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Research Article

The Existence of Solutions to the Nonhomogeneous A-Harmonic Equations with Variable Exponent

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We first discuss the existence and uniqueness of weak solution for the obstacle problem of the nonhomogeneous A-harmonic equation with variable exponent, and then we obtain the existence of the solutions of the equation $d^\star A(x,d\omega) = B(x,d\omega)$ in the weighted variable exponent Sobolev space $W_d^{p(x)}(\Omega,\Lambda^l,\mu)$.

1. Introduction

In [1–5], the nonhomogeneous A-harmonic equation $d^\star A(x,d\omega) = B(x,d\omega)$ for differential forms has received much investigation. In [6], the obstacle problem of the A-harmonic equation for differential forms has been discussed. However, most of these results are developed in the $L^p(\Omega,\Lambda^l)$ space or $W^{1,p}(\Omega,\Lambda^l)$ space. Meanwhile, in the past few years the subject of variable exponent space has undergone a vast development; see [7–11]. For example, [8–10] discuss the weighted $L^{p(x)}$ and $W^{k,p(x)}$ spaces and the weak solution for obstacle problem with variable growth has been studied in [10, 11].

In this paper, we are interested in the following obstacle problem:

$$\int_\Omega (A(x,du)\cdot d(v-u) + B(x,du)\cdot(v-u))\,dx \geq 0 \quad (1)$$

for v belonging to

$$\mathfrak{K}_{\psi,\theta} = \Big\{v \in W_d^{p(x)}\left(\Omega,\Lambda^l,\mu\right) : v \geq \psi, \text{ a.e. } x\in\Omega, v-\theta \in W_{0d}^{p(x)}\left(\Omega,\Lambda^l,\mu\right)\Big\}, \quad (2)$$

where $\psi(x) = \sum \psi_I(x)dx_I \in \Lambda^l(\mathbb{R}^n)$, $v(x) = \sum v_I(x)dx_I \in \Lambda^l(\mathbb{R}^n)$, v_I, $\psi_I : \Omega \to [-\infty,+\infty]$; $v \geq \psi$, a.e. $x\in\Omega$ means that, for any I, we have $v_I \geq \psi_I$, a.e. $x\in\Omega$; $\theta \in W_d^{p(x)}(\Omega,\Lambda^l,\mu)$, $l = 0,1,\ldots,n-1$, and the variable exponent $p(x)\in\mathscr{P}(\Omega)$ satisfies

$$1 < p^- \leq p(x) \leq p^+ < \infty \quad \text{for a.e. } x\in\Omega. \quad (3)$$

The operators $A(x,\xi) : \Omega\times\Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R})$ and $B(x,\xi) : \Omega\times\Lambda^l(\mathbb{R}^n) \to \Lambda^{l-1}(\mathbb{R})$ satisfy the following growth conditions on a bounded domain Ω:

(H1) $A(x,\xi)$ and $B(x,\xi)$ are measurable for all ξ with respect to x and continuous for a.e. $x\in\Omega$ with respect to ξ,

(H2) $|A(x,\xi)| \leq C_1 w(x)|\xi|^{p(x)-1}$,

(H3) $A(x,\xi)\cdot\xi \geq C_2 w(x)|\xi|^{p(x)}$,

(H4) $|B(x,\xi)| \leq C_3 w(x)|\xi|^{p(x)-1}$,

(H5) $B(x,d\xi)\cdot\xi \geq C_4 w(x)|\xi|^{p(x)}$,

(H6) $(A(x,d\xi)-A(x,d\eta))\cdot(d\xi-d\eta)+(B(x,d\xi)-B(x,d\eta))\cdot(\xi-\eta) \geq 0$ for $\xi\neq\eta$,

where C_1, C_2, C_3, and C_4 are nonnegative constants. $w(x)\in L^1(\Omega)$ nonnegative and $w(x)^{1/(p(x)-1)} \in L^1(\Omega)$. We will discuss the existence and uniqueness of the solution $u\in\mathfrak{K}_{\psi,\theta}$ for the abovementioned obstacle problem.

Now, we introduce the existing results and related definitions.

