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Research Article

On Regularity Criteria for the Two-Dimensional Generalized Liquid Crystal Model

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We establish the regularity criteria for the two-dimensional generalized liquid crystal model. It turns out that the global existence results satisfy our regularity criteria naturally.

1. Introduction

In this paper we consider the following two-dimensional (2D) liquid crystal model:

$$u_t + u \cdot \nabla u + \nabla p + \Lambda^{2\alpha} u = -\nabla d \cdot \Delta d, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{1}$$

$$d_t + u \cdot \nabla d + \Lambda^{2\beta} d = -f(d), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{2}$$

$$\nabla \cdot u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{3}$$

$$(u,d)(x,0) = (u_0, d_0)(x), \quad x \in \mathbb{R}^2, \tag{4}$$

where $u(x,t) \in \mathbb{R}^2$ represents the velocity field, $d(x,t) \in \mathbb{R}^2$ is a vectorial field modeling the orientation of the crystal molecules, and p is the scalar pressure, while $\alpha \geq 0$, $\beta \geq 0$ are two real parameters. $f(d) := (|d|^2 - 1)d$ and the operator $\Lambda = (-\Delta)^{1/2}$ is defined by $\widehat{\Lambda f}(\xi) = |\xi| \widehat{f}(\xi)$, and here $\widehat{f}(\xi)$ denotes the Fourier transform of $f(x)$. We identify the cases $\alpha = 0$ or $\beta = 0$ as the 2D generalized liquid crystal model with zero velocity diffusion or zero orientation diffusion, respectively.

We say our system is a generalized form of liquid crystal model. When $\alpha = 0$, $\beta = 1$, the system is a simplified version of the Ericksen-Leslie system modeling the hydrodynamics of nematic liquid crystals which was developed during the period of 1958 through 1968 [1–3]. We notice that if $d \equiv 0$, $\alpha = 1$, then the system (1)–(4) becomes the Navier-Stokes equations. In this sense, the study of the system (1)–(4) can be valuable and interesting in both mathematical sense and physical sense.

The existence and uniqueness of the weak and smooth solutions for system (1)–(4) are given in [4–6] when $\alpha = \beta = 1$. Local existence of classical solutions for the nematic liquid crystal flows was established in [7].

Now, we mention some known results about regularity theory for the system. In 2010, Zhou and Fan established a regularity criterion for it as $\int_0^T \|\nabla u\|_{L^p}^r / (1 + \ln(e + \|\nabla u\|_{L^p}))dt < +\infty$ with $2/r + 3/p = 2$, $2 \leq p \leq 3$ in [8]. Later, some regularity criteria are proved for the system with zero dissipation in [9]. In [10], Fan et al. established a global regularity for this system with mixed partial viscosity. Recently, in [11, 12], it is proved that smooth solutions are global in the following three cases: $0 < \beta < 1$, $\alpha + \beta \geq 2$; $\alpha > 0$, $\beta = 1$; $\alpha = 0$, $\beta > 1$. Moreover, global strong solution to the density-dependent 2D liquid crystal flows was studied in [13].

This paper is devoted to obtain some regularity criteria for the generalized system (1)–(4). Our main results are the following Theorems.

In our Theorems we set $\rho = \max\{2/\alpha, 2\}$, $\varrho = \max\{2/\beta, 2\}$. The first one is for large α and β.

Theorem 1. *Let $\alpha, \beta \geq 1/2$. Suppose $(u_0, d_0)(x) \in H^2(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$ and $(u,d)(x,t)$ is a local smooth solution of the*

2D generalized liquid crystal model (1)–(4). *If* $u(x,t)$, $d(x,t)$ *satisfy*

$$\int_0^T \|\nabla u(\cdot,t)\|_{L^p}^{1/(1-\theta_\beta)} dt < \infty, \quad (5)$$

or

$$\int_0^T \|\Delta d(\cdot,t)\|_{L^p}^2 dt < \infty, \quad (6)$$

or

$$\int_0^T \|\Delta d(\cdot,t)\|_{L^\varrho}^2 dt < \infty, \quad (7)$$

then $(u(x,t),b(x,t))$ *is a regular solution in* $[0,T]$. *Here* $\theta_\beta = 1/p\beta$, $p\beta \geq 1$.

If in addition, $1 > \beta \geq 1/2$, $\alpha > 0$, $\Lambda^\beta b$ *satisfies*

$$\int_0^T \left\|\Lambda^{\beta+1} d(\cdot,t)\right\|_{L^s}^{2/(1-\theta_\alpha)} dt < \infty, \quad (8)$$

then $(u(x,t),b(x,t))$ *is a regular solution in* $[0,T]$, *where* $\theta_\alpha = ((2/s)-(2\beta-1))/\alpha$, $(2\beta-1)/2 \leq 1/s \leq (2\beta-1+\alpha)/2$.

The following theorems are established for the cases α or β small.

Theorem 2. *Let* $0 < \alpha < 1/2$, $\beta > 0$ *or* $0 < \beta < 1/2$, $\alpha > 0$. *Suppose* $(u_0,d_0)(x) \in H^2(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$ *and* $(u,d)(x,t)$ *is a local smooth solution of the 2D generalized liquid crystal model* (1)–(4). *If* $u(x,t)$, $d(x,t)$ *satisfy*

$$\int_0^T \|\nabla u(\cdot,t)\|_{L^p}^{1/(1-\sigma)} dt < \infty, \qquad \int_0^T \|\Delta d(\cdot,t)\|_{L^p}^{1/(1-\sigma)} dt < \infty, \quad (9)$$

then the solution remains smooth on $(0,T]$. *Here* $\sigma = \max\{\theta_\alpha,\theta_\beta\}$, $\theta_\alpha = 1/p\alpha$, $p\alpha \geq 1$, $\theta_\beta = 1/p\beta$, $p\beta \geq 1$.

