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Research Article

Multiplicity of Positive Solutions for a Second-Order Elliptic System of Kirchhoff Type

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We study elliptic problems of Kirchhoff type in $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$. Using variational tools, we establish the existence of at least two nontrivial and nonnegative solutions.

1. Introduction and Preliminaries

In this paper, we are concerned with the following problem:

$$\begin{aligned} -\left(a_1 + b_1\left(\int_\Omega |\nabla u|^2 + a(x)|u|^2\right)dx\right)(\Delta u + a(x)u) &= F_u(x,u,v) + \lambda b(x)|u|^{q-2}u \quad \text{in } \Omega, \\ -\left(a_2 + b_2\left(\int_\Omega |\nabla v|^2 + a(x)|v|^2\right)dx\right)(\Delta v + a(x)v) &= F_v(x,u,v) + \mu c(x)|v|^{q-2}v \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with the smooth boundary $\partial\Omega$ such that $0 \in \Omega$, $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian operator, $1 < q < 2$, $\lambda, \mu > 0$, $a_i, b_i > 0$ $(i = 1, 2)$, and $a, b, c \in C(\Omega, \mathbb{R}^+)$, the function $F \in C^1(\overline{\Omega} \times (\mathbb{R}^+)^2, \mathbb{R}^+)$, is positively homogeneous of degree $\alpha = 4N/(N-2)$ which is the Sobolev critical exponent; that is, $F(x, tu, tv) = t^\alpha F(x, u, v)$, $(t > 0)$ holds for all $(x, u, v) \in \overline{\Omega} \times (\mathbb{R}^+)^2$, $(F_u, F_v) = \nabla F$.

In recent years, there have been many papers concerned with the existence of the positive solutions for Kirchhoff equation

$$\begin{aligned} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u &= \lambda f(x,u) \quad \text{in } \Omega \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

which is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{3}$$

where $M(s) = a + bs$, $a, b > 0$. It was proposed by Kirchhoff [1] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings.

Some interesting studies on these problems by variational methods can be found in [2–6]. As for perturbed fourth-order Kirchhoff-type elliptic problems, in [7] the following equation,

$$\begin{aligned} \Delta\left(|\Delta u|^{p-2}\Delta u\right) - \left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u + \rho|u|^{p-2}u &= \lambda f(x,u) \quad \text{in } \Omega \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

where $p > \max\{1, N/2\}$, $\lambda > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, and $M : [0, +\infty[\to \mathbb{R}$ is a continuous function, has been investigated. The authors proved (4) has multiple nontrivial weak solutions.

In [8] the authors established the existence of a weak solution for the following system equation:

$$-\left[M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\right]^{p-1}\Delta_p u = f(u,v)+\rho_1(x) \quad \text{in } \Omega$$
$$-\left[M_2\left(\int_\Omega |\nabla v|^p\,dx\right)\right]^{p-1}\Delta_p v = f(u,v)+\rho_2(x) \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial\eta}=\frac{\partial v}{\partial\eta}=0 \quad \text{on } \partial\Omega, \tag{5}$$

where $M_1(t), M_2(t) \ge m_0 > 0$.

Motivated by the results of the above cited papers, we will attempt to treat problem (1) and extend the results for our problem.

In this paper we make the following assumptions.

Let S be the best Sobolev embedding constant defined by

$$S=\inf_{u\in W_0^{1,2}(\Omega)\setminus\{0\}}\frac{\int_\Omega\left(|\nabla u|^2+a(x)|u|^2\right)dx}{\left(\int_\Omega |u|^\alpha\,dx\right)^{2/\alpha}} \tag{6}$$

and let $|\Omega|$ be the Lebesgue measure of Ω; $|\cdot|_\infty$ denotes the $L^\infty(\Omega)$ norm, $\beta=\min\{b_1,b_2\}$, and

$$C(q,N,K,S,|\Omega|,a_1,a_2) = \frac{A(\alpha-2)S^{q/2}}{(\alpha-q)|\Omega|^{(\alpha-q)/\alpha}}\left(\frac{A(2-q)S^{\alpha/2}}{K(\alpha-q)}\right)^{(2-q)/(\alpha-2)}, \tag{7}$$
$$C_0=\left(\frac{q}{2}\right)C(q,N,K,S,|\Omega|,a_1,a_2),$$

where $A=\min\{a_1,a_2\}$.

Also the following hold:

(F1) $F:\overline{\Omega}\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a C^1 function and $F(x,tu,tv)=t^\alpha F(x,u,v)$,

(F2) $F(x,u,0)=F(x,0,v)=F_u(x,u,0)=F_v(x,0,v)=0$ where $u,v\in\mathbb{R}^+$,

(F3) $F_u(x,u,v), F_v(x,u,v)$ are strictly increasing function about u, v for all $u>0$, $v>0$.