Throughout this paper, we assume that Ω is a bounded domain in $\mathbb{R}^n$. Let $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ be the set of all l-forms in

$\mathbb{R}^n$. A differential l-form $u(x)$ is generated by $\{dx_{i1} \wedge dx_{i2} \wedge \cdots \wedge dx_{il}\}$, $l = 1, 2, \ldots, n$; that is, $u(x) = \sum_I u_I(x) dx_I = \sum u_{i1,i2,\ldots,il}(x) dx_{i1} \wedge dx_{i2} \wedge \cdots \wedge dx_{il}$, where $u_I(x)$ is differential function, $I = (i_1, i_2, \ldots, i_l)$, and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Let $D'(\Omega, \Lambda^l)$ be the space of all differential l-forms on Ω. For $\alpha(x) = \Sigma \alpha_I(x) dx_I \in \Lambda^l$ and $\beta(x) = \Sigma \beta_I(x) dx_I \in \Lambda^l$, then the inner product is obtained by $\alpha \cdot \beta = \star(\alpha \wedge \star\beta) = \sum_I \alpha_I(x)\beta_I(x)$. We write $|u| = (u \cdot u)^{1/2} = (\sum_I |u_I(x)|^2)^{1/2}$. We denote the exterior derivative by $du = \sum_{i=1}^n \sum_I (\partial u_I(x)/\partial x_i) dx_I \wedge dx_i : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n-1$. Its formal adjoint operator $d^\star : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$, $l = 0, 1, 2, \ldots, n-1$; here $\star$ is the well-known Hodge star operator. Denote the class of infinitely differential l-forms on Ω by $C^\infty(\Omega, \Lambda^l)$. A differential l-form $u \in D'(\Omega, \Lambda^l)$ is called a closed form if $du = 0$ in Ω.

Next we will introduce some basic properties of weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega, \mu)$ and weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega, \mu)$, and we define $\mathscr{P}(\Omega)$ to be the set of all n-dimensioned Lebesgue measurable functions $p : \Omega \to [1, \infty]$. Functions $p \in \mathscr{P}(\Omega)$ are called variable exponents on Ω. We define $p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$, $p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$. If $p^+ < \infty$, then we call p a bounded variable exponent. If $p \in \mathscr{P}(\Omega)$, then we define $p' \in \mathscr{P}(\Omega)$ by $(1/p(x)) + (1/p'(x)) = 1$, where $1/\infty := 0$. The function p' is called the dual variable exponent of p. We denote w as a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e.; also in general $d\mu = w dx$. From [7, 10], we know that if $p \in \mathscr{P}(\Omega)$ satisfies (3), the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega, \mu) = \{f : \int_\Omega |\lambda f(x)|^{p(x)} d\mu < \infty, \lambda > 0\}$ with the norm $\|f\|_{L^{p(x)}(\Omega,\mu)} = \inf\{\lambda > 0 : \int_\Omega |f(x)/\lambda|^{p(x)} d\mu \leq 1\}$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega, \mu) = \{f \in L^{p(x)}(\Omega, \mu) : \nabla f \in L^{p(x)}(\Omega, \mu)\}$ with the norm $\|f\|_{W^{1,p(x)}(\Omega,\mu)} = \|f\|_{L^{p(x)}(\Omega,\mu)} + \|\nabla f\|_{L^{p(x)}(\Omega,\mu)}$ are Banach space and reflexive and uniformly convex. On the set of all differential forms on Ω, we define the weighted variable exponent Lebesgue spaces of differential l-forms $L^{p(x)}(\Omega, \Lambda^l, \mu)$ and the weighted variable exponent Sobolev spaces of differential forms $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$.

Definition 1. We denote the weighted variable exponent Lebesgue spaces of differential l-forms by $L^{p(x)}(\Omega, \Lambda^l, \mu) = \{u = \sum_I u_I(x) dx_I \in \Lambda^l : u_I(x) \in L^{p(x)}(\Omega, \mu)\}$ $l = 0, 1, 2, \ldots, n$ and we endow $L^{p(x)}(\Omega, \Lambda^l, \mu)$ with the following norm:

$$\|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} d\mu \leq 1\right\}. \quad (4)$$

And the spaces $W_d^{p(x)}(\Omega, \Lambda^l, \mu) = \{u \in L^{p(x)}(\Omega, \Lambda^l, \mu) : du \in L^{p(x)}(\Omega, \Lambda^{l+1}, \mu)\}$ with the norm

$$\|u\|_{W_d^{p(x)}(\Omega,\Lambda^l,\mu)} = \|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)} + \|du\|_{L^{p(x)}(\Omega,\Lambda^{l+1},\mu)} \quad (5)$$

are the weighted variable exponent Sobolev spaces of differential l-forms; $l = 0, 1, 2, \ldots, n-1$. $W_{0d}^{p(x)}(\Omega, \Lambda^l, \mu)$ is the completion of $C_0^\infty(\Omega, \Lambda^l, \mu)$ in $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$. We need the following Hölder inequalities; see [7, 10].