Theorem 3. *Let* $\alpha = 0$, $\beta > 0$ *or* $\alpha > 0$, $\beta = 0$. *Suppose* $(u_0,d_0)(x) \in H^2(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$ *and* $(u,d)(x,t)$ *is a local smooth solution of the 2D generalized liquid crystal model* (1)–(4). *If* $u(x,t)$, $d(x,t)$ *satisfy*

$$\int_0^T \|\nabla u(\cdot,t)\|_{\mathrm{BMO}} + \|\nabla u(\cdot,t)\|_{L^\varrho}^2 + \|\Delta d(\cdot,t)\|_{L^\varrho}^2 dt < \infty, \quad (10)$$

or

$$\int_0^T \|\nabla u(\cdot,t)\|_{L^\infty} + \|\Delta d(\cdot,t)\|_{L^p}^{2/(2-\theta)} + \|d(\cdot,t)\|_{\mathrm{BMO}}^2 dt < \infty \quad (11)$$

respectively, then the solution remains smooth on $(0,T]$.

Remark 4. The results in this paper are motivated by the recent works on 2D incompressible generalized MHD equations (refer [14] for details). It turns out that our regularity criteria imply previous global existence results naturally. If $0 < \beta < 1, \alpha + \beta \geq 2$ or $\alpha > 0$, $\beta = 1$ it is proved in [12] $u \in L^\infty(0,T;H^2(\mathbb{R}^2))$, $d \in L^2(0,T;H^3(\mathbb{R}^2))$, the regularity criteria in Theorem 2 are satisfied naturally. If $\alpha = 0$, $\beta > 1$, one can prove $\omega \in L^\infty(0,T;L^p(\mathbb{R}^2))$, $2 \leq p \leq \infty$, $\Delta d \in L^2(0,T;L^p(\mathbb{R}^2))$, $2 \leq p < \infty$ (refer [11] for details); the regularity criteria (10) in Theorem 3 are satisfied naturally.

2. Proof of Theorem 1

In this section, we are devoted to prove our main Theorem 1. Under the assumption in Theorem 1, if $u \in L^\infty(0,T;H^1(\mathbb{R}^2))$ and $d \in L^\infty(0,T;H^2(\mathbb{R}^2))$, we can deduce $u \in L^\infty(0,T;H^2(\mathbb{R}^2))$ and $d \in L^\infty(0,T;H^3(\mathbb{R}^2))$. So we only have to give the regularity criteria to guarantee the H^1 estimation for $u, \nabla d$.

Proof. Firstly, we give the following priori estimates.

Multiplying (2) by $p|d|^{p-2}d$, integrating over $\mathbb{R}^2$, after integrating by parts, and using the following property,

$$\int \Lambda^{2\beta} d \cdot |d|^{p-2} \cdot d > 0, \quad (12)$$

we obtain

$$\frac{d}{dt}\|d\|_{L^p}^p + p\|d\|_{L^{p+2}}^{p+2} = p\|d\|_{L^p}^p \quad (13)$$

for any $2 \leq p < \infty$. Applying Gronwall's inequality, we deduce that

$$\|d\|_{L^p}^p + p\int_0^T \|d\|_{L^{p+2}}^{p+2} d\tau \leq e^{Tp}\|d_0\|_{L^p}^p. \quad (14)$$

Multiplying (1) and (2) by u and $-\Delta d$, respectively, integrating over $\mathbb{R}^2$, and adding the resulting equations together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \left\|\Lambda^\alpha u\right\|_{L^2}^2 + \left\|\Lambda^{\beta+1} d\right\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^2} \nabla f(d)\,\nabla d dx \\
&\quad \leq C\int_{\mathbb{R}^2} |d|^2|\nabla d|^2 dx + \|\nabla d\|_{L^2}^2 \\
&\quad \leq C\|d\|_{L^{2p}}^2\|\nabla d\|_{L^{2q}}^2 + \|\nabla d\|_{L^2}^2 \\
&\quad \leq C\|d\|_{L^{2p}}^2\|\nabla d\|_{L^2}^{2-2\theta}\left\|\Lambda^{\beta+1} d\right\|_{L^2}^{2\theta} + \|\nabla d\|_{L^2}^2 \\
&\quad \leq C\|d\|_{L^{2p}}^{2/(1-\theta)}\|\nabla d\|_{L^2}^2 + \frac{1}{2}\left\|\Lambda^{\beta+1} d\right\|_{L^2}^2 + \|\nabla d\|_{L^2}^2,
\end{aligned} \quad (15)$$

where $1/p + 1/q = 1$, $p > 1/\beta$. Here we have used the following Galiardo-Nirenberg inequality:

$$\|f\|_{L^{2q}} \leq C\|f\|_{L^2}^{1-\theta}\left\|\Lambda^\beta f\right\|_{L^2}^\theta, \qquad \frac{1}{2q} = \left(\frac{1}{2}-\frac{\beta}{2}\right)\theta + \frac{1-\theta}{2}, \quad 0 < \theta < 1, \quad (16)$$

in the above inequality

$$\theta = \frac{1/2 - 1/2q}{\beta/2} = \frac{1/2p}{\beta/2} = \frac{1}{p\beta}. \quad (17)$$

Applying Gronwall's inequality and (14), we obtain

$$\left(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \int_0^T \left\|\Lambda^\alpha u\right\|_{L^2}^2 + \left\|\Lambda^{\beta+1} d\right\|_{L^2}^2 d\tau \leq C(T). \quad (18)$$

Now, we are ready to give the H^1 estimation for $(u, \nabla d)$. Multiplying (1) by $-\Delta u$, applying Δ to (2), and testing it by Δd, then by using (14), (18), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2+\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2\\
&\quad=\int_{\mathbb{R}^2}\nabla d\cdot\Delta d\cdot\Delta u dx-\int_{\mathbb{R}^2}\Delta\left(u\cdot\nabla d\right)\cdot\Delta d dx\\
&\qquad-\int_{\mathbb{R}^2}\Delta f\left(d\right)\cdot\Delta d dx\\
&\quad\le C\int_{\mathbb{R}^2}|\nabla u|\cdot|\Delta d|^2 dx\\
&\qquad+C\int_{\mathbb{R}^2}|d|^2|\Delta d|^2+|\Delta d|^2+|d|\,|\nabla d|^2\,|\Delta d|\,dx\\
&\quad\le C\|\nabla u\|_{L^p}\|\Delta d\|_{L^{2q}}^2+C\|d\|_{L^{2p}}^2\|\Delta d\|_{L^{2q}}^2\\
&\qquad+\|d\|_{L^{2p}}\|\Delta d\|_{L^{2q}}\|\nabla d\|_{L^4}^2+\|\Delta d\|_{L^2}^2\\
&\quad\le C\|\nabla u\|_{L^p}\|\Delta d\|_{L^2}^{2-2\theta}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^{2\theta}\\
&\qquad+C\|d\|_{L^{2p}}^2\|\Delta d\|_{L^2}^{2-2\theta}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^{2\theta}\\
&\qquad+C\|d\|_{L^{2p}}\|\Delta d\|_{L^2}^{1-\theta}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^{\theta}\|\nabla d\|_2\|\Delta d\|_2+\|\Delta d\|_{L^2}^2\\
&\quad\le C\left(\|\nabla u\|_{L^p}^{1/(1-\theta)}+1\right)\|\Delta d\|_{L^2}^2+\frac{1}{2}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2.
\end{aligned}\tag{19}$$