In addition, using assumption (F1), we have the so-called Euler identity

$$(u,v)\cdot\nabla F(x,u,v)=\alpha F(x,u,v) \tag{8}$$

and, for a positive constant K,

$$F(x,u,v)\le K\left(|u|^2+|v|^2\right)^{\alpha/2}. \tag{9}$$

Let $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega|\nabla u|^2)^{1/2}$.

It is easy to show that, for every $u\in W_0^{1,2}(\Omega)$, the above norm is equivalent with $\|u\| = (\int_\Omega|\nabla u|^2+a(x)|u|^2)^{1/2}$. Problem (1) is posed in the framework of the Sobolev space

$$E=\left\{(u,v)\in\left(W_0^{1,2}(\Omega)\right)^2 \mid \int_\Omega a(x)\left(|u|^2+|v|^2\right)dx<+\infty\right\}, \tag{10}$$

with the standard norm

$$\|(u,v)\|_E = \left(\int_\Omega\left(|\nabla u|^2+a(x)|u|^2\right)dx + \int_\Omega\left(|\nabla v|^2+a(x)|v|^2\right)dx\right)^{1/2}. \tag{11}$$

We will look for solutions of (1) by finding critical points of the energy functional $J_{\lambda,\mu}:E\to\mathbb{R}$ given by

$$J_{\lambda,\mu}(u,v)=\frac{a_1}{2}\|u\|^2+\frac{a_2}{2}\|v\|^2+\frac{b_1}{4}\|u\|^4 +\frac{b_2}{4}\|v\|^4-\frac{1}{\alpha}\int_\Omega F(x,u,v)\,dx-\frac{1}{q}K_{\lambda,\mu}(u,v), \tag{12}$$

where $K_{\lambda,\mu}:E\to\mathbb{R}$ is the functional defined by

$$K_{\lambda,\mu}=\int_\Omega\left(\lambda b(x)|u|^q+\mu c(x)|v|^q\right)dx. \tag{13}$$

It is well known that the functional $J_{\lambda,\mu}\in C^1(E,\mathbb{R})$. For any $(\varphi_1,\varphi_2)\in E$, there holds

$$\begin{aligned}\left\langle J'_{\lambda,\mu}(u,v),(u,v)\right\rangle &= \left(a_1+b_1\left(\int_\Omega\left(|\nabla u|^2+a(x)|u|^2\right)dx\right)\right)\\ &\quad\times\left(\int_\Omega\left(\nabla u\nabla\varphi_1+a(x)u\varphi_1\right)dx\right)\\ &\quad+\left(a_2+b_2\left(\int_\Omega\left(|\nabla v|^2+a(x)|v|^2\right)dx\right)\right)\\ &\quad\times\left(\int_\Omega\left(\nabla v\nabla\varphi_2+a(x)u\varphi_2\right)dx\right)\\ &\quad-\frac{1}{\alpha}\int_\Omega\left(F_u(x,u,v)\varphi_1+F_v(x,u,v)\varphi_2\right)dx\\ &\quad-\lambda\int_\Omega b(x)|u|^{q-2}u\varphi_1dx-\mu\int_\Omega c(x)|v|^{q-2}v\varphi_2dx.\end{aligned} \tag{14}$$

Consider the Nehari manifold

$$N_{\lambda,\mu}=\left\{(u,v)\in E\setminus\{0,0\}\left\langle J'_{\lambda,\mu}(u,v),(u,v)\right\rangle=0\right\}. \tag{15}$$

Note that $(u,v)\in N_{\lambda,\mu}$ if and only if

$$a_1\|u\|^2+a_2\|v\|^2+b_1\|u\|^4+b_2\|v\|^4-\int_\Omega F(x,u,v)\,dx - K_{\lambda,\mu}(u,v)=0. \tag{16}$$

So $N_{\lambda,\mu}$ contains all nontrivial weak solutions of (1).

Define $\Phi_{\lambda,\mu}(u,v) = \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle$. Then, for $(u,v) \in N_{\lambda,\mu}$,

$$\begin{aligned}
&\left\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\right\rangle \\
&\quad = 2a_1\|u\|^2 + 2a_2\|v\|^2 + 4b_1\|u\|^4 \\
&\qquad + 4b_2\|v\|^4 - \alpha\int_\Omega F(x,u,v)\,dx - qK_{\lambda,\mu}(u,v) \\
&\quad = 2b_1\|u\|^4 + 2b_2\|v\|^4 - (\alpha-2)\int_\Omega F(x,u,v)\,dx \\
&\qquad - (q-2)K_{\lambda,\mu}(u,v) \\
&\quad = -2a_1\|u\|^2 - 2a_2\|v\|^2 - (\alpha-4)\int_\Omega F(x,u,v)\,dx \\
&\qquad - (q-4)K_{\lambda,\mu}(u,v) \\
&\quad = (2-\alpha)\left(a_1\|u\|^2 + a_2\|v\|^2\right) \\
&\qquad + (4-\alpha)\left(b_1\|u\|^4 + b_2\|v\|^4\right) - (q-\alpha)K_{\lambda,\mu}(u,v) \\
&\quad = (2-q)\left(a_1\|u\|^2 + a_2\|v\|^2\right) \\
&\qquad + (4-q)\left(b_1\|u\|^4 + b_2\|v\|^4\right) - (\alpha-q)\int_\Omega F(x,u,v)\,dx.
\end{aligned} \tag{17}$$