Proposition 2. *Let $p, q \in \mathscr{P}(\Omega)$ be such that $1 = (1/p(x)) + (1/q(x))$ for μ-almost every $x \in \Omega$. Then*

$$\int_\Omega |fg|\, d\mu \leq \int_\Omega \frac{1}{p(x)} |f|^{p(x)} d\mu + \int_\Omega \frac{1}{q(x)} |g|^{q(x)} d\mu, \quad (6)$$

$$\int_\Omega |fg|\, d\mu \leq \left(\left(\frac{1}{p}\right)^+ + \left(\frac{1}{q}\right)^+\right) \|f\|_{L^{p(x)}(\Omega,\mu)} \|g\|_{L^{q(x)}(\Omega,\mu)}, \quad (7)$$

for all $f \in L^{p(x)}(\Omega, \mu)$ and $g \in L^{q(x)}(\Omega, \mu)$.

Lemma 3 (see [7]). *Let $(D, \sum, \mu)$ be a σ-finite, complete measure space; if $f \in L^{p(\cdot)}(D, \mu)$, $g \in L^0(D, \mu)$, and $0 \leq |g| \leq |f|$ μ-almost everywhere, then $g \in L^{p(\cdot)}(D, \mu)$ and $\|g\|_{L^{p(x)}(D,\mu)} \leq \|f\|_{L^{p(x)}(D,\mu)}$.*

By the inequality $(\sum_{i=1}^n (a_i)^2)^{1/2} \leq \sum_{i=1}^n a_i \leq n^{1/2}(\sum_{i=1}^n (a_i)^2)^{1/2}$ for any $a_i \geq 0$, using Lemma 3, we can easily have the following lemma.

Lemma 4. *If $u = \Sigma_I u_I(x) dx_I \in D'(\Omega, \Lambda^l)$ and $|u| = (\sum_I |u_I|^2)^{1/2}$, then $u \in L^{p(x)}(\Omega, \Lambda^l, \mu)$ and $|u| \in L^{p(x)}(\Omega, \mu)$ are equivalent, and $\|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)} = \||u|\|_{L^{p(x)}(\Omega,\mu)}$.*

2. Main Results

In this section, we will obtain the existence and uniqueness of weak solution for obstacle problem of the nonhomogeneous A-harmonic equation in space $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$.

Theorem 5. *Suppose $\mathfrak{K}_{\psi,\theta}$ is not empty, under conditions (H1)–(H6), and there exists a unique solution u to the obstacle problem (1)-(2). That is, there is a differential form u in $\mathfrak{K}_{\psi,\theta}$ such that*

$$\int_\Omega (A(x, du) \cdot d(v - u) + B(x, du) \cdot (v - u))\, dx \geq 0, \quad (8)$$

whenever $v \in \mathfrak{K}_{\psi,\theta}$.

We deduce Theorem 5 from a proposition of Kinderlehrer and Stampacchia.

Proposition 6 (see [12]). *Let K be a nonempty closed convex subset of X and let $\mathfrak{A} : K \to X'$ be monotone, coercive, and weakly continuous on K. Then there exists an element u in K such that $\langle \mathfrak{A}u, v - u\rangle \geq 0$ whenever $v \in K$.*

Now let $X = W_d^{p(x)}(\Omega, \Lambda^l, \mu)$ and $\langle\cdot,\cdot\rangle$ be the usual pairing between X and X', $\langle f, g\rangle = \int_\Omega f \cdot g\, d\mu$, where g is in X and f in $X' = W_d^{p'(x)}(\Omega, \Lambda^l, \mu)$. We will take $\mathfrak{K}_{\psi,\theta}$ as K. We define a mapping $\mathfrak{A} : \mathfrak{K}_{\psi,\theta} \to X'$ by

$$\langle \mathfrak{A}v, u\rangle = \int_\Omega (A(x, dv) \cdot du + B(x, dv) \cdot u)\, dx \quad (9)$$

for $u \in W_d^{p(x)}(\Omega, \Lambda^l, \mu)$.

