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Research Article

Stability of a Mathematical Model of Malaria Transmission with Relapse

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A more realistic mathematical model of malaria is introduced, in which we not only consider the recovered humans return to the susceptible class, but also consider the recovered humans return to the infectious class. The basic reproduction number R_0 is calculated by next generation matrix method. It is shown that the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, and the system is uniformly persistence if $R_0 > 1$. Some numerical simulations are also given to explain our analytical results. Our results show that to control and eradicate the malaria, it is very necessary for the government to decrease the relapse rate and increase the recovery rate.

1. Introduction

Malaria is caused by a parasite called *Plasmodium*, which is transmitted via the bites of infected mosquitoes. Approximately half of the world's population is at risk of malaria. Most malaria cases and deaths occur in Sub-Saharan Africa. In 2011, 99 countries and territories had ongoing malaria transmission [1]. Recently, the incidence of malaria has been rising due to drug resistance. Various control strategies have been taken to reduce malaria transmissions.

Many epidemic models have been analyzed mathematically and applied to specific diseases [2, 3]. Since the first mathematical model of malaria transmission is introduced by Ross [4], quite a few mathematical models have been formulated to investigate the transmission dynamics of malaria [5–12]. Ngwa and Shu [5] analyze a deterministic differential equation model for endemic malaria involving variable human and mosquito populations. Ngwa [6] also analyzes a mathematical model for endemic malaria involving variable human and mosquito populations and uses a perturbation analysis to approximate the endemic equilibrium in the important case where the disease related death rate is nonzero, small but significant. Furthermore, in quasistationarity, the stochastic process undergoes oscillations about a mean population whose size can be approximated by the stable endemic deterministic equilibrium. Chitnis et al. [7, 8] study a model that both human and vector species follow a logistic population, and human have immigration and disease-induced death. They present a bifurcation analysis and analyze a periodically-forced difference equation model for malaria in mosquitoes that captures the effects of seasonality and allows the mosquitoes to feed on a heterogeneous population of hosts. Chamchod and Britton [9] incorporate a vector-bias term into a malaria transmission model to account for the greater attractiveness of infectious humans to mosquitoes in terms of differing probabilities that a mosquito arriving at a human depending on whether he is infectious or susceptible. To take account of the incubation periods of parasites within the human and the mosquito, a delayed Ross-Macdonald model is taken by Ruan et al. [10]. Further, Xiao and Zou [11] use mathematical models to explored a natural concern of possible epidemics caused by multiple species of malaria parasites in one region. They find that epidemics involving both species in a single region are possible. Li [12] provides a basic analysis for the stage-structured malaria model and shows that both the baseline and the stage-structured malaria models undergo backward bifurcations.

Recently, Li et al. [13] consider a fast and slow dynamics of malaria model with relapse, and analyse the global dynamics

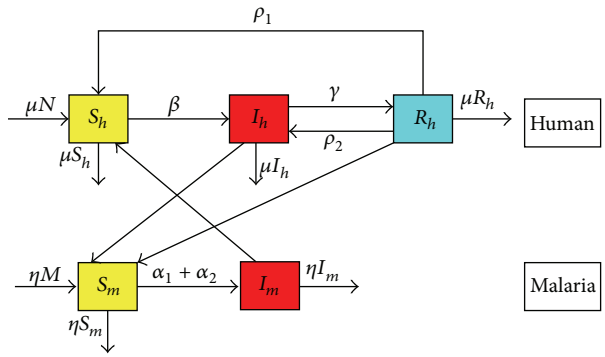

FIGURE 1: Transfer diagram of the model (1).

by using the geometric singular perturbation theory. They find that a treatment should be given to symptomatic patients completely and adequately rather than asymptomatic infection. On the other hand, for the asymptomatic patients, their results strongly suggest that to control and eradicate the malaria, it is very necessary for the government to control the relapse rate strictly. Nadjm and Behrens [14] state that relapse is when symptoms reappear after the parasites had been eliminated from blood but persist as dormant hypnozoites in liver cells. This commonly occurs between 8–24 weeks and is commonly seen with *P. vivax* and *P. ovale* infections. Other papers also consider the inluence of relapse in giving up smoking or quitting drinking, please see [15, 16] and references cited therein.

Chitnis et al. [7] assume that the recovered humans have some immunity to the disease and do not get clinically ill, but they still harbor low levels of parasite in their blood streams and can pass the infection to mosquitoes. After some period of time, they lose their immunity and return to the susceptible class. Unfortunately, they do not consider that the recovered humans will return to their infectious state because of incomplete treatment. Li et al. [13] consider the relapse but not that the recovered humans may return to the susceptible class.

Motivated by these works, in this paper, we propose a more realistic mathematical model of malaria, in which we assume that the recovered humans return to the susceptible class and relapse. The basic reproductive number R_0 is calculated and the persistence theory is used to analyze the uniformly persistence of the system.