Now, we split $N_{\lambda,\mu}$ into three parts:

$$\begin{aligned}
N^+_{\lambda,\mu} &= \left\{(u,v)\in N_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle > 0\right\}; \\
N^0_{\lambda,\mu} &= \left\{(u,v)\in N_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\right\}; \\
N^-_{\lambda,\mu} &= \left\{(u,v)\in N_{\lambda,\mu} \mid \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle < 0\right\}.
\end{aligned} \tag{18}$$

2. Statement of the Main Results

Let us first define $\varrho_{\lambda,\mu} = (\lambda|b|_\infty)^{2/(2-q)} + (\mu|c|_\infty)^{2/(2-q)}$ and the main results read as follows.

Theorem 1. *If (λ,μ) satisfy $0 < \varrho_{\lambda,\mu} < C(q,N,K,S,|\Omega|,a_1,a_2)$ and (F1)–(F3) hold, then problem (1) has at least one positive solution.*

Theorem 2. *If (λ,μ) satisfy $0 < \varrho_{\lambda,\mu} < C_0^*$ and (F1)–(F3) hold, then problem (1) has at least two positive solutions.*

Note that, using assumption (F3), we have that $F_u, F_v \in C(\overline{\Omega}\times(\mathbb{R}^+)^2,\mathbb{R}^+)$ are positively homogeneous of degree $\alpha-1$. This implies that

$$\begin{gathered}
\left|F_u(x,u,v)\right| \le M\left(|u|^{\alpha-1} + |v|^{\alpha-1}\right), \\
\left|F_v(x,u,v)\right| \le M\left(|u|^{\alpha-1} + |v|^{\alpha-1}\right), \quad \forall x\in\overline{\Omega},\ u,v\in\mathbb{R}^+,
\end{gathered} \tag{19}$$

for some positive constant M. Similar to Willem [9, Theorem A.2], we consider the continuity of the superposition operator

$$A : L^2(\Omega) \longrightarrow L^q(\Omega) : (u,v) \longmapsto f(x,u,v). \tag{20}$$

Lemma 3. *Assume that $|\Omega| < \infty$, $r < \infty$, $f \in C(\overline{\Omega}\times\mathbb{R}^2,\mathbb{R})$, and*

$$\left|f(x,u,v)\right| \le c\left(1 + |u|^{2/r} + |v|^{2/r}\right). \tag{21}$$

Then, for every $(u,v) \in L^2(\Omega)$, $f(\cdot,u,v) \in L^r(\Omega)$ and the operator $A : L^p(\Omega) \to L^r(\Omega) : (u,v) \mapsto f(x,u,v)$ is continuous.

Now, we consider the functional $\psi(u,v) = \int_\Omega F(x,u,v)dx$; then we have the following result.

Lemma 4. *Assume that $|\Omega| < \infty$, $F \in C(\overline{\Omega}\times(\mathbb{R}^+)^2,\mathbb{R}^+)$ satisfying (F3); then the functional ψ is of class $C^1(E,\mathbb{R}^+)$ and*

$$\langle \psi'(u,v),(\varphi_1,\varphi_2)\rangle = \int_\Omega \left(F_u(x,u,v)\varphi_1 + F_v(x,u,v)\varphi_2\right)dx, \tag{22}$$

where $(u,v),(\varphi_1,\varphi_2) \in E$.

Proof. The proof is almost the same as that in [10]. □

Lemma 5. *The energy functional $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$.*

Proof. If $(u,v) \in N_{\lambda,\mu}$, then by the Hölder inequality and the Sobolev embedding theorem

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &= \frac{\alpha-2}{2\alpha}\left(a_1\|u\|^2 + a_2\|v\|^2\right) \\
&\quad + \frac{\alpha-4}{4\alpha}\left(b_1\|u\|^4 + b_2\|v\|^4\right) - \frac{\alpha-q}{\alpha q}K_{\lambda,\mu}(u,v) \\
&\ge \frac{\alpha-2}{2\alpha}\left(a_1\|u\|^2 + a_2\|v\|^2\right) + \frac{\alpha-4}{4\alpha}\left(b_1\|u\|^4 + b_2\|v\|^4\right) \\
&\quad - \frac{\alpha-q}{\alpha q}\varrho_{\lambda,\mu}^{(2-q)/2} S^{-q/2}|\Omega|^{(\alpha-q)/\alpha}\|(u,v)\|_E^q.
\end{aligned} \tag{23}$$

Thus, $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. □

Lemma 6. *Suppose that (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ and that $(u_0,v_0) \notin N^0_{\lambda,\mu}$. Then $J'_{\lambda,\mu}(u_0,v_0) = 0$ in E^{-1} (the dual space of the Sobolev space E).*