Lemma 7. *If $p(x)$ satisfies (3), then spaces $L^{p(x)}(\Omega, \Lambda^l, \mu)$ and $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$ are complete and convex.*

Proof. From [7], we know that if p satisfies (3) and $w(x)^{1/(p(x)-1)} \in L^1(\Omega)$, then let $L^{p(x)}(\Omega, \mu)$ be Banach space and uniformly convex. If ω_1 and ω_2 are two l-forms: $\omega_1 = \sum_I a_I dx_I$ and $\omega_2 = \sum_I b_I dx_I$, we can easily have $\omega_1 + \omega_2 = \sum_I (a_I + b_I) dx_I$ and $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$, so we can immediately obtain the convexity of spaces $L^{p(x)}(\Omega, \Lambda^l, \mu)$ and $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$.

Let $u_j = \sum_I u_{jI} dx_I \in L^{p(x)}(\Omega, \Lambda^l, \mu)$ be a Cauchy sequence in $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$. Then for any I, $u_{jI}(x)$ converges in $L^{p(x)}(\Omega, \mu)$. Suppose that $u_{jI}(x) \to u_I(x)$ in $L^{p(x)}(\Omega, \mu)$. Now let $u = \sum_I u_I dx_I \in L^{p(x)}(\Omega, \Lambda^l, \mu)$, we have

$$\left|u_j - u\right| = \left(\sum_I \left|u_{jI} - u_I\right|^2\right)^{1/2} \leq \sum_I \left|u_{jI} - u_I\right|, \tag{10}$$

using Lemmas 3 and 4, and we know the sequence u_j converges to u in $L^{p(x)}(\Omega, \Lambda^l, \mu)$.

For the sequence $\{du_j\}$, we suppose $du_j \to v$ in $L^{p(x)}(\Omega, \Lambda^{l+1}, \mu)$, and then $v \in L^{p(x)}(\Omega, \Lambda^{l+1}, \mu)$. So (u_j, du_j) converges to (u, v) in the normed space $L^{p(x)}(\Omega, \Lambda^l, \mu) \times L^{p(x)}(\Omega, \Lambda^{l+1}, \mu)$. From $du_j - du = \sum_I (du_{jI} - du_I) \wedge dx_I$, we have

$$\begin{aligned} \left|du_j - du\right| &= \left|\sum_I \sum_{i=1}^n \frac{\partial u_{jI} - \partial u_I}{\partial x_i} dx_i \wedge dx_I\right| \\ &\leq \left(\sum_I \sum_{i=1}^n \left|\frac{\partial u_{jI} - \partial u_I}{\partial x_i}\right|^2\right)^{1/2} \\ &\leq \sum_I \sum_{i=1}^n \left|\frac{\partial u_{jI} - \partial u_I}{\partial x_i}\right| \\ &\leq \sum_I n^{1/2} \left(\sum_{i=1}^n \left|\frac{\partial u_{jI} - \partial u_I}{\partial x_i}\right|^2\right)^{1/2} \\ &= n^{1/2} \sum_I \left|\nabla u_{jI} - \nabla u_I\right|. \end{aligned} \tag{11}$$

In view of [10], for any I, we know that $u_{jI} \to u_I$ in $L^{p(x)}(\Omega, \mu)$ and $\nabla u_{jI} \to v_I$ in $L^{p(x)}(\Omega, \mu)$, and then $\nabla u_I = v_I \in L^{p(x)}(\Omega, \mu)$. Using Lemmas 3 and 4, we get the sequence du_j that converges to du in $L^{p(x)}(\Omega, \Lambda^l, \mu)$; it follows that $v = du \in L^{p(x)}(\Omega, \Lambda^{l+1}, \mu)$. So, we prove $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$ is a closed subspace of $L^{p(x)}(\Omega, \Lambda^l, \mu) \times L^{p(x)}(\Omega, \Lambda^{l+1}, \mu)$. This ends the proof of Lemma 7. □

Using Lemma 7, we can immediately obtain the following lemma.