The organization of this paper is as follows. In the next section, a mathematical model of malaria with relapse is formulated. In Section 3, the basic reproduction number and the stability of disease-free equilibria are investigated. The existence of endemic equilibrium and uniformly persistence are proved in Section 4, and some numerical simulations are given in Section 5. In the last section, we give some brief discussions.

2. The Model

2.1. System Description. In this section, we introduce a mathematical model of malaria with relapse. Because hosts might get repeatedly infected due to not acquiring complete immunity so the population is assumed to be described by the SIRS model. Mosquitoes are assumed not to recover from the parasites so the mosquito population can be described by the SI model. The total number of population at time t is given by $N = S_h(t) + I_h(t) + R_h(t)$ and $M = S_m(t) + I_m(t)$. The structure of model is shown in Figure 1. The transfer diagram leads to the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dS_h(t)}{dt} &= \mu N - \frac{\beta S_h I_m}{N} + \rho_1 R_h - \mu S_h,\\
\frac{dI_h(t)}{dt} &= \frac{\beta S_h I_m}{N} + \rho_2 R_h - (\gamma + \mu) I_h,\\
\frac{dR_h(t)}{dt} &= \gamma I_h - (\rho_1 + \rho_2 + \mu) R_h,\\
\frac{dS_m(t)}{dt} &= \eta M - \frac{\alpha_1 S_m I_h}{N} - \frac{\alpha_2 S_m R_h}{N} - \eta S_m,\\
\frac{dI_m(t)}{dt} &= \frac{\alpha_1 S_m I_h}{N} + \frac{\alpha_2 S_m R_h}{N} - \eta I_m,
\end{aligned} \tag{1}$$

where S_h, I_h, R_h, S_m, I_m, N, and M represent the number of susceptible humans, infectious humans, recovered humans, susceptible mosquitoes, infectious mosquitoes, the total size of the human population, and the total size of the mosquitoes population, respectively. μ is the natural birth and death rate of humans, η is the natural birth and death rate of mosquitoes, β is from an infectious mosquito to a susceptible human transmission rate in humans, α_1 and α_2 represent both infectious and recovered human to a susceptible mosquito transmission rate in mosquitoes, γ is treatment rate, ρ_1 is recovery rate (individuals from recovered class could back to susceptible class again because they had a very small amount of parasites, which would be cleared quickly by their own immune system), ρ_2 is relapse rate, and q is the number of mosquitoes per individual. All the parameters can be found in Table 1. In the model, N and M are constant, so we introduce the new variables in terms of proportion as follows:

$$\begin{aligned}
s_h &= \frac{S_h}{N}, \qquad x_1 = \frac{I_h}{N}, \qquad x_2 = \frac{R_h}{N},\\
s_m &= \frac{S_m}{M}, \qquad y = \frac{I_m}{M},
\end{aligned} \tag{2}$$

Table 1: The parameters description of malaria model.

α_1	From an infectious human to a susceptible mosquito, transmission rate in mosquitoes
α_2	From a recovered human to a susceptible mosquito, transmission rate in mosquitoes.
β	From an infectious mosquito to a susceptible human, transmission rate in humans
N	The total size of human population
M	The total size of mosquito population
μ	Natural birth and death rate of humans
γ	Treatment rate
ρ_1	Recovery rate
ρ_2	Relapse rate
η	Natural birth and death rate of mosquitoes
q	The number of mosquitoes per individual

with $s_h + x_1 + x_2 = 1$, $s_m + y = 1$. Then the system (1) becomes

$$\begin{aligned} \frac{dx_1(t)}{dt} &= q\beta(1 - x_1 - x_2)y + \rho_2 x_2 - (\gamma + \mu)x_1, \\ \frac{dx_2(t)}{dt} &= \gamma x_1 - (\rho_1 + \rho_2 + \mu)x_2, \\ \frac{dy(t)}{dt} &= \alpha_1(1 - y)x_1 + \alpha_2(1 - y)x_2 - \eta y. \end{aligned} \tag{3}$$

2.2. Basic Properties

2.2.1. Invariant Region. Notice that from (1) we have

$$\frac{dN(t)}{dt} = 0, \qquad \frac{dM(t)}{dt} = 0. \tag{4}$$

Thus, the total human population N and mosquitoes' population M are constant. Since the system (3) monitor human population, it is plausible to assume that all its state variables and parameters are nonnegative for all $t \geq 0$. Further, it can be shown that the region

$$\begin{aligned} \Omega = \{&(x_1(t), x_2(t), y(t)) \in R_+^3 : \\ &x_1(t) + x_2(t) \leq 1\} \end{aligned} \tag{5}$$

is positively-invariant. Thus, each solution of the system (3), with initial conditions in Ω, remains there for $t \geq 0$. Therefore, the ω-limit sets of solutions of the system (3), are contained in Ω. Furthermore, in Ω, the usual existence, uniqueness, and continuation results hold for the system, so that the system (3), is well-posed mathematically and epidemiologically. So we consider dynamics of system (3) on the set Ω in this paper.