Proof. If (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$, then (u_0,v_0) is a solution of the optimization problem minimizer $J_{\lambda,\mu}(u,v)$ subject to $\Phi_{\lambda,\mu}(u,v) = 0$. Hence, by the theory of Lagrange multipliers, there exists $\xi_1 \in R$, such that

$$J'_{\lambda,\mu}(u_0,v_0) = \xi_1\Phi'_{\lambda,\mu}(u_0,v_0) \quad \text{in } E^{-1}(\Omega), \tag{24}$$

and thus,

$$\langle J'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle = \xi_1\langle \Phi'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle. \tag{25}$$

Since $(u_0, v_0) \in N_{\lambda,\mu}$, we have $\langle J'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle = 0$. Moreover, $\langle \Phi'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle \neq 0$, so $\xi_1 = 0$. This completes the proof. □

Lemma 7. *If* $0 < \varrho_{\lambda,\mu} < C(q, N, K, S, |\Omega|, a_1, a_2)$, *then* $N^0_{\lambda,\mu} = \emptyset$.

Proof. Suppose otherwise that $0 < \varrho_{\lambda,\mu} < C(q, N, K, S, |\Omega|, a_1, a_2)$ such that $N^0_{\lambda,\mu} \neq \emptyset$.

Then, for $(u, v) \in N^0_{\lambda,\mu}$,

$$\begin{aligned} 0 &= \langle \Phi'_{\lambda,\mu}(u, v), (u, v)\rangle \\ &= (2-\alpha)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) + (4-\alpha)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) \\ &\quad - (q-\alpha) K_{\lambda,\mu}(u, v) \\ &= (2-q)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) + \left(b_1 \|u\|^4 + b_2 \|v\|^4\right) \\ &\quad - (\alpha - q)\int_\Omega F(x, u, v)\, dx. \end{aligned} \tag{26}$$

By the Holder inequality and the Sobolev embedding theorem

$$\begin{aligned} &(\alpha-2)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) + (\alpha-4)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) \\ &\quad = (\alpha - q) K_{\lambda,\mu}(u, v) \\ &\quad \le (\alpha - q)\, \varrho_{\lambda,\mu}^{(2-q)/2} S^{-q/2} |\Omega|^{(\alpha-q)/\alpha} \|(u, v)\|_E^q. \end{aligned} \tag{27}$$

So,

$$\begin{aligned} A(\alpha-2)\|(u, v)\|_E^2 &\le (\alpha-2)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) \\ &\le (q-\alpha)\, \varrho_{\lambda,\mu}^{(2-q)/2} S^{-q/2} |\Omega|^{(\alpha-q)/\alpha} \|(u, v)\|_E^q. \end{aligned} \tag{28}$$

Thus,

$$\|(u, v)\|_E \le \left(\frac{(\alpha - q)\, \varrho_{\lambda,\mu}^{(2-q)/2} S^{-q/2} |\Omega|^{(\alpha-q)/\alpha}}{A(\alpha-2)}\right)^{1/(2-q)} \tag{29}$$

and, by the Minkowski inequality, the Sobolev embedding theorem, and (9),

$$\begin{aligned} \int_\Omega F(x, u, v)\, dx &\le K\left(\int_\Omega \left(|u|^2 + |v|^2\right)^{\alpha/2} dx\right)^{(2/\alpha)\cdot(\alpha/2)} \\ &\le K\left(\left(\int_\Omega |u|^\alpha dx\right)^{2/\alpha} + \left(\int_\Omega |v|^\alpha dx\right)^{2/\alpha}\right)^{\alpha/2} \\ &\le K S^{-\alpha/2} \|(u, v)\|_E^\alpha. \end{aligned} \tag{30}$$

Thus,

$$\|(u, v)\|_E \ge \left(\frac{A(2-q) S^{\alpha/2}}{K(\alpha - q)}\right)^{1/(\alpha-2)}. \tag{31}$$

This implies that

$$\varrho_{\lambda,\mu} \ge C(q, N, K, S, |\Omega|, a_1, a_2), \tag{32}$$

which is a contradiction. Thus, we can conclude that if

$$0 < \varrho_{\lambda,\mu} < C(q, N, K, S, |\Omega|, a_1, a_2), \tag{33}$$

we have $N^0_{\lambda,\mu} = \emptyset$. □

By Lemma 7, we write $N_{\lambda,\mu} = N^+_{\lambda,\mu} \cup N^-_{\lambda,\mu}$ and define

$$\begin{aligned} \theta_{\lambda,\mu} &= \inf_{(u,v)\in N_{\lambda,\mu}} J_{\lambda,\mu}(u, v); \\ \theta^+_{\lambda,\mu} &= \inf_{(u,v)\in N^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v); \\ \theta^-_{\lambda,\mu} &= \inf_{(u,v)\in N^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v). \end{aligned} \tag{34}$$

Then we have the following result.

Lemma 8. *Consider the following.*

(i) *If* $0 < \varrho_{\lambda,\mu} < C(q, N, K, S, |\Omega|, a_1, a_2)$, *then one has* $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$.