Here we used (16) and the following Galiardo-Nirenberg inequality:

$$\|f\|_{L^4}\le C\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2}.\tag{20}$$

Thanks to Gronwall's inequality and (5), we deduce

$$\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\int_0^T\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2+\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 d\tau\le C\left(T\right).\tag{21}$$

If $\beta>0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2+\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2\\
&\quad\le C\int_{\mathbb{R}^2}|\nabla u|\cdot|\Delta d|^2 dx\\
&\qquad+C\int_{\mathbb{R}^2}|d|^2|\Delta d|^2+|\Delta d|^2+|d|\,|\nabla d|^2\,|\Delta d|\,dx\\
&\quad\le\begin{cases}C\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)\left(\|\Delta d\|_{L^{2/\beta}}^2+1\right)+\dfrac{1}{2}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2,\\ \qquad 0<\beta<1,\\ C\left(\|\nabla u\|_{L^2}^2+1\right)\|\Delta d\|_{L^2}^2+\dfrac{1}{2}\left\|\Lambda^3 d\right\|_{L^2}^2,\\ \qquad\beta\ge 1.\end{cases}
\end{aligned}\tag{22}$$

Here we have used

$$\begin{aligned}
&\int_{\mathbb{R}^2}|\nabla u|\cdot|\Delta d|^2 dx\\
&\qquad+\int_{\mathbb{R}^2}|d|^2|\Delta d|^2+|\Delta d|^2+|d|\,|\nabla d|^2\,|\Delta d|\,dx\\
&\quad\le C\|\nabla u\|_{L^2}\|\Delta d\|_{L^4}^2+C\|d\|_{L^{2p}}^2\|\Delta d\|_{L^{2q}}^2\\
&\qquad+\|d\|_{L^{2p}}\|\Delta d\|_{L^{2q}}\|\nabla d\|_{L^4}^2+\|\Delta d\|_{L^2}^2\\
&\quad\le C\|\nabla u\|_2\|\Delta d\|_{2/\beta}\left\|\Lambda^{\beta+2}d\right\|_2+C\|d\|_{L^{2p}}^2\|\Delta d\|_{L^2}^{2-2\theta}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^{2\theta}\\
&\qquad+C\|d\|_{L^{2p}}\|\Delta d\|_{L^2}^{1-\theta}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^{\theta}\|\nabla d\|_2\|\Delta d\|_2+\|\Delta d\|_{L^2}^2\\
&\quad\le C\|\nabla u\|_{L^2}^2\|\Delta d\|_{L^{2/\beta}}^2+C\|\Delta d\|_{L^2}^2+\frac{1}{2}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2,\\
&\qquad\qquad 0<\beta<1,\\
&\int_{\mathbb{R}^2}|\nabla u|\cdot|\Delta d|^2 dx\\
&\qquad+\int_{\mathbb{R}^2}|d|^2|\Delta d|^2+|\Delta d|^2+|d|\,|\nabla d|^2\,|\Delta d|\,dx\\
&\quad\le\|\nabla u\|_{L^2}\|\Delta d\|_{L^4}^2+C\|d\|_{L^4}^2\|\Delta d\|_{L^4}^2\\
&\qquad+\|d\|_{L^4}\|\Delta d\|_{L^4}\|\nabla d\|_{L^4}^2+\|\Delta d\|_{L^2}^2\\
&\quad\le C\|\nabla u\|_{L^2}\|\Delta d\|_{L^2}\left\|\Lambda^3 d\right\|_{L^2}+C\|d\|_{L^4}^2\|\Delta d\|_{L^2}\left\|\Lambda^3 d\right\|_{L^2}\\
&\qquad+C\|d\|_{L^4}\|\Delta d\|_{L^2}^{3/2}\left\|\Lambda^3 d\right\|_{L^2}^{1/2}\|\nabla d\|_{L^2}+\|\Delta d\|_{L^2}^2\\
&\quad\le C\|\nabla u\|_{L^2}^2\|\Delta d\|_{L^2}^2+C\|\Delta d\|_{L^2}^2+\frac{1}{2}\left\|\Lambda^3 d\right\|_{L^2}^2,\quad\beta\ge 1.
\end{aligned}\tag{23}$$

In the above estimation we have used (16), (20), and the following Galiardo-Nirenberg inequality:

$$\|\Delta d\|_{L^4}^2\le C\left\|\Lambda^{\beta+2}d\right\|_{L^2}\|\Delta d\|_{L^{2/\beta}}.\tag{24}$$

Using the condition (6) we get the H^1 estimation (21).