Lemma 8. *$\mathfrak{K}_{\psi,\theta}$ is a closed convex set.*

Lemma 9. *For each $v \in \mathfrak{K}_{\psi,\theta}$, $\mathfrak{A}v \in [W_d^{p(x)}(\Omega, \Lambda^l, \mu)]'$.*

Proof. Using Hölder inequality (7) with $1 = (1/p(x)) + (1/p'(x))$ and (H2), we get

$$\begin{aligned} &\left|\int_\Omega A(x, v(x)) \cdot u(x)\, dx\right| \\ &\quad \leq C_1 \int_\Omega |v(x)|^{p(x)-1} |u(x)|\, w(x)\, dx \\ &\quad \leq C_1 \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \left\| |v(x)|^{p(x)-1} \right\|_{L^{p'(x)}(\Omega,\Lambda^l,\mu)} \|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)} \\ &\quad \leq 2C_1 \|v(x)\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)}^{p(x)-1} \|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)}. \end{aligned} \tag{12}$$

Similarly, for the operator $B(x, \xi)$, we have

$$\begin{aligned} &\left|\int_\Omega B(x, v(x)) \cdot u(x)\, dx\right| \\ &\quad \leq 2C_3 \|v(x)\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)}^{p(x)-1} \|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)}. \end{aligned} \tag{13}$$

Using (12) and (13) we can easily prove

$$\begin{aligned} &|\langle \mathfrak{A}v, u\rangle| \\ &\quad = \left|\int_\Omega (A(x, dv) \cdot du + B(x, dv) \cdot u)\, dx\right| \\ &\quad \leq 2C_1 \|dv\|_{L^{p(x)}(\Omega,\Lambda^{l+1},\mu)}^{p(x)-1} \|du\|_{L^{p(x)}(\Omega,\Lambda^{l+1},\mu)} \\ &\qquad + 2C_3 \|dv\|_{L^{p(x)}(\Omega,\Lambda^{l+1},\mu)}^{p(x)-1} \|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)} \\ &\quad \leq C_5 \|dv\|_{L^{p(x)}(\Omega,\Lambda^{l+1},\mu)}^{p(x)-1} \left(\|du\|_{L^{p(x)}(\Omega,\Lambda^{l+1},\mu)} + \|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)}\right) \\ &\quad \leq C_5 \|v(x)\|_{W_d^{p(x)}(\Omega,\Lambda^l,\mu)}^{p(x)-1} \|u\|_{W_d^{p(x)}(\Omega,\Lambda^l,\mu)}. \end{aligned} \tag{14}$$

So we get $\mathfrak{A}v \in [W_d^{p(x)}(\Omega, \Lambda^l, \mu)]'$, whenever $v \in \mathfrak{K}_{\psi,\theta}$, $u \in W_d^{p(x)}(\Omega, \Lambda^l, \mu)$, and C_1 and C_3 are constants fixed to (H2) and (H4). □

Lemma 10. *$\mathfrak{A}$ is monotone and coercive on $\mathfrak{K}_{\psi,\theta}$.*

Proof. It follows from (H6) that $\mathfrak{A}$ is monotone. To show that $\mathfrak{A}$ is coercive on $\mathfrak{K}_{\psi,\theta}$, fix $\varphi \in \mathfrak{K}_{\psi,\theta}$ and using the conditions (H2)–(H5), (12), (13), and (6), then