(ii) *If* $0 < \varrho_{\lambda,\mu} < C_0$, *then* $\theta^-_{\lambda,\mu} > d_0$ *for some constant*

$$d_0 = d_0\left(q, N, K, S, |\Omega|, \varrho_{\lambda,\mu}, a_1, a_2, b_1, b_2\right) > 0. \tag{35}$$

Proof. (i) Let $(u, v) \in N^+_{\lambda,\mu}$. Then

$$\begin{aligned} &(2-q)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) + (4-q)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) \\ &\quad - (\alpha - q)\int_\Omega F(x, u, v)\, dx > 0 \end{aligned} \tag{36}$$

and so

$$\begin{aligned} J_{\lambda,\mu}(u, v) &= \left(\frac{1}{2} - \frac{1}{q}\right)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) \\ &\quad + \left(\frac{1}{4} - \frac{1}{q}\right)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) \\ &\quad + \left(\frac{1}{q} - \frac{1}{\alpha}\right)\int_\Omega F(x, u, v)\, dx \\ &< \frac{q-2}{q}\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) \\ &\quad + \frac{q-4}{q}\left(\frac{1}{4} - \frac{1}{\alpha}\right)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) < 0. \end{aligned} \tag{37}$$

Thus, from the definition of $\theta_{\lambda,\mu}$ and $\theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$.

(ii) Let $(u, v) \in N^{-}_{\lambda,\mu}$. Then

$$(2-q)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) + (4-q)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) - (\alpha - q) \int_\Omega F(x,u,v)\, dx < 0. \tag{38}$$

By (8) we have

$$\begin{aligned}(2-q)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) &+ (4-q)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) \\ &< (\alpha - q) \int_\Omega F(x,u,v)\, dx \\ &\le (\alpha - q) K S^{-\alpha/q} \|(u,v)\|_E^\alpha.\end{aligned} \tag{39}$$

This implies that

$$\|(u,v)\|_E > \left(\frac{A(2-q) S^{\alpha/q}}{K(\alpha - q)}\right)^{1/(\alpha-2)}. \tag{40}$$

By (23) in the proof of Lemma 5,

$$\begin{aligned}J_{\lambda,\mu}(u,v) &\ge A\left(\frac{1}{2} - \frac{1}{\alpha}\right)\|(u,v)\|_E^2 \\ &\quad - \left(\frac{1}{q} - \frac{1}{\alpha}\right) \varrho_{\lambda,\mu}^{(2-q)/2} S^{-q/2} |\Omega|^{(\alpha-q)/\alpha} \|(u,v)\|_E^q \\ &> A\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left(\frac{A(q-2)S^{\alpha/q}}{K(\alpha-q)}\right)^{2/(\alpha-2)} \\ &\quad - \left(\frac{1}{q} - \frac{1}{\alpha}\right) \varrho_{\lambda,\mu}^{(2-q)/2} S^{(N-q)/2} |\Omega|^{(\alpha-q)/\alpha} \\ &\quad \times \left(\frac{A(q-2)S^{\alpha/q}}{K(\alpha-q)}\right)^{q/(\alpha-2)}.\end{aligned} \tag{41}$$

Thus, if $0 < \lambda < C_0$, then

$$J_{\lambda,\mu}(u,v) > d_0 \quad \forall (u,v) \in N^{-}_{\lambda,\mu}, \tag{42}$$

for some $d_0 = d_0(\lambda, \mu, q, N, K, S, |\Omega|, |b|_\infty, |c|_\infty, a_1, a_2, b_1, b_2) > 0$. This completes the proof. □

For each $(u, v) \in E$ with $\int_\Omega F(x,u,v)dx > 0$, set

$$\begin{aligned}t_{\max} = \Bigg(\Big((2-q)\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) &+ (4-q)\left(b_1 \|u\|^4 + b_2 \|v\|^4\right)\Big) \\ &\times \left((\alpha - q)\int_\Omega F(x,u,v)\,dx\right)^{-1}\Bigg)^{1/(\alpha-2)} \\ &> 0.\end{aligned} \tag{43}$$

Then we have the following.

Lemma 9 (see [11, Lemma 2.6]). *For each $(u, v) \in E$ with $\int_\Omega F(x,u,v)dx > 0$, there are unique $0 < t^+ < t_{\max} < t^-$ such that $(t^+u, t^+v) \in N^{+}_{\lambda,\mu}$, $(t^-u, t^-v) \in N^{-}_{\lambda,\mu}$ and $J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} J_{\lambda,\mu}(tu, tv)$; $J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv)$.*

3. Proof of the Main Theorems

We will need the following lemma.