2.2.2. Positivity of Solutions. For system (3), to ensure the solutions of the system with positive initial conditions remain positive for all $t > 0$, it is necessary to prove that all the state variables are nonnegative, so we have the following lemma.

Lemma 1. *If $x_1(0) > 0, x_2(0) > 0, y(0) > 0$, the solutions $x_1(t)$, $x_2(t)$, and $y(t)$ of system (3) are positive for all $t \geq 0$.*

Proof. Under the given initial conditions, it is easy to prove that the solutions of the system (3) are positive; if not, we assume a contradiction: that there exists a first time t_1 such that

$$\begin{aligned} &x_1(t_1) = 0, \qquad x_1'(t_1) \leq 0, \qquad x_2(t_1) \geq 0, \\ &y(t_1) \geq 0, \qquad x_2(t_1) + y(t_1) > 0, \\ &x_2(t) > 0, \qquad y(t) > 0, \\ &t \in (0, t_1); \end{aligned} \tag{6}$$

there exists a t_2, such that

$$\begin{aligned} &x_2(t_2) = 0, \qquad x_2'(t_2) \leq 0, \qquad x_1(t_2) \geq 0, \\ &y(t_2) \geq 0, \qquad x_1(t_2) + y(t_2) > 0, \\ &x_1(t) > 0, \qquad y(t) > 0, \\ &t \in (0, t_2); \end{aligned} \tag{7}$$

there exists a t_3, such that

$$\begin{aligned} &y(t_3) = 0, \qquad y'(t_3) \leq 0, \qquad x_1(t_3) \geq 0, \\ &x_2(t_3) \geq 0, \qquad x_1(t_3) + x_2(t_3) > 0, \\ &x_1(t) > 0, \qquad x_2(t) > 0, \\ &t \in (0, t_3). \end{aligned} \tag{8}$$

In the first case, we have

$$x_1'(t_1) = q\beta(1 - x_2)y + \rho_2 x_2 > 0, \tag{9}$$

which is a contradiction meaning that $x_1(t) > 0$, $t \geq 0$.

In the second case, we have

$$x_2'(t_2) = \gamma x_1 > 0, \tag{10}$$

which is a contradiction meaning that $x_2(t) > 0, t \geq 0$.

In the third case, we have

$$y'(t_3) = \alpha_1 x_1 + \alpha_2 x_2 > 0, \tag{11}$$

which is a contradiction meaning that $y(t) > 0$, $t \geq 0$. Thus, the solutions $x_1(t)$, $x_2(t)$, and $y(t)$ of system (3) remain positive for all $t > 0$. □

3. Analysis of the Model

The model (3) has one disease-free equilibrium E_0 and one endemic equilibrium E^*.

3.1. Disease-Free Equilibrium and the Basic Reproduction Number. The model has a disease-free equilibrium given by

$$E_0 = (0, 0, 0). \tag{12}$$

In the following, the basic reproduction number of system (3) will be obtained by the next generation matrix method formulated in [17].

Let $X = (x_1, x_2, y)^T$, then system (3) can be written as

$$\frac{dX}{dt} = \mathscr{F}(X) - \mathscr{V}(X), \tag{13}$$

where

$$\mathscr{F}(X) = \begin{pmatrix} q\beta(1 - x_1 - x_2)\, y \\ 0 \\ \alpha_1(1 - y)\, x_1 + \alpha_2(1 - y)\, x_2 \end{pmatrix},$$
$$\mathscr{V}(X) = \begin{pmatrix} -\rho_2 x_2 + (\gamma + \mu)\, x_1 \\ -\gamma x_1 + (\rho_1 + \rho_2 + \mu)\, x_2 \\ \eta y \end{pmatrix}. \tag{14}$$

The Jacobian matrices of $\mathscr{F}(X)$ and $\mathscr{V}(X)$ at the disease-free equilibrium E_0 are, respectively,

$$D\mathscr{F}(E_0) = \begin{pmatrix} 0 & 0 & q\beta \\ 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 \end{pmatrix},$$
$$D\mathscr{V}(E_0) = \begin{pmatrix} \gamma + \mu & -\rho_2 & 0 \\ -\gamma & \rho_1 + \rho_2 + \mu & 0 \\ 0 & 0 & -\eta \end{pmatrix}. \tag{15}$$

The model reproduction number denoted by R_0 is thus given by $R_0 = \sqrt{q\beta[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma]/\eta[(\mu + \gamma)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2]}$. Here R_0 is associated with disease transmission by infected humans as well as the infection of susceptible humans by infected mosquitoes. Susceptible mosquitoes acquire malaria infection from infected humans in two ways, namely, by infected or recoveries. Susceptible humans acquire infection following effective contacts with infected mosquitoes.

