_0|^2 d\xi \\ &\quad + C\int_0^t \varrho'(s)\int_{\sigma(s)} |\widehat{u}(\xi,s)|^2 d\xi\, ds \\ &= I_1 + I_2.\end{aligned} \tag{28}$$

For I_1, by means of the initial condition in Theorem 1 and Parseval equality, it is easy to check that

$$I_1 = \int_{\mathbf{R}^3} |u_0|^2 dx \le C. \tag{29}$$

For I_2, by employing Lemma 4 and direct computation, we have

$$\begin{aligned}I_2 &\le C\int_0^t \varrho'(s)\int_0^{(\varrho'(s)/\varrho(s))^{1/2}} r^2 dr\, ds \\ &\le C\int_0^t \varrho'(s)\left(\frac{\varrho'(s)}{\varrho(s)}\right)^{3/2} ds.\end{aligned} \tag{30}$$

We now choose $\varrho(t) = (1+t)^3$ to get

$$\begin{aligned}&\int_0^t \varrho'(s)\left(\frac{\varrho'(s)}{\varrho(s)}\right)^{3/2} ds \\ &\quad \le C\int_0^t (1+s)^2(1+s)^{-3/2} ds \\ &\quad \le C(1+t)^{3/2}.\end{aligned} \tag{31}$$

Thus,

$$I_2 \le C(1+t)^{3/2}. \tag{32}$$

Inserting the estimates of I_1 and I_2 into (28) becomes, note that $\varrho(t) = (1+t)^3$

$$(1+t)^3\int_{\mathbf{R}^3} |\widehat{u}|^2 d\xi \le C + C(1+t)^{3/2} \tag{33}$$

from which, and together with, Parseval equality implies that

$$\|u(t)\| \le C(1+t)^{-3/4}. \tag{34}$$

Hence we complete the proof of Theorem 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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