Lemma 10 (see [12]). *Consider the following.*

(i) *If $0 < \varrho_{\lambda,\mu}^{(2-q)/2} < C(q, N, K, S, |\Omega|, \varrho_{\lambda,\mu}, a_1, a_2)$, then there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$.*

(ii) *If $0 < \varrho_{\lambda,\mu} < C_0$, then there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N^{-}_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$.*

Theorem 11. *If $0 < \varrho_{\lambda,\mu} < C(q, N, K, S, |\Omega|, \varrho_{\lambda,\mu}, a_1, a_2)$ and (F1)–(F3) hold, then $J_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $N^{+}_{\lambda,\mu}$ and it satisfies the following:*

(i) *$J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu}$;*

(ii) *(u_0^+, v_0^+) is a positive solution of (1).*

Proof. By Lemma 10(i), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o(1), \qquad J'_{\lambda,\mu}(u_n, v_n) = o(1). \tag{44}$$

Then by Lemma 5 and the compact imbedding theorem, there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+) \in E$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^+ \quad \text{weakly in } W_0^{1,2}, \\ u_n &\longrightarrow u_0^+ \quad \text{strongly in } L^q(\Omega), \\ v_n &\rightharpoonup v_0^+ \quad \text{weakly in } W_0^{1,2}, \\ v_n &\longrightarrow v_0^+ \quad \text{strongly in } L^q(\Omega). \end{aligned} \tag{45}$$

This implies that $K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+)$ as $n \to \infty$. By (44) and (45), it is easy to prove that (u_0^+, v_0^+) is *s* weak solution of (1). Since

$$\begin{aligned}J_{\lambda,\mu}(u_n, v_n) &= \frac{(N+2)}{4N}\left(a_1 \|u\|^2 + a_2 \|v\|^2\right) \\ &\quad + \frac{1}{2N}\left(b_1 \|u\|^4 + b_2 \|v\|^4\right) \\ &\quad - \frac{\alpha - q}{\alpha q} K_{\lambda,\mu}(u_n, v_n) \\ &\ge -\frac{\alpha - q}{\alpha q} K_{\lambda,\mu}(u_n, v_n)\end{aligned} \tag{46}$$

and by Lemma 8(i),

$$J_{\lambda,\mu}(u_n, v_n) \longrightarrow \theta_{\lambda,\mu} < 0 \quad \text{as } n \longrightarrow \infty. \tag{47}$$

Letting $n \to \infty$, we see that $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$. Thus, (u_0^+, v_0^+) is a nontrivial solution of problem (1). Now it follows that $u_n \to u_0^+$ strongly in $W_0^{1,2}$ and $v_n \to v_0^+$ strongly in $W_0^{1,2}$ and $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}$. By $(u_0^+, v_0^+) \in N_{\lambda,\mu}$ and applying Fatou's lemma, we get

$$\begin{aligned}\theta_{\lambda,\mu} \le J_{\lambda,\mu}(u_0^+, v_0^+) &= \frac{N+2}{4N}\left(a_1\|u_0^+\|^2 + a_2\|v_0^+\|^2\right) \\ &\quad \times \frac{1}{2N}\left(b_1\|u_0^+\|^4 + b_2\|v_0^+\|^4\right) \\ &\quad - \frac{\alpha - q}{\alpha q}K_{\lambda,\mu}(u_0^+, v_0^+) \\ &\le \liminf_{n\to\infty}\left(\frac{N+2}{4N}\left(a_1\|u_n\|^2 + a_2\|v_n\|^2\right)\right. \\ &\quad + \frac{1}{2N}\left(b_1\|u\|^4 + b_2\|v\|^4\right) \\ &\quad \left.-\frac{\alpha - q}{\alpha q}K_{\lambda,\mu}(u_n, v_n)\right) \\ &\le \liminf_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}.\end{aligned} \tag{48}$$

This implies that

$$\begin{gathered} J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}; \quad \lim_{n\to\infty}\|u_n\|^2 = \|u_0^+\|^2, \\ \lim_{n\to\infty}\|v_n\|^2 = \|v_0^+\|^2. \end{gathered} \tag{49}$$

Let $\tilde{u}_n = u_n - u_0^+$, $\tilde{v}_n = v_n - v_0^+$; then by the Brezis-Lieb lemma [13], this implies

$$\begin{gathered} \|\tilde{u}_n\|^2 = \|u_n\|^2 - \|u_0^+\|^2, \\ \|\tilde{v}_n\|^2 = \|v_n\|^2 - \|v_0^+\|^2. \end{gathered} \tag{50}$$

Therefore, $u_n \to u_0^+$ strongly in $W_0^{1,2}$ and $v_n \to v_0^+$ strongly in $W_0^{1,2}$. Moreover, we have $(u_0^+, v_0^+) \in N_{\lambda,\mu}^+$. In fact, if $(u_0^+, v_0^+) \in N_{\lambda,\mu}^-$, by Lemma 9, there are unique t_0^+ and t_0^- such that $(t_0^+u_0^+, t_0^+v_0^+) \in N_{\lambda,\mu}^+$ and $(t_0^-u_0^+, t_0^-v_0^+) \in N_{\lambda,\mu}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) = 0, \qquad \frac{d^2}{dt^2}J_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) > 0, \tag{51}$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) < J_{\lambda,\mu}(\bar{t}u_0^+, \bar{t}v_0^+)$. By Lemma 9,

$$\begin{aligned} J_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) < J_{\lambda,\mu}(\bar{t}u_0^+, \bar{t}v_0^+) &\le J_{\lambda,\mu}(t_0^-u_0^+, t_0^-v_0^+) \\ &= J_{\lambda,\mu}(u_0^+, v_0^+) \end{aligned} \tag{52}$$

which is a contradiction. It follows from the maximum principle that (u_0^+, v_0^+) is a positive solution of problem (1). This completes the proof. □

The following two lemmas are similar to those in [14].