3.2. Global Stability of E_0

Theorem 2. *For system* (3)*, the disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$.*

Proof. The linearised system (3) at the disease-free equilibrium is given by

$$\frac{dx_1}{dt} = -(\gamma + \mu)\, x_1 + \rho_2 x_2 + q\beta y,$$
$$\frac{dx_2}{dt} = \gamma x_1 - (\rho_1 + \rho_2 + \mu)\, x_2, \tag{16}$$
$$\frac{dy}{dt} = \alpha_1 x_1 + \alpha_2 x_2 - \eta y.$$

Therefore, the characteristic equation is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3\left(1 - R_0^2\right) = 0. \tag{17}$$

with $A_1 = \gamma + 2\mu + \rho_1 + \rho_2 + \eta$, $A_2 = [(\gamma + \mu)(\rho_1 + \rho_2 + \mu)(1 + \eta) - q\beta\alpha_1]$, and $A_3 = [(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2]\eta$. We use the Routh-Hurwitz criterion [18] to prove that when $R_0 < 1$, all roots of (17) have negative real part. From (17), we see that $H_1 = \gamma + 2\mu + \rho_1 + \rho_2 + \eta > 0$ and

$$\begin{aligned} H_2 &= A_1A_2 - A_3\left(1 - R_0^2\right) \\ &= (\gamma + 2\mu + \rho_1 + \rho_2 + \eta) \\ &\quad \times [(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2](1 + \eta) \\ &\quad - q\beta\alpha_1 - \eta(\gamma + \mu)(\rho_1 + \rho_2 + \mu) \\ &\quad + \eta\gamma\rho_2 + q\beta[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma]. \end{aligned} \tag{18}$$

For ease of notation, we introduce $B_1 = \gamma + \mu$ and $B_2 = \rho_1 + \rho_2 + \mu$, so that

$$\begin{aligned} H_2 &= (B_1 + B_2)(B_1B_2 - \gamma\rho_2) + \eta(B_1 + B_2 + \eta) \\ &\quad \times \left[1 - \frac{(B_1 + \eta)\, q\beta\alpha_1}{\eta(B_1 + B_2)(B_1 + B_2 + \eta)}\right] + q\beta\alpha_2\gamma \\ &\geq (B_1 + B_2)(B_1B_2 - \gamma\rho_2) + \eta(B_1 + B_2 + \eta) \\ &\quad \times \left[1 - \frac{q\beta\alpha_1}{\eta(B_1 + B_2)}\right] + q\beta\alpha_2\gamma \\ &\geq (B_1 + B_2)(B_1B_2 - \gamma\rho_2) + \eta(B_1 + B_2 + \eta) \\ &\quad \times \left[1 - \frac{q\beta\alpha_1B_2 + q\beta\alpha_2\gamma}{\eta(B_1B_2 + B_2^2)}\right] + q\beta\alpha_2\gamma \\ &\geq (B_1 + B_2)(B_1B_2 - \gamma\rho_2) + \eta(B_1 + B_2 + \eta) \\ &\quad \times \left[1 - \frac{q\beta\alpha_1B_2 + q\beta\alpha_2\gamma}{\eta(B_1B_2 - \gamma\rho_2)}\right] + q\beta\alpha_2\gamma \\ &= (B_1 + B_2)(B_1B_2 - \gamma\rho_2) + \eta(B_1 + B_2 + \eta) \\ &\quad \times \left[1 - R_0^2\right] + q\beta\alpha_2\gamma. \end{aligned} \tag{19}$$

Thus, for $R_0 < 1$, $H_2 > 0$. Lastly, $H_3 = H_2A_3(1 - R_0^2)$. Thus, for $R_0 < 1$, all roots of (17) have negative real parts. The disease-free equilibrium point E_0, is locally asymptotically stable if $R_0 < 1$. □

In the following, we prove that when $R_0 \leq 1$, E_0 is globally asymptotically stable in Ω.

Theorem 3. *For system* (3)*, the disease-free equilibrium E_0 is globally asymptotically stable if $R_0 \leq 1$.*

Proof. We introduce the following Lyapunov function [19, 20]:

$$V = ax_1 + bx_2 + cy, \tag{20}$$

where $a = \alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma$, $b = \alpha_2(\gamma + \mu) + \alpha_1\rho_2$, and $c = (\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2$. It is easy to see that a, b, and c are all positive. The derivative of V is given by

$$\begin{aligned}\dot{V} &= a\dot{x}_1 + b\dot{x}_2 + c\dot{y}\\ &= a\left[q\beta\left(1 - x_1 - x_2\right)y + \rho_2 x_2 - (\gamma + \mu)x_1\right]\\ &\quad + b\left[\gamma x_1 - (\rho_1 + \rho_2 + \mu)x_2\right]\\ &\quad + c\left[\alpha_1(1 - y)x_1 + \alpha_2(1 - y)x_2 - \eta y\right]\\ &= (aq\beta - c\eta)y - \left[a(\gamma + \mu) - b\gamma - c\alpha_1\right]x_1\\ &\quad + \left[a\rho_2 - b(\rho_1 + \rho_2 + \mu) + c\alpha_2\right]x_2\\ &\quad - (aq\beta + c\alpha_1)x_1 y - (aq\beta + c\alpha_2)x_2 y\\ &= -(aq\beta + c\alpha_1)x_1 y - (aq\beta + c\alpha_2)x_2 y\\ &\quad + \frac{\left(R_0^2 - 1\right)y}{\left[(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2\right]\eta}.\end{aligned} \tag{21}$$