If $\alpha>0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2+\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2\\
&\quad\le C\int_{\mathbb{R}^2}|\nabla u|\cdot|\Delta d|^2 dx\\
&\qquad+C\int_{\mathbb{R}^2}|d|^2|\Delta d|^2+|\Delta d|^2+|d|\,|\nabla d|^2\,|\Delta d|\,dx\\
&\quad\le\begin{cases}C\left(\|\Delta d\|_{L^{2/\alpha}}^2+1\right)\|\Delta d\|_{L^2}^2\\ \quad+\dfrac{1}{2}\left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2+\dfrac{1}{2}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2,\quad 0<\alpha<1,\\ C\|\Delta d\|_{L^2}^4+C\|\Delta d\|_{L^2}^2\\ \quad+\dfrac{1}{2}\|\Delta u\|_{L^2}^2+\dfrac{1}{2}\left\|\Lambda^{\beta+2}d\right\|_{L^2}^2,\qquad\alpha\ge 1.\end{cases}
\end{aligned}\tag{25}$$

Here we have used

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla u| \cdot |\Delta d|^2 dx \\
&\quad + \int_{\mathbb{R}^2} |d|^2 |\Delta d|^2 + |\Delta d|^2 + |d| \, |\nabla d|^2 \, |\Delta d| \, dx \\
&\le C\|\nabla u\|_{L^{2/(1-\alpha)}} \|\Delta d\|^2_{L^{4/(1+\alpha)}} + C\|d\|^2_{L^{2p}} \|\Delta d\|^{2-2\theta}_{L^2} \left\|\Lambda^{\beta+2} d\right\|^{2\theta}_{L^2} \\
&\quad + C\|d\|_{L^{2p}} \|\Delta d\|^{1-\theta}_{L^2} \left\|\Lambda^{\beta+2} d\right\|^{\theta}_{L^2} \|\nabla d\|_{L^2} \|\Delta d\|_{L^2} + \|\Delta d\|^2_{L^2} \\
&\le C\left\|\Lambda^{\alpha+1} u\right\|_{L^2} \|\Delta d\|_{L^2} \|\Delta d\|_{L^{2/\alpha}} + C\|\Delta d\|^2_{L^2} + \frac{1}{2}\left\|\Lambda^{\beta+2} d\right\|^2_{L^2} \\
&\le C\left(\|\Delta d\|^2_{L^{2/\alpha}} + 1\right) \|\Delta d\|^2_{L^2} + \frac{1}{2}\left\|\Lambda^{\alpha+1} u\right\|^2_{L^2} + \frac{1}{2}\left\|\Lambda^{\beta+2} d\right\|^2_{L^2}, \\
&\qquad 0 < \alpha < 1, \\
&\int_{\mathbb{R}^2} |\nabla u| \cdot |\Delta d|^2 dx \\
&\quad + \int_{\mathbb{R}^2} |d|^2 |\Delta d|^2 + |\Delta d|^2 + |d| \, |\nabla d|^2 \, |\Delta d| \, dx \\
&\le C\left(\|\nabla u\|_{\mathrm{BMO}} \|\Delta d\|_{L^2} + \|\nabla d\|_{\mathrm{BMO}} \|\Delta u\|_{L^2}\right) \|\Delta d\|_{L^2} \\
&\quad + C\|d\|^2_{L^{2p}} \|\Delta d\|^{2-2\theta}_{L^2} \left\|\Lambda^{\beta+2} d\right\|^{2\theta}_{L^2} \\
&\quad + C\|d\|_{L^{2p}} \|\Delta d\|^{1-\theta}_{L^2} \left\|\Lambda^{\beta+2} d\right\|^{\theta}_{L^2} \|\nabla d\|_{L^2} \|\Delta d\|_{L^2} + \|\Delta d\|^2_{L^2} \\
&\le C\|\Delta d\|^4_{L^2} + C\|\Delta d\|^2_{L^2} + \frac{1}{2}\|\Delta u\|^2_{L^2} + \frac{1}{2}\left\|\Lambda^{\beta+2} d\right\|^2_{L^2}, \\
&\qquad \alpha \ge 1.
\end{aligned} \tag{26}$$

Here and in the following we use the embeddings $H^s(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for $s < n/2$ and $1/p = 1/2 - s/n$, $H^1(\mathbb{R}^2) \hookrightarrow \mathrm{BMO}$ and the commutator estimate given in [15]

$$\left\|\partial^\alpha f \cdot \partial^\beta g\right\|_r \le C\left(\|f\|_{\mathrm{BMO}} \left\|\Lambda^{|\alpha|+|\beta|} g\right\|_r + \|g\|_{\mathrm{BMO}} \left\|\Lambda^{|\alpha|+|\beta|} f\right\|_r\right), \tag{27}$$

usually. Thanks to condition (7) we have the H^1 estimation (21).

If in addition, $1 > \beta \ge 1/2$, $\alpha > 0$, use (16),

$$\begin{aligned}
&\|\nabla u\|_{L^r} \le \|\nabla u\|^{1-\theta_\alpha}_{L^2} \left\|\Lambda^{\alpha+1} u\right\|^{\theta_\alpha}_{L^2}, \\
&\frac{1}{r} = \left(\frac{1}{2} - \frac{\alpha}{2}\right)\theta_\alpha + \frac{1-\theta_\alpha}{2}, \\
&\qquad 0 \le \theta_\alpha \le 1,
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\|\Delta d\|_{L^{2l}} \le C\left\|\Lambda^{\beta+1} d\right\|^{1/2}_{L^s} \left\|\Lambda^{2+\beta} d\right\|^{1/2}_{L^2}, \\
&\frac{1}{2l} = \frac{1-\beta}{2} + \frac{1}{2s},
\end{aligned}$$

where

$$\begin{aligned}
&\theta_\alpha = \frac{1/2 - 1/r}{\alpha/2} = \frac{r-2}{r\alpha}, \quad 2 \le r \le \frac{2}{1-\alpha}, \\
&s = \frac{r}{r\beta - 1}, \qquad \frac{1}{r} + \frac{1}{l} = 1.
\end{aligned} \tag{29}$$