Lemma 12. *If $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u, v)$ in E, then $J'_{\lambda,\mu}(u, v) = 0$, and there exists a positive constant Λ, such that $J_{\lambda,\mu}(u, v) \ge -\Lambda\varrho_{\lambda,\mu}$.*

Lemma 13. *If $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$, then $\{(u_n, v_n)\}$ is bounded in E.*

Define

$$S_F := \inf_{(u,v)\in E}\left\{\frac{a_1\|u\|^2 + a_2\|v\|^2 + b_1\|u\|^4 + b_2\|v\|^4}{\left(\int_\Omega F(x,u,v)\,dx\right)^{2/\alpha}} : \int_\Omega F(x,u,v)\,dx > 0\right\}. \tag{53}$$

In addition, we need the following version of the Brezis-Lieb lemma [13].

Lemma 14. *Consider $F \in C^1(\overline{\Omega}, (\mathbb{R}^+)^2)$ with $F(x,0,0) = 0$ and*

$$\left|\frac{\partial F(x,u,v)}{\partial u}\right|, \left|\frac{\partial F(x,u,v)}{\partial v}\right| \le C_1(|u| + |v|) \tag{54}$$

for some $C_1 > 0$. Let $\{(u_k, v_k)\}$ be a bounded sequence in $L^2(\overline{\Omega}, (\mathbb{R}^+)^2)$ such that $(u_k, v_k) \rightharpoonup (u, v)$ weakly in E. Then as $k \to \infty$,

$$\begin{aligned}\int_\Omega F(x, u_k, v_k)\,dx \longrightarrow &\int_\Omega F(x, u_k - u, v_k - v)\,dx \\ &+ \int_\Omega F(x,u,v)\,dx.\end{aligned} \tag{55}$$

Lemma 15. *$J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition with c satisfying*

$$-\infty < c < c_\infty = \left(\frac{1}{4} - \frac{1}{\alpha}\right)S_F^{2N/(N+2)} - \Lambda\varrho_{\lambda,\mu}. \tag{56}$$

Proof. Let $\{(u_n, v_n)\} \subset E$ be a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $c \in (-\infty, c_\infty)$. It follows from Lemma 13 that $\{(u_n, v_n)\}$ is bounded in E, and then $(u_n, v_n) \rightharpoonup (u, v)$ up to a subsequence, where (u, v) is a critical point of $J_{\lambda,\mu}$. Furthermore, we may assume

$$\begin{aligned} &u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{in } W_0^{1,2} \\ &u_n \longrightarrow u, \quad v_n \longrightarrow v \quad \text{in } L^q(\Omega) \\ &u_n \longrightarrow u, \quad v_n \longrightarrow v \quad \text{a.e. on } \Omega. \end{aligned} \tag{57}$$

Hence we have that $J'_{\lambda,\mu}(u, v) = 0$ and

$$K_{\lambda,\mu}(u_n, v_n) \longrightarrow K_{\lambda,\mu}(u, v). \tag{58}$$

Let $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. Then by the Brezis-Lieb lemma [13], we obtain

$$\|\tilde{u}_n\|^2 \longrightarrow \|u_n\|^2 - \|u\|^2, \quad \|\tilde{v}_n\|^2 \longrightarrow \|v_n\|^2 - \|v\|^2 \quad \text{as } n \longrightarrow \infty, \tag{59}$$

and by Lemma 14,

$$\int_\Omega F(x,\tilde{u}_n,\tilde{v}_n)\,dx \longrightarrow \int_\Omega F(x,u_n,v_n)\,dx - \int_\Omega F(x,u,v)\,dx. \tag{60}$$

Since $J_{\lambda,\mu}(u_n,v_n) = c+o(1)$, $J'_{\lambda,\mu}(u_n,v_n) = o(1)$ and (58)–(60), we can deduce that

$$\begin{aligned}&\frac{a_1}{2}\|\tilde{u}_n\|^2 + \frac{a_2}{2}\|\tilde{v}_n\|^2 + \frac{b_1}{4}\|\tilde{u}_n\|^4 + \frac{b_2}{4}\|\tilde{v}_n\|^4\\ &\quad - \frac{1}{\alpha}\int_\Omega F(x,\tilde{u}_n,\tilde{v}_n)\,dx\\ &= c - J_{\lambda,\mu}(u,v) + o(1).\end{aligned} \tag{61}$$