If $R_0 \le 1$, then $(R_0^2 - 1)/[(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2]\eta \le 0$. As we know that $-(aq\beta + c\alpha_1) < 0$ and $-(aq\beta + c\alpha_2) < 0$, so we obtain $\dot{V} \le 0$. Furthermore, $\dot{V} = 0$ only if $y = 0$ or $R_0 = 1$. The maximum invariant set in $\{(x_1, x_2, y) : \dot{V} = 0\}$ is the singleton E_0. By LaSalle's Invariance Principle [21], E_0 is globally asymptotically stable in Ω. □

3.3. Endemic Equilibrium

3.3.1. Existence of the Endemic Equilibrium

Theorem 4. *If $R_0 > 1$, system (3) has a unique endemic equilibrium $E^* = (x_1^*, x_2^*, y^*)$, where*

$$\begin{aligned}x_1^* &= \Big(\left(\rho_1 + \rho_2 + \mu\right)\left[(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2\right]\\ &\qquad \times\eta\left(R_0^2 - 1\right)\Big)\\ &\quad \times\left(\left[(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2\right]\right.\\ &\qquad \times\left[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma\right]\\ &\qquad \left. + (\rho_1 + \rho_2 + \mu + \gamma)\right)^{-1},\\ x_2^* &= \frac{\gamma}{\rho_1 + \rho_2 + \mu}x_1,\\ y^* &= \frac{\left[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma\right]x_1}{\left[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma\right]x_1 + \eta(\rho_1 + \rho_2 + \mu)}.\end{aligned} \tag{22}$$

Proof. It follows from system (3) that

$$\begin{aligned}q\beta\left(1 - x_1^* - x_2^*\right)y^* + \rho_2 x_2^* - (\gamma + \mu)x_1^* &= 0,\\ \gamma x_1^* - (\rho_1 + \rho_2 + \mu)x_2^* &= 0,\\ \alpha_1(1 - y^*)x_1^* + \alpha_2(1 - y^*)x_2^* - \eta y &= 0.\end{aligned} \tag{23}$$

From the second equation of (23), we obtain

$$x_2^* = \frac{\gamma}{\rho_1 + \rho_2 + \mu}x_1^*. \tag{24}$$

Substituting x_2^* into the third equation of (23), we have

$$y^* = \frac{\left[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma\right]x_1^*}{\left[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma\right]x_1^* + \eta(\rho_1 + \rho_2 + \mu)}. \tag{25}$$

Then substituting (24) and (25) into first equation of (23), we get

$$\begin{aligned}x_1^* &= \Big(\left(\rho_1 + \rho_2 + \mu\right)\left[(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2\right]\\ &\qquad \times\eta\left(R_0^2 - 1\right)\Big)\\ &\quad \times\left(\left[(\gamma + \mu)(\rho_1 + \rho_2 + \mu) - \gamma\rho_2\right]\right.\\ &\qquad \times\left[\alpha_1(\rho_1 + \rho_2 + \mu) + \alpha_2\gamma\right]\\ &\qquad \left. + (\rho_1 + \rho_2 + \mu + \gamma)\right)^{-1}.\end{aligned} \tag{26}$$

Hence, if $R_0 \le 1$, there is no positive root of (26), while if $R_0 > 1$ there is one positive root. □

3.3.2. Uniform Persistence of the Disease. We using the persistence theory of dynamical system to show the uniform persistence of the disease when $R_0 > 1$. Let E be a closed positively invariant subset of Ω, on which a continuous flow $\mathcal{F}$ is defined. We denote the restriction $\mathcal{F}$ to ∂E by $\partial\mathcal{F}$ and note that ∂E is in general not positively invariant. Let N be the maximal invariant set of $\partial\mathcal{F}$ on ∂E. Suppose N is a closed invariant set and there exists a cover $\{N_\alpha\}_{\alpha\in A}$ of N, where A is a nonempty index set. $N_\alpha \subset \partial E$, $N \subset \bigcup_{\alpha\in A} N_\alpha$, and $\{N_\alpha\}(\alpha \in A)$ are pairwise disjoint closed invariant sets. Furthermore, we propose the following hypothesis and Lemma. (H_1) All N_α are isolated invariant sets of the flow $\mathcal{F}$. (H_2), $\{N_\alpha\}_{\alpha\in A}$ is acyclic; that is, any finite subset of $\{N_\alpha\}_{\alpha\in A}$ does not form a cycle. (H_3) Any compact subset of ∂E contains, at most, finitely many sets of $\{N_\alpha\}_{\alpha\in A}$ [22].