$$\begin{aligned}
&\langle \mathfrak{A}u - \mathfrak{A}\varphi, u-\varphi\rangle \\
&\quad = \int_\Omega ((A(x,du) - A(x,d\varphi)) \cdot (du - d\varphi) \\
&\qquad + (B(x,du) - B(x,d\varphi)) \cdot (u-\varphi))\, dx \\
&\quad = \int_\Omega (A(x,du)\cdot du)\, dx + \int_\Omega (A(x,d\varphi)\cdot d\varphi)\, dx \\
&\qquad - \int_\Omega (A(x,du)\cdot d\varphi)\, dx - \int_\Omega (A(x,d\varphi)\cdot du)\, dx \\
&\qquad + \int_\Omega (B(x,du)\cdot u)\, dx + \int_\Omega (B(x,d\varphi)\cdot \varphi)\, dx \\
&\qquad - \int_\Omega (B(x,du)\cdot \varphi)\, dx - \int_\Omega (B(x,d\varphi)\cdot u)\, dx \\
&\quad \geq C_2 \left(\int_\Omega |du|^{p(x)} w(x)\, dx + \int_\Omega |d\varphi|^{p(x)} w(x)\, dx \right) \\
&\qquad - C_1 \int_\Omega |du|^{p(x)-1} |d\varphi|\, w(x)\, dx - 2C_1 \|d\varphi\|^{p(x)-1} \|du\| \\
&\qquad + C_4 \left(\int_\Omega |u|^{p(x)} w(x)\, dx + \int_\Omega |\varphi|^{p(x)} w(x)\, dx \right) \\
&\qquad - C_3 \int_\Omega |du|^{p(x)-1} |\varphi|\, w(x)\, dx - 2C_3 \|d\varphi\|^{p(x)-1} \|u\| \\
&\quad \geq C_2 \int_\Omega |du|^{p(x)} d\mu - C_1 \int_\Omega \frac{1}{p'(x)} \varepsilon |du|^{p(x)} d\mu \\
&\qquad - C_1 \int_\Omega \frac{1}{p(x)} \varepsilon^{1-p(x)} |d\varphi|^{p(x)} d\mu \\
&\qquad - C_3 \int_\Omega \frac{1}{p'(x)} \varepsilon |du|^{p(x)} d\mu \\
&\qquad - C_3 \int_\Omega \frac{1}{p(x)} \varepsilon^{1-p(x)} |\varphi|^{p(x)} d\mu + C_4 \int_\Omega |u|^{p(x)} d\mu \\
&\qquad - 2\max(C_1, C_3) \|d\varphi\|^{p(x)-1} (\|du\| + \|u\|) \\
&\qquad + C(\varphi, p(x), C_2, C_4),
\end{aligned} \tag{15}$$

where $\|\cdot\|$ is the $L^{p(x)}(\Omega, \Lambda^l, \mu)$ norm; taking $\varepsilon = C_2 (p')^- / 2(C_1 + C_3)$, we have

$$\begin{aligned}
&\langle \mathfrak{A}u - \mathfrak{A}\varphi, u-\varphi\rangle \\
&\quad \geq \frac{C_2}{2} \int_\Omega |du|^{p(x)} d\mu + C_4 \int_\Omega |u|^{p(x)} d\mu \\
&\qquad - C_5 \|d\varphi\|^{p(x)-1} (\|du\| + \|u\|) + C_6 \\
&\quad \geq \frac{C_2}{2} \int_\Omega \left(|du - d\varphi|^{p(x)} + |d\varphi|^{p(x)} \right) d\mu \\
&\qquad + C_4 \int_\Omega \left(|u-\varphi|^{p(x)} + |\varphi|^{p(x)} \right) d\mu \\
&\qquad - C_5 \|d\varphi\|^{p(x)-1} (\|du - d\varphi\| + \|d\varphi\| \\
&\qquad\qquad + \|u-\varphi\| + \|\varphi\|) + C_6 \\
&\quad \geq C_7 \left(\int_\Omega |du - d\varphi|^{p(x)} d\mu + \int_\Omega |u-\varphi|^{p(x)} d\mu \right) \\
&\qquad - C_5 \|d\varphi\|^{p(x)-1} (\|du - d\varphi\| + \|u-\varphi\|) + C_8.
\end{aligned} \tag{16}$$

Let $\rho_{p(\cdot)}(t) = \int_\Omega |t|^{p(x)} d\mu$; from [7, pages 24, 73], we know that if the variable exponent $p \in \mathscr{P}(\Omega)$ satisfied $p^+ < \infty$, then

$$\begin{aligned}
&\min\left\{ \left(\varrho_{p(\cdot)}(f)\right)^{1/p^-}, \left(\varrho_{p(\cdot)}(f)\right)^{1/p^+} \right\} \\
&\quad \leq \|f\|_{L^{p(x)}(\omega,\mu)} \leq \max\left\{ \left(\varrho_{p(\cdot)}(f)\right)^{1/p^-}, \left(\varrho_{p(\cdot)}(f)\right)^{1/p^+} \right\}
\end{aligned} \tag{17}$$

holds. Whenever $\|u\|_{W_d^{p(x)}(\Omega,\Lambda^l,\mu)} \to \infty$, we have $\|du - d\varphi\|_{L^{p(x)}(\Omega,\Lambda^{l+1},\mu)} > 1$ or $\|u-\varphi\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)} > 1$, and using (17) we have

$$\begin{aligned}
&\int_\Omega |du - d\varphi|^{p(x)} d\mu + \int_\Omega |u-\varphi|^{p(x)} d\mu \\
&\quad \geq \max\left\{ \|du - d\varphi\|^{p^-}, \|du - d\varphi\|^{p^+} \right\} \\
&\qquad + \max\left\{ \|u-\varphi\|^{p^-}, \|u-\varphi\|^{p^+} \right\}.
\end{aligned} \tag{18}$$