We have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}\right) + \left\|\Lambda^{\alpha+1} u\right\|^2_{L^2} + \left\|\Lambda^{\beta+2} d\right\|^2_{L^2} \\
&\quad \le C\|\nabla u\|_{L^r} \|\Delta d\|^2_{L^{2l}} + C\|d\|^2_{L^{2p}} \|\Delta d\|^2_{L^{2q}} \\
&\qquad + \|d\|_{L^{2p}} \|\Delta d\|_{L^{2q}} \|\nabla d\|^2_{L^4} + \|\Delta d\|^2_{L^2} \\
&\quad \le C\|\nabla u\|^{1-\theta_\alpha}_{L^2} \left\|\Lambda^{\alpha+1} u\right\|^{\theta_\alpha}_{L^2} \left\|\Lambda^{\beta+1} d\right\|_{L^s} \left\|\Lambda^{\beta+2} d\right\|_{L^2} \\
&\qquad + C\|\Delta d\|^{2-2\theta}_{L^2} \left\|\Lambda^{\beta+2} d\right\|^{2\theta}_{L^2} \\
&\qquad + C\|\Delta d\|^{1-\theta}_{L^2} \left\|\Lambda^{\beta+2} d\right\|^{\theta}_{L^2} \|\nabla d\|_{L^2} \|\Delta d\|_{L^2} + \|\Delta d\|^2_{L^2} \\
&\quad \le C\left(\|\nabla u\|^2_{L^2} + \|\Delta d\|^2_{L^2}\right)\left(\left\|\Lambda^{\beta+1} d\right\|^{2/(1-\theta_1)}_{L^s} + 1\right) \\
&\qquad + \frac{1}{2}\left\|\Lambda^{\alpha+1} u\right\|^2_{L^2} + \frac{1}{2}\left\|\Lambda^{\beta+2} d\right\|^2_{L^2}.
\end{aligned} \tag{30}$$

Thanks to condition (8) we have the H^1 estimation (21). Now we complete the H^1 estimation.

Since $H^1 \hookrightarrow \mathrm{BMO}$ in 2D, if $\nabla u, \Delta d \in L^\infty(0,T;H^1(\mathbb{R}^2))$, then we can deduce that $\nabla u, \Delta d \in L^\infty(0,T;\mathrm{BMO})$.

Applying Δ to (1) and testing by Δu, applying Δ to (1), and testing by $-\Delta\Delta d$, after suitable integration by parts we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta u\|^2_2 + \|\nabla\Delta d\|^2_2\right) + \left\|\Lambda^{\alpha+2} u\right\|^2_2 + \left\|\Lambda^{\beta+3} d\right\|^2_2 \\
&\quad = -\int_{R^2} \Delta(u \cdot \nabla u) \cdot \Delta u dx - \int_{R^2} \Delta(\nabla d \cdot \Delta d) \cdot \Delta u dx \\
&\qquad + \int_{R^2} \Delta(u \cdot \nabla d) \cdot \Delta\Delta d dx + \int_{R^2} \Delta f(d) \cdot \Delta\Delta d dx \\
&\quad \le C\int_{R^2} |\nabla u| \cdot |\Delta u|^2 dx + C\int_{R^2} |\Delta u| \cdot |\Delta d| \cdot |\nabla\Delta d| \, dx \\
&\qquad + C\int_{R^2} |\nabla u| \cdot |\nabla\Delta d|^2 dx + \int_{R^2} \left|\nabla\Delta f(d)\right| \cdot |\nabla\Delta d| \, dx \\
&\quad := I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{31}$$

If $\alpha \ge 1/2$, $\beta \ge 1/2$, we estimate the right hand side of (31) one by one:

$$\begin{aligned}
I_1 &= C\int_{R^2} |\nabla u| \cdot |\Delta u|^2 dx \\
&\le C\|\nabla u\|_{L^3} \|\Delta u\|^2_{L^3} \\
&\le C\|\nabla u\|_{L^3} \|\Delta u\|^{4/3}_{L^3} \|\Delta u\|^{2/3}_{L^3} \\
&\le C\|\nabla u\|^{7/9}_{L^2} \left\|\Lambda^{1/2}\Delta u\right\|^{2/9}_{L^2} \left\|\Lambda^{1/2}\nabla u\right\|^{2/9}_{L^2} \|\Delta u\|^{2/9}_{L^2} \left\|\Lambda^{1/2}\Delta u\right\|^{14/9}_{L^2}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|\nabla u\|_{L^2}^{7/9}\left\|\Lambda^{1/2}\nabla u\right\|_{L^2}^{2/9}\|\Delta u\|_{L^2}^{2/9}\left\|\Lambda^{1/2}\Delta u\right\|_{L^2}^{16/9}\\
&\leq C(\epsilon)\|\nabla u\|_{L^2}^{7}\left\|\Lambda^{1/2}\nabla u\right\|_{L^2}^{2}\|\Delta u\|_{L^2}^{2}+\epsilon\left\|\Lambda^{1/2}\Delta u\right\|_{L^2}^{2},\\
I_2 &= C\int_{R^2}|\Delta u|\cdot|\Delta d|\cdot|\nabla\Delta d|\,dx\\
&\leq C\|\Delta d\|_{L^3}\|\nabla\Delta d\|_{L^3}\|\Delta u\|_{L^3}\\
&\leq C\|\Delta d\|_{L^2}^{7/9}\left\|\Lambda^{1/2}\Delta d\right\|_{L^2}^{1/9}\|\nabla\Delta d\|_{L^2}^{1/9}\left\|\Lambda^{1/2}\nabla\Delta d\right\|_{L^2}\\
&\quad\times\left\|\Lambda^{1/2}\nabla u\right\|_{L^2}^{1/9}\|\Delta u\|_{L^2}^{1/9}\left\|\Lambda^{1/2}\Delta u\right\|_{L^2}^{7/9}\\
&\leq C(\epsilon)\|\Delta d\|_{L^2}^{7}\left(\left\|\Lambda^{1/2}\Delta d\right\|_{L^2}^{2}\|\nabla\Delta d\|_{L^2}^{2}+\left\|\Lambda^{1/2}\nabla u\right\|_{L^2}^{2}\|\Delta u\|_{L^2}^{2}\right)\\
&\quad+\epsilon\left(\left\|\Lambda^{1/2}\nabla\Delta d\right\|_{L^2}^{2}+\left\|\Lambda^{1/2}\Delta u\right\|_{L^2}^{2}\right),\\
I_3 &= C\int_{R^2}|\nabla u|\cdot|\nabla\Delta d|^2dx\\
&\leq C\|\nabla u\|_{L^3}\|\nabla\Delta d\|_{L^3}^{2}\\
&\leq C\|\nabla u\|_{L^3}\|\nabla\Delta d\|_{L^3}^{4/3}\|\nabla\Delta d\|_{L^3}^{2/3}\\
&\leq C(\epsilon)\|\nabla u\|_{L^2}^{7}\left\|\Lambda^{1/2}\Delta d\right\|_{L^2}^{2}\|\nabla\Delta d\|_{L^2}^{2}\\
&\quad+\epsilon\left(\left\|\Lambda^{1/2}\nabla\Delta d\right\|_{L^2}^{2}+\left\|\Lambda^{1/2}\Delta u\right\|_{L^2}^{2}\right).
\end{aligned}\tag{32}$$