Lemma 5 (see [22, Theorem 4.3]). *Let E be a closed positively invariant subset of Ω on which a continuous flow $\mathcal{F}$ is defined. Suppose there is a constant $\varepsilon > 0$ such that $\mathcal{F}$ is point dissipative on $S[\partial E, \varepsilon] \bigcap E^0$ and the assumption $(H_1$–$H_3)$ holds. Then the flow $\mathcal{F}$ is uniformly persistent, if and only if $W^+(N_\alpha) \bigcap S[\partial E, \alpha] \bigcap E^0 = \phi$. For any $\alpha \in A$, where $W^+(N_\alpha) = \{y \in \Omega, \omega(y) \subset N_\alpha\}$, $S[\partial E, \varepsilon] = \{x : x \in \Omega, d(x, \partial E) \le \varepsilon\}$, and E^0 is interior of set E.*

By this lemma, we can show the uniform persistence of disease when $R_0 > 1$, and similar to the proof of Theorem 2.3 in [13], we have the following.

Theorem 6. *In system (3), assume that $R_0 > 1$, and the disease is initially present, then the disease is uniformly persistent; that is, there is a constant $k > 0$ such that $\liminf_{t\to+\infty} x_1(t) \ge k$, $\liminf_{t\to+\infty} x_2(t) \ge k$, and $\liminf_{t\to+\infty} y(t) \ge k$.*

Proof. We set $E = \{(x_1, x_2, y) \in \mathbf{R}_+^3 \mid 0 \le x_1 + x_2 \le 1, 0 \le y \le 1\}, \partial E = \{(x_1, x_2, y) \in E \mid x_1 = 0\}$; we will prove below that the conditions of Lemma 5 are satisfied. Clearly $N_\alpha = E_0 = (0, 0, 0)$ is isolated. Hence, the covering is simply $N = E_0$, which is acyclic. Thus, the condition $(H_1$–$H_3)$ holds.

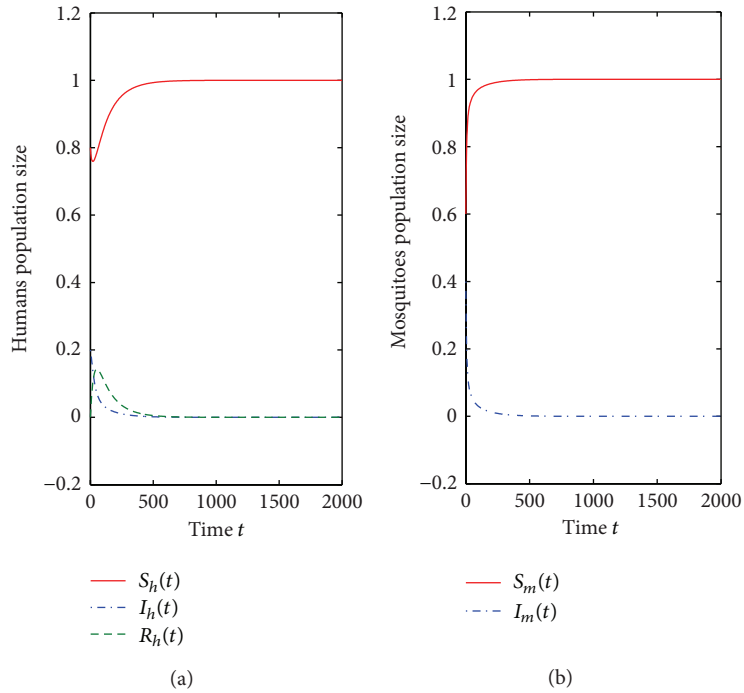

FIGURE 2: $R_0 < 1$, the disease-free equilibrium, E_0, is globally asymptotically stable.

We also can obtain $\mathcal{F}$ is point dissipative by Lemma 1. Now we show that $W^+(E_0) \bigcap E^0 = \phi$; suppose this is not true, then there exists a solution $(x_1(t), x_2(t), y(t)) \in E^0$ such that: $\lim_{t\to+\infty} x_1(t) = 0, \lim_{t\to+\infty} x_2(t) = 0, \lim_{t\to+\infty} y(t) = 0$. For any sufficiently small constant $\varepsilon > 0$, there exists a positive constant $T = T(\varepsilon)$ such that $x_1(t) < \varepsilon, x_2(t) < \varepsilon, y(t) < \varepsilon$, for all $t \geq T$.