Substituting (18) in (16) we obtain

$$\begin{aligned}
&\langle \mathfrak{A}u - \mathfrak{A}\varphi, u-\varphi\rangle \\
&\quad \geq C_7 \left(\int_\Omega |du - d\varphi|^{p(x)} d\mu + \int_\Omega |u-\varphi|^{p(x)} d\mu \right) \\
&\qquad - C_5 \|d\varphi\|^{p(x)-1} (\|du - d\varphi\| + \|u-\varphi\|) + C_8 \\
&\quad \geq C_9 \left(\max\left\{ \|du - d\varphi\|^{p^-}, \|du - d\varphi\|^{p^+} \right\} \right. \\
&\qquad \left. + \max\left\{ \|u-\varphi\|^{p^-}, \|u-\varphi\|^{p^+} \right\} \right) \\
&\qquad - C_5 \|d\varphi\|^{p(x)-1} (\|du - d\varphi\| + \|u-\varphi\|) + C_8.
\end{aligned} \tag{19}$$

Then it is easy to obtain

$$\frac{\langle \mathfrak{A}u - \mathfrak{A}\varphi, u-\varphi\rangle}{\|u-\varphi\|_{W_d^{p(x)}(\Omega,\Lambda^l,\mu)}} \longrightarrow \infty \tag{20}$$

as $\|u-\varphi\|_{W_d^{p(x)}(\Omega,\Lambda^l,\mu)} \to \infty$. It follows that $\mathfrak{A}$ is coercive on $\mathfrak{K}_{\psi,\theta}$. This completes the proof of Lemma 10. □

Lemma 11. $\mathfrak{A}$ *is weakly continuous on* $\mathfrak{K}_{\psi,\theta}$.

Proof. Let $u_i \in \mathfrak{K}_{\psi,\theta}$ be a sequence that converges to an element $u \in \mathfrak{K}_{\psi,\theta}$ in $W_d^{p(x)}(\Omega, \Lambda^l, \mu)$. Pick a subsequence u_{ij} such that $u_{ij} \to u$ a.e. in Ω. Since the mapping $\xi \to A(x,\xi)$ and $\xi \to B(x,\xi)$ are continuous for a.e. x in Ω, we have $A(x, u_{ij}(x))w^{-1/p(x)} \to A(x, u(x))w^{-1/p(x)}$ a.e. in Ω. Under the conditions (H2) and (H4), we know that $A(x, du_{ij})w^{-1/p(x)}$ and $B(x, du_{ij})w^{-1/p(x)}$ are uniformly bounded in $L^{p(x)}(\Omega, \Lambda^{l+1})$, and we have $A(x, du_{ij})w^{-1/p(x)} \to A(x, du)w^{-1/p(x)}$ weakly in $L^{p(x)}(\Omega, \Lambda^{l+1}, \mu)$ and $B(x, du_{ij})w^{-1/p(x)} \to B(x, du_{ij})w^{-1/p(x)}$ weakly in $L^{p(x)}(\Omega, \Lambda^l, \mu)$.

Since the weak limit is independent of the choice of the subsequence, it follows that

$$A(x, du_i)\, w^{-1/p(x)} \longrightarrow A(x, du)\, w^{-1/p(x)}, \qquad B(x, du_i)\, w^{-1/p(x)} \longrightarrow B(x, du)\, w^{-1/p(x)} \tag{21}$$

for all $u \in L^{p(x)}(\Omega, \Lambda^l, \mu)$, $duw^{1/p(x)} \in L^{p(x)}(\Omega, \Lambda^{l+1})$.