Here we used the following Galiardo-Nirenberg inequalities:

$$\begin{aligned}
&\|\nabla f\|_{L^3}\leq C\left\|\Lambda^{1/2}f\right\|_{L^2}^{1/6}\left\|\Lambda^{1/2}\nabla f\right\|_{L^2}^{5/6},\\
&\|\nabla f\|_{L^3}\leq C\|\nabla f\|_{L^2}^{1/3}\left\|\Lambda^{1/2}\nabla f\right\|_{L^2}^{2/3},\\
&\|\nabla f\|_{L^3}=\|\nabla f\|_{L^3}^{2/3}\|\nabla f\|_{L^3}^{1/3}\leq C\left\|\Lambda^{1/2}f\right\|_{L^2}^{1/9}\|\nabla f\|_{L^2}^{1/9}\left\|\Lambda^{1/2}\nabla f\right\|_{L^2}^{7/9},\\
&\|f\|_{L^3}\leq C\|f\|_{L^2}^{7/9}\left\|\Lambda^{1/2}\nabla f\right\|_{L^2}^{2/9}.
\end{aligned}\tag{33}$$

Using (14), (16), (20), (21), and the following Galiardo-Nirenberg inequality,

$$\|\nabla d\|_{L^4}\leq\|d\|_{L^2}^{1/2}\|\nabla\Delta d\|_{L^2}^{1/2},\tag{34}$$

we have

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^2}\left|\nabla\Delta f(d)\right|\cdot|\nabla\Delta d|\,dx\\
&\leq C\int_{R^2}\left(|d|^2|\nabla\Delta d|^2+|d|\,|\nabla d|\,|\Delta d|\,|\nabla\Delta d|\right)dx+\|\nabla\Delta d\|_{L^2}^{2}\\
&\leq C\|d\|_{L^{2p}}^{2}\|\nabla\Delta d\|_{L^{2q}}^{2}+C\|d\|_{L^{2p}}\|\nabla d\|_{L^4}\|\Delta d\|_{L^4}\|\nabla\Delta d\|_{L^{2q}}\\
&\quad+\|\nabla\Delta d\|_{L^2}^{2}\\
&\leq C\|d\|_{L^{2p}}\|d\|_{L^2}^{1/2}\|\nabla\Delta d\|_{L^2}^{1/2}\|\Delta d\|_{L^2}^{1/2}\|\nabla\Delta d\|_{L^2}^{1/2}\|\nabla\Delta d\|_{L^2}^{1-\theta}\\
&\quad\times\left\|\Lambda^{\beta+3}d\right\|_{L^2}^{\theta}+C\|d\|_{L^{2p}}^{2}\|\nabla\Delta d\|_{L^2}^{2-2\theta}\left\|\Lambda^{\beta+3}d\right\|_{L^2}^{2\theta}+\|\nabla\Delta d\|_{L^2}^{2}\\
&\leq C\|\nabla\Delta d\|_{L^2}^{2}+\epsilon\left\|\Lambda^{\beta+3}d\right\|_{L^2}^{2}.
\end{aligned}\tag{35}$$

Finally, putting the above results together, we deduce that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\Delta u\|_{L^2}^{2}+\|\nabla\Delta d\|_{L^2}^{2}\right)+\left\|\Lambda^{\alpha+2}u\right\|_{L^2}^{2}+\left\|\Lambda^{\beta+3}d\right\|_{L^2}^{2}\\
&\quad\leq C\left(\|\nabla u\|_{L^2}^{7}+\|\Delta d\|_{L^2}^{7}\right)\\
&\qquad\times\left(\left\|\Lambda^{1/2}\nabla u\right\|_{L^2}^{2}\|\Delta u\|_{L^2}^{2}+\left\|\Lambda^{1/2}\Delta d\right\|_{L^2}^{2}\|\nabla\Delta d\|_{L^2}^{2}\right.\\
&\qquad\quad\left.+\|\nabla\Delta d\|_{L^2}^{2}\right).
\end{aligned}\tag{36}$$

By using Gronwall's inequality and (21), we obtain

$$\left(\|\Delta u\|_{L^2}^{2}+\|\nabla\Delta d\|_{L^2}^{2}\right)+\int_0^t\left\|\Lambda^{\alpha+2}u\right\|_{L^2}^{2}+\left\|\Lambda^{\beta+3}d\right\|_{L^2}^{2}dt\leq C(T).\tag{37}$$

The proof of Theorem 1 is finished. □

3. Proof of Theorem 2

Now, we give the proof of Theorem 2.

Proof. This section focuses on the case with $0<\alpha<1/2$, $\beta>0$ or $0<\beta<1/2$, $\alpha>0$. Since we cannot estimate (37) as that in Theorem 1, so we should give the estimation of $I_i, i=1,2,3,4,5$ which is defined in Section 2. Using (16), we have

$$\begin{aligned}
I_1 &= C\int_{\mathbb{R}^2}|\nabla u|\cdot|\Delta u|^2dx\leq C\|\nabla u\|_{L^p}\|\Delta u\|_{L^{2q}}^{2}\\
&\leq C\|\nabla u\|_{L^p}\|\Delta u\|_{L^2}^{2(1-\theta_\alpha)}\left\|\Lambda^{2+\alpha}u\right\|_{L^2}^{2\theta_\alpha}\\
&\leq C\|\nabla u\|_{L^p}^{1/(1-\theta_\alpha)}\|\Delta u\|_{L^2}^{2}+\epsilon\left\|\Lambda^{2+\alpha}u\right\|_{L^2}^{2},
\end{aligned}\tag{38}$$

where $1/p+1/q=1$ and $\theta_\alpha=1/p\alpha$. We use the same way to estimate I_2 and I_3.