Noting that

$$\begin{aligned} \frac{dx_1}{dt} &\geq -(\gamma+\mu)x_1 + \rho_2 x_2 + q\beta y, \\ \frac{dx_2}{dt} &= \gamma x_1 - (\rho_+\rho_2 + \mu)x_2, \\ \frac{dy}{dt} &\geq \alpha_1 x_1 + \alpha_2 x_2 - \eta y. \end{aligned} \tag{27}$$

Therefore, if $x_1, x_2, y \to 0$, as $t \to \infty$, then by a standard comparison argument and the nonnegativity, the solution x_1, x_2, y of

$$\begin{aligned} \frac{dx_1}{dt} &= -(\gamma+\mu)x_1 + \rho_2 x_2 + q\beta y, \\ \frac{dx_2}{dt} &= \gamma x_1 - (\rho_+\rho_2 + \mu)x_2, \\ \frac{dy}{dt} &= \alpha_1 x_1 + \alpha_2 x_2 - \eta y, \end{aligned} \tag{28}$$

with initial data $x_1(T) = x_1(T), x_2(T) = x_2(T), y(T) = y(T)$, converges to $(0,0,0)$ as well. Thus $\lim W(t) = 0$, where $W(t) > 0$, is defined by

$$\begin{aligned} \frac{dW}{dt} &= k_1\left[-(\gamma+\mu)x_1 + \rho_2 x_2 + q\beta y\right] \\ &\quad + k_2\left[\gamma x_1 - (\rho_+\rho_2 + \mu)x_2\right] \\ &\quad + k_3\left[\alpha_1 x_1 + \alpha_2 x_2 - \eta y\right]. \end{aligned} \tag{29}$$

Here, $k_1 = \alpha_1(\rho_1+\rho_2+\mu) + \alpha_2\gamma$, $k_2 = \alpha_2(\gamma+\mu) + \rho_2\rho$, $k_3 = (\gamma+\mu)(\rho_1+\rho_2+\mu) - \gamma\rho_2$. The derivative of $W(t)$ is given by

$$\frac{dW}{dt} = \left[(\gamma+\mu)(\rho_1+\rho_2+\mu) - \gamma\rho_2\right]\left(R_0^2 - 1\right)y \geq 0. \tag{30}$$

Therefore, $W(t)$ goes to either infinity or some positive number as $t \to \infty$, which is a contradiction to $\lim_{t\to+\infty} W(t) = 0$. Thus, we have $W^+(E_0) \bigcap E^0 = \phi$. Then, we obtain $\liminf_{t\to+\infty} x_1(t) \geq k_1$, for some constant $k_1 > 0$. By the second and third equations of (3) and the use of Lemma 1, we have $k_2 = \gamma k_1/(\rho_1+\rho_2+\mu)$, $k_3 = (\alpha_1 k_1 + \alpha_2 k_2)/\eta$, such that $\liminf_{t\to+\infty} x_2(t) \geq k_2, \liminf_{t\to+\infty} y(t) \geq k_3$. Denote $k = \min\{k_1, k_2, k_3\}$, $\liminf_{t\to+\infty} x_1(t) \geq k$, $\liminf_{t\to+\infty} x_2(t) \geq k$, $\liminf_{t\to+\infty} y(t) \geq k$. Then the proof of Theorem 6 is completed. □

4. Numerical Simulation

To illustrate the analytical results obtained above, we give some simulations using the parameter values in Table 2. Numerical results are displayed in Figures 2–5. First, we

Table 2: The parameters values of malaria model.

α_1	From an infectious human to a susceptible mosquito, transmission rate in mosquitoes	0.8333 (day^{-1})	[7]
α_2	From a recovered human to a susceptible mosquito, transmission rate in mosquitoes	$8.333 * 10^{-2}$ (day^{-1})	[7]
β	From an infectious mosquito to a susceptible human, transmission rate in humans	$2.000 * 10^{-2}$ (day^{-1})	[7]
N	The total size of human population	Estimated	
M	The total size of mosquito population	qN	[9]
μ	Natural birth and death rate of humans	1/70 (year^{-1})	[9]
γ	Treatment rate	$3.704 * 10^{-3}$ (day^{-1})	[7]
ρ_1	Recovery rate	Estimated	
ρ_2	Relapse rate	Estimated	
η	Natural birth and death rate of mosquitoes	0.1429 (day^{-1})	[7]
q	The number of mosquitoes per individual	1-2	[9]