Then we have

$$\begin{aligned}
\langle \mathfrak{A}u_i, v\rangle &= \int_\Omega \left(A(x, du_i)\, w^{-1/p(x)} \cdot dvw^{1/p(x)}\right) dx \\
&\quad + \int_\Omega \left(B(x, du_i)\, w^{-1/p(x)} \cdot vw^{1/p(x)}\right) dx \\
&\longrightarrow \int_\Omega \left(A(x, du)\, w^{-1/p(x)} \cdot dvw^{1/p(x)}\right) dx \\
&\quad + \int_\Omega \left(B(x, du)\, w^{-1/p(x)} \cdot vw^{1/p(x)}\right) dx = \langle \mathfrak{A}u, v\rangle .
\end{aligned} \tag{22}$$

Hence $\mathfrak{A}$ is weakly continuous on $\mathfrak{K}_{\psi,\theta}$. This ends the proof of Lemma 11. □

Proof of Theorem 5. We can apply Proposition 6 and the above lemmas to obtain the existence. If there are two weak solutions $u_1, u_2 \in \mathfrak{K}_{\psi,\theta}$ to obstacle problem (1)-(2), then we have

$$\begin{aligned}
&\int_\Omega \left(A(x, du_1) \cdot d(u_2 - u_1) + B(x, du_1) \cdot (u_2 - u_1)\right) dx \ge 0, \\
&\int_\Omega \left(A(x, du_2) \cdot d(u_1 - u_2) + B(x, du_2) \cdot (u_1 - u_2)\right) dx \ge 0,
\end{aligned} \tag{23}$$

so

$$\begin{aligned}
&\int_\Omega \left(\left(A(x, du_2) - A(x, du_1)\right) \cdot d(u_2 - u_1)\right. \\
&\qquad \left. + \left(B(x, du_2) - B(x, du_1)\right) \cdot (u_2 - u_1)\right) dx \le 0.
\end{aligned} \tag{24}$$

In view of (H6), we can further infer that

$$\begin{aligned}
&\int_\Omega \left(\left(A(x, du_2) - A(x, du_1)\right) \cdot d(u_2 - u_1)\right. \\
&\qquad + \left(B(x, du_2) - B(x, du_1)\right) \\
&\qquad \left. \cdot (u_2 - u_1)\right) dx = 0 \quad \text{a.e. on } \Omega,
\end{aligned} \tag{25}$$

which means that $u_1 = u_2$ a.e. on Ω, and now we complete the proof. □

Corollary 12. *Let* Ω *be a bounded domain and* $\theta \in W_d^{p(x)}(\Omega, \Lambda^{l-1}, \mu)$. *Under the conditions (H1)–(H6), there is a differential form* $u \in W_d^{p(x)}(\Omega, \Lambda^{l-1}, \mu)$ *with* $u - \theta \in W_d^{p(x)}(\Omega, \Lambda^{l-1}, \mu)$ *such that*

$$d^\star A(x, du) = B(x, du), \quad \textit{weakly in } \Omega; \tag{26}$$

that is, $\int_\Omega (A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi) dx = 0$, *whenever* $\varphi \in W_{0d}^{p(x)}(\Omega, \Lambda^{l-1}, \mu), l = 1, 2, \ldots, n$.

Proof. Choose $\psi_I \equiv -\infty$ and let u be the solution to the obstacle problem (1)-(2) in $\mathfrak{K}_{\psi,\theta}$. For any $\varphi \in W_{0d}^{p(x)}(\Omega, \Lambda^{l-1}, \mu)$, since $u + \varphi$ and $u - \varphi$ both belong to $\mathfrak{K}_{\psi,\theta}$, we have

$$\begin{aligned}
&-\left(\int_\Omega \left(A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi\right) dx\right) \ge 0, \\
&\int_\Omega \left(A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi\right) dx \ge 0.
\end{aligned} \tag{27}$$

Thus

$$\int_\Omega \left(A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi\right) dx = 0, \tag{28}$$

as desired. □

Remark 13. If $p(x) = p$, then $\|u\|_{L^{p(x)}(\Omega,\Lambda^l,\mu)} = \|u\|_{L^p(\Omega,\Lambda^l,\mu)} = \|u\|_{p,\Omega,\mu} = (\int_\Omega |u|^p d\mu)^{1/p}$ and the Luxemburg norm reduces to the L^p norm. So (26) is the extension of the equation in [2–5].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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