$$\begin{aligned}
I_2 &= C\int_{\mathbb{R}^2}|\Delta u|\cdot|\Delta d|\cdot|\nabla\Delta d|\,dx\\
&\leq C\|\Delta d\|_{L^p}\|\nabla\Delta d\|_{L^{2q}}\|\Delta u\|_{L^{2q}}\\
&\leq C\|\Delta d\|_{L^p}^{1/2}\|\nabla\Delta d\|_{L^{2q}}\|\Delta d\|_{L^p}^{1/2}\|\Delta u\|_{L^{2q}}\\
&\leq C\left(\|\Delta d\|_{L^p}^{1/(1-\theta_\beta)}\|\nabla\Delta d\|_{L^2}^{2}+\|\Delta d\|_{L^p}^{1/(1-\theta_\alpha)}\|\Delta u\|_{L^2}^{2}\right)\\
&\quad+\epsilon\left(\left\|\Lambda^{2+\alpha}u\right\|_{L^2}^{2}+\left\|\Lambda^{3+\beta}d\right\|_{L^2}^{2}\right);
\end{aligned}$$

$$\begin{aligned} I_3 &= C \int_{\mathbb{R}^2} |\nabla u| \cdot |\nabla \Delta d|^2 dx \\ &\leq C \|\nabla u\|_{L^p} \|\nabla \Delta d\|_{L^{2q}}^2 \\ &\leq C \|\nabla u\|_{L^p} \|\nabla \Delta d\|_{L^2}^{2-2\theta_\beta} \left\|\Lambda^{3+\beta} d\right\|_{L^2}^{2\theta_\beta} \\ &\leq C \|\nabla u\|_{L^p}^{1/(1-\theta_\beta)} \|\nabla \Delta d\|_{L^2}^2 + \epsilon \left\|\Lambda^{3+\beta} d\right\|_{L^2}^2, \end{aligned} \tag{39}$$

The same as that in Section 2, we have

$$\begin{aligned} I_4 &= \int_{\mathbb{R}^2} \left|\nabla \Delta f(d)\right| \cdot |\nabla \Delta d|\, dx \\ &\leq C \|d\|_{L^{2p}}^2 \|\nabla \Delta d\|_{L^{2q}}^2 \\ &\quad + C \|d\|_{L^{2p}} \|\nabla d\|_{L^4} \|\Delta d\|_{L^4} \|\nabla \Delta d\|_{L^{2q}} + \|\nabla \Delta d\|_{L^2}^2 \\ &\leq C \|\nabla \Delta d\|_{L^2}^2 + \epsilon \left\|\Lambda^{\beta+3} d\right\|_{L^2}^2. \end{aligned} \tag{40}$$

Finally, putting the above results together, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2\right) + \left\|\Lambda^{2+\alpha} u\right\|_{L^2}^2 + \left\|\Lambda^{3+\beta} d\right\|_{L^2}^2 \\ &\quad \leq C\Big(\|\Delta d\|_{L^p}^{1/(1-\theta_\beta)} + \|\nabla u\|_{L^p}^{1/(1-\theta_\beta)} + \|\Delta d\|_{L^p}^{1/(1-\theta_\alpha)} \\ &\qquad\qquad + \|\nabla u\|_{L^p}^{1/(1-\theta_\alpha)} + 1\Big) \\ &\qquad \times \left(\|\nabla \Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right) + \epsilon \left(\left\|\Lambda^{2+\alpha} u\right\|_{L^2}^2 + \left\|\Lambda^{3+\beta} d\right\|_{L^2}^2\right). \end{aligned} \tag{41}$$

Using Gronwall's inequality and (9) we can deduce (37).

The proof is finished. □

4. Proof of Theorem 3

Proof. For $\alpha = 0$, $\beta > 0$, firstly we give the H^1 estimation for $(u, \nabla d)$:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) + \left\|\Lambda^{\beta+2} d\right\|_{L^2}^2 \\ &\quad \leq \begin{cases} C\left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right)\left(\|\Delta d\|_{L^{2/\beta}}^2 + 1\right) + \frac{1}{2}\left\|\Lambda^{\beta+2} d\right\|_{L^2}^2, & 0 < \beta < 1, \\ C\left(\|\nabla u\|_{L^2}^2 + 1\right)\|\Delta d\|_{L^2}^2 + \frac{1}{2}\left\|\Lambda^3 d\right\|_{L^2}^2, & \beta \geq 1. \end{cases} \end{aligned} \tag{42}$$

Then by using Gronwall's inequality and (10) we obtain (21).

Now, we give the H^1 estimation for $(\nabla u, \Delta d)$. We should give the estimation of $I_i, i = 1,2,3,4$ which is defined in Section 2:

$$\begin{aligned} I_1 &= C \int_{\mathbb{R}^2} |\nabla u| \cdot |\Delta u|^2 dx \leq C \|\nabla u\|_{\mathrm{BMO}} \|\Delta u\|_{L^2}^2, \\ I_2 &= C \int_{\mathbb{R}^2} |\Delta u| \cdot |\Delta d| \cdot |\nabla \Delta d|\, dx \\ &\leq \begin{cases} C\,\|\Delta d\|_{L^{2/\beta}} \left\|\Lambda^{\beta+3} d\right\|_{L^2} \|\Delta u\|_{L^2} \\ \quad \leq C \|\Delta d\|_{L^{2/\beta}}^2 \|\Delta u\|_{L^2}^2 + \epsilon \left\|\Lambda^{\beta+3} d\right\|_{L^2}^2, & 0 < \beta < 1, \\ C \|\Delta d\|_{L^2}^2 \|\Delta u\|_{L^2}^2 + \epsilon \left\|\nabla^2 \Delta d\right\|_{L^2}^2, & \beta \geq 1. \end{cases} \\ I_3 &= C \int_{\mathbb{R}^2} |\nabla u| \cdot |\nabla \Delta d|^2 dx \\ &\leq \begin{cases} \|\nabla u\|_{L^{2/\beta}} \left\|\Lambda^{\beta+3} d\right\|_{L^2} \|\nabla \Delta d\|_{L^2} \\ \quad \leq C \|\nabla u\|_{L^{2/\beta}}^2 \|\nabla \Delta d\|_{L^2}^2 + \epsilon \left\|\Lambda^{\beta+3} d\right\|_{L^2}^2, & 0 < \beta < 1, \\ C \|\nabla u\|_{L^2}^2 \|\nabla \Delta d\|_{L^2}^2 + \epsilon \left\|\nabla^2 \Delta d\right\|_{L^2}^2, & \beta \geq 1. \end{cases} \end{aligned} \tag{43}$$