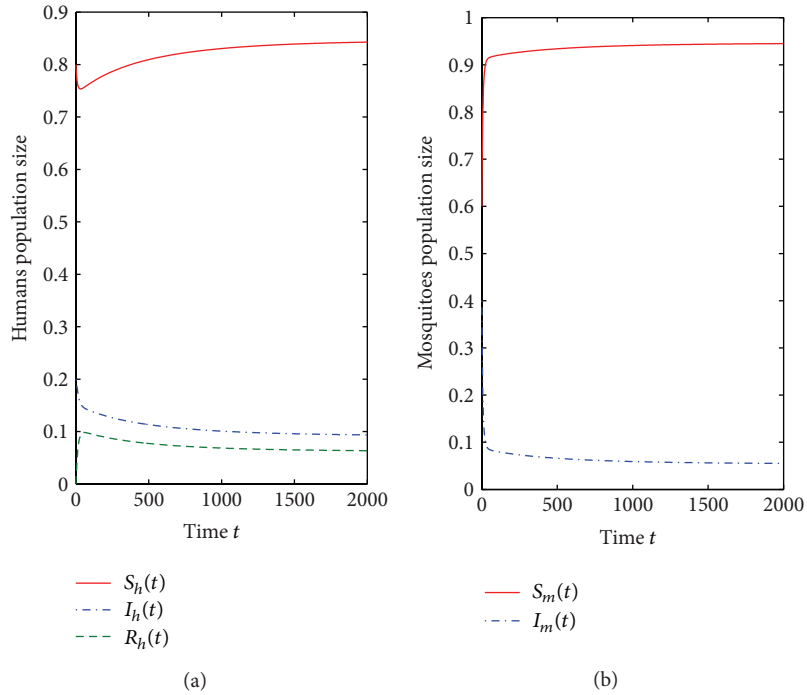

Figure 3: $R_0 > 1$, the disease is uniformly persistent.

choose $\rho_2 = 0.004$, $\rho_1 = 0.0146$, and $q = 1.5$, numerical simulation gives $R_0 = 0.6940 < 1$, then the disease-free equilibrium E_0 is globally asymptotically stable (Figure 2). Second, we choose $\rho_2 = 0.04$, $\rho_1 = 0.0146$, and $q = 1.5$, numerical simulation gives $R_0 = 1.1254 > 1$, the disease is uniformly persistent (Figure 3).

Finally, for showing the effect of relapse and recover rate to the basic reproduction number, we give the relation between R_0 and ρ_2 (Figure 4), and the relation between R_0 and ρ_1 (Figure 5) in the numerical simulation. From Figures 4 and 5, we know that R_0 is increasing with respect to the relapse rate, while it is decreasing with respect to the recovery rate.

5. Discussion

An ordinary differential equation for the transmission of malaria is formulated in this paper. The model exhibits two equilibria, that is, the disease-free equilibrium and endemic equilibrium. By constructing Lyapunov function and persistence theory of dynamical system, it is shown that if $R_0 \leq 1$, then the disease-free equilibrium point E_0 is globally stable, and if $R_0 > 1$, the disease is uniformly persistent. Some numerical simulations for R_0 in terms of relapse rate and recover rate are performed. R_0 is increasing with respect to the relapse rate while it is decreasing with respect to the recovery rate.

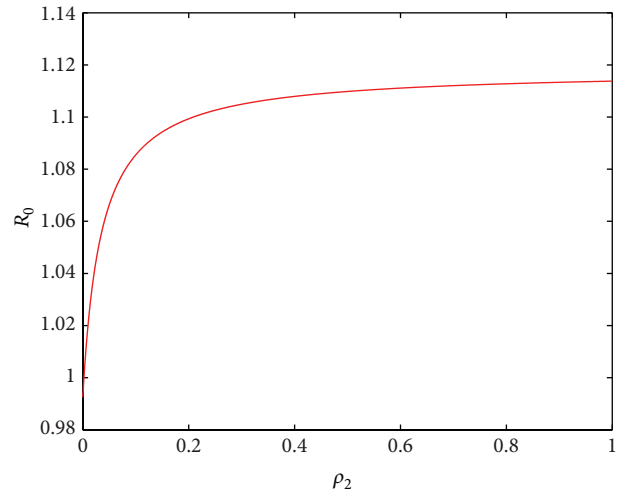

Figure 4: The relationship between R_0 and ρ_2.

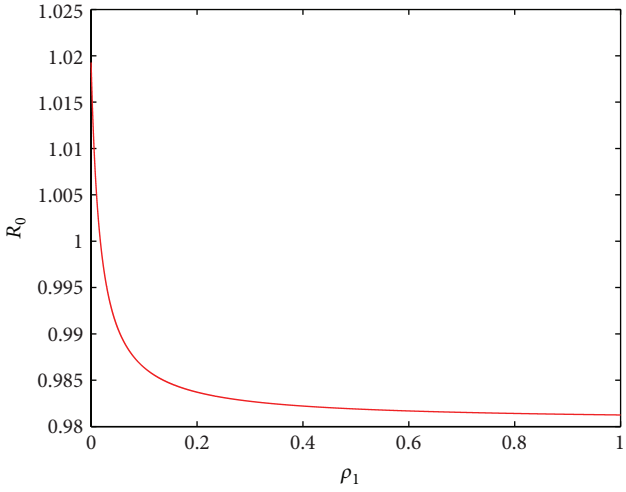

Figure 5: The relationship between R_0 and ρ_1.

Our results strongly suggest that to control and eradicate the malaria, it is very necessary for the government to decrease the relapse rate and increase the recovery rate.

Conflict of Interests

On behalf of all the authors, Hai-Feng Huo declares that there is no conflict of interests regarding the publication of this paper.

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