The same as that in Section 2, we have

$$\begin{aligned} I_4 &= \int_{\mathbb{R}^2} \left|\nabla \Delta f(d)\right| \cdot |\nabla \Delta d|\, dx \\ &\leq C \|\nabla \Delta d\|_{L^2}^2 + \frac{1}{2}\left\|\Lambda^{\beta+3} d\right\|_{L^2}^2. \end{aligned} \tag{44}$$

Putting the above results together, we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\Delta u\|_{L^2}^2 + \|\nabla \Delta u\|_{L^2}^2\right) + \left\|\Lambda^{2+\alpha} u\right\|_{L^2}^2 + \left\|\Lambda^{3+\beta} d\right\|_{L^2}^2 \\ &\quad \leq C\left(\|\nabla u\|_{\mathrm{BMO}} + \|\Delta d\|_{L^\varrho}^2 + \|\nabla u\|_{L^\varrho}^2 + 1\right) \\ &\qquad \times \left(\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2\right) \\ &\qquad + \epsilon \left(\left\|\Lambda^{2+\alpha} u\right\|_{L^2}^2 + \left\|\Lambda^{3+\beta} d\right\|_{L^2}^2\right). \end{aligned} \tag{45}$$

Applying Gronwall's inequality and (10) we obtain (37).

For $\alpha > 0$, $\beta = 0$, firstly we give the estimation for $(u, \nabla d)$:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \left\|\Lambda^{\beta+1} d\right\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^2} \nabla f(d)\, \nabla d dx \\ &\quad \leq C \int_{\mathbb{R}^2} |d|^2 |\nabla d|^2 dx + \|\nabla d\|_{L^2}^2 \\ &\quad \leq C \|d\|_{\mathrm{BMO}}^2 \|\nabla d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \end{aligned} \tag{46}$$

and the H^1 estimation for $(u, \nabla d)$:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) + \left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 &\le C\|\nabla u\|_{L^\infty}\|\Delta d\|_{L^2}^2 \\ &\quad + C\int_{\mathbb{R}^2} |d|^2|\Delta d|^2 + |d|\,|\nabla d|\,|\nabla d|\,|\Delta d|\,dx + \|\Delta d\|_{L^2}^2 \\ &\le C\|\nabla u\|_{L^\infty}\|\Delta d\|_{L^2}^2 + \|d\|_{\mathrm{BMO}}^2\|\Delta d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \\ &\le C\left(\|\nabla u\|_{L^\infty} + \|d\|_{\mathrm{BMO}}^2 + 1\right)\|\Delta d\|_{L^2}^2. \end{aligned} \tag{47}$$

The regularity criteria (11) can guarantee estimations (18) and (21).

Now, we give the estimation of $I_i, i = 1, 2, 3, 4$ which is defined in Section 2:

$$\begin{aligned} I_1 &= C\int_{\mathbb{R}^2} |\nabla u|\cdot|\Delta u|^2 dx \le C\|\nabla u\|_{L^\infty}\|\Delta u\|_{L^2}^2, \\ I_3 &= C\int_{\mathbb{R}^2} |\nabla u|\cdot|\nabla\Delta d|^2 dx \le C\|\nabla u\|_{L^\infty}\|\nabla\Delta d\|_{L^2}^2, \\ I_4 &= \int_{\mathbb{R}^2}\left|\nabla\Delta f(d)\right|\cdot|\nabla\Delta d|\,dx \\ &\le C\int_{\mathbb{R}^2}\left(|d|^2|\nabla\Delta d|^2 + |d|\,|\nabla d|\,|\Delta d|\,|\nabla\Delta d|\right)dx + \|\nabla\Delta d\|_{L^2}^2 \\ &\le C\left(\|d\|_{\mathrm{BMO}}^2 + 1\right)\|\nabla\Delta d\|_{L^2}^2. \end{aligned} \tag{48}$$

Using the following Galiardo-Nirenberg inequality,

$$\begin{gathered} \|\Delta u\|_{L^q} \le C\|\Delta u\|_{L^2}^{1-\theta}\left\|\Lambda^{\alpha+2}u\right\|_{L^2}^{\theta}, \\ \frac{1}{q} = \left(\frac{1}{2} - \frac{\alpha}{2}\right)\theta + \frac{1-\theta}{2}, \\ 0 \le \theta \le 1, \end{gathered} \tag{49}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \qquad \theta = \frac{1/2 - 1/q}{\alpha/2} = \frac{2}{p\alpha}, \qquad p\alpha \ge 2, \tag{50}$$

we can deduce

$$\begin{aligned} I_2 &= C\int_{\mathbb{R}^2} |\Delta u|\cdot|\Delta d|\cdot|\nabla\Delta d|\,dx \\ &\le C\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^p}\|\Delta u\|_{L^q} \\ &\le C\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^p}\|\Delta u\|_{L^2}^{1-\theta}\left\|\Lambda^{\alpha+2}u\right\|_{L^2}^{\theta} \\ &\le C\|\Delta d\|_{L^p}^{2/(2-\theta)}\left(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2\right) + \epsilon\left\|\Lambda^{\alpha+2}u\right\|_{L^2}^2. \end{aligned} \tag{51}$$

Finally, putting the above results together, we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\Delta u\|_2^2 + \|\nabla\Delta d\|_2^2\right) + \left\|\Lambda^{\alpha+2}u\right\|_{L^2}^2 + \left\|\Lambda^{\beta+3}d\right\|_{L^2}^2 &\le C\left(\|\nabla u\|_{L^\infty} + \|\Delta d\|_{L^p}^{2/(2-\theta)} + \|d\|_{\mathrm{BMO}}^2 + 1\right) \\ &\quad \times\left(\|\nabla\Delta d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right) \\ &\quad + \epsilon\left(\left\|\Lambda^{\alpha+2}u\right\|_{L^2}^2 + \left\|\Lambda^{\beta+3}d\right\|_{L^2}^2\right). \end{aligned} \tag{52}$$

By using Gronwall's inequality and (11) we obtain (37). Now we complete our proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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