So

$$\begin{aligned}&\frac{1}{4}\left(a_1\|\tilde{u}_n\|^2 + a_2\|\tilde{v}_n\|^2 + b_1\|\tilde{u}_n\|^4 + b_2\|\tilde{v}_n\|^4\right)\\ &\quad - \frac{1}{\alpha}\int_\Omega F(x,\tilde{u}_n,\tilde{v}_n)\,dx\\ &\le \frac{a_1}{2}\|\tilde{u}_n\|^2 + \frac{a_2}{2}\|\tilde{v}_n\|^2 + \frac{b_1}{4}\|\tilde{u}_n\|^4\\ &\quad + \frac{b_2}{4}\|\tilde{v}_n\|^4 - \frac{1}{\alpha}\int_\Omega F(x,\tilde{u}_n,\tilde{v}_n)\,dx\\ &= c - J_{\lambda,\mu}(u,v) + o(1),\\ &a_1\|\tilde{u}_n\|^2 + a_2\|\tilde{v}_n\|^2 + b_1\|\tilde{u}_n\|^4 + b_2\|\tilde{v}_n\|^4\\ &\quad - \int_\Omega F(x,\tilde{u}_n,\tilde{v}_n)\,dx = o(1).\end{aligned} \tag{62}$$

Hence, we may assume that

$$\begin{aligned}&a_1\|\tilde{u}_n\|^2 + a_2\|\tilde{v}_n\|^2 + b_1\|\tilde{u}_n\|^4 + b_2\|\tilde{v}_n\|^4 \longrightarrow l,\\ &\int_\Omega F(x,\tilde{u}_n,\tilde{v}_n)\,dx \longrightarrow l.\end{aligned} \tag{63}$$

If $l=0$, the proof is complete. Assume $l>0$; then from (63), we obtain

$$\begin{aligned}S_F l^{2/\alpha} &= S_F \lim_{n\to\infty}\left(\int_\Omega F(x,\tilde{u}_n,\tilde{v}_n)\,dx\right)^{2/\alpha}\\ &\le \lim_{n\to\infty}\left(a_1\|\tilde{u}_n\|^2 + a_2\|\tilde{v}_n\|^2 + b_1\|\tilde{u}_n\|^4 + b_2\|\tilde{v}_n\|^4\right) = l\end{aligned} \tag{64}$$

which implies that $l \ge S_F^{2N/(N+2)}$. In addition, from Lemma 12, (61), and (63), we get

$$c \ge \left(\frac{1}{4}-\frac{1}{\alpha}\right)l + J_{\lambda,\mu}(u,v) \ge \left(\frac{1}{4}-\frac{1}{\alpha}\right)S_F^{2N/(N+2)} - \Lambda\varrho_{\lambda,\mu}, \tag{65}$$

which contradicts $c < (1/4-1/\alpha)S_F^{2N/(N+2)} - \Lambda\varrho_{\lambda,\mu}$. □

Lemma 16. *There exists a nonnegative function $(u,v) \in E\setminus\{(0,0)\}$ and $C^*>0$ such that, for $\varrho_{\lambda,\mu}\in(0,C^*)$, one has*

$$\sup_{t\ge 0} J_{\lambda,\mu}(tu,tv) < c_\infty. \tag{66}$$

In particular, $\theta^-_{\lambda,\mu} < c_\infty$ for all $\varrho_{\lambda,\mu}\in(0,C^*)$.

Proof. Since $0\in\Omega$, there is $\rho_0>0$ such that $B^N(0;2\rho_0)\subset\Omega$. Now, we consider the functional $I: E\to R$ defined by

$$I(u,v) = \frac{a_1}{2}\|u\|^2 + \frac{a_2}{2}\|v\|^2 - \frac{1}{\alpha}\int_\Omega F(x,u,v)\,dx \quad \forall (u,v)\in E \tag{67}$$

and define a cut-off function $\eta(x)\in C_0^\infty(\Omega)$ such that $\eta(x)=1$ for $|x|<\rho_0$, $\eta(x)=0$ for $|x|>2\rho_0$, $0\le\eta\le 1$, and $|\nabla\eta|\le C$. For $\varepsilon>0$, let

$$u_\varepsilon(x) = \frac{\eta(x)}{(\varepsilon+|x|^2)^{(N-2)/2}}. \tag{68}$$

From [14], we have the following estimates:

$$\begin{aligned}&\left(\int_\Omega |u_\varepsilon|^\alpha dx\right)^{2/\alpha} = \varepsilon^{-(N-2)/2}\|U\|^2_{L^\alpha(R^N)} + O(\varepsilon),\\ &\int_\Omega |\nabla u_\varepsilon|^2 dx = \varepsilon^{-(N-2)/2}\|\nabla U\|^2_{L^2(R^N)} + O(1),\\ &\frac{\int_\Omega\left(|\nabla u_\varepsilon|^2 + a(x)|u_\varepsilon|^2\right)dx}{\left(\int_\Omega |u_\varepsilon|^\alpha dx\right)^{2/\alpha}} = S + O\left(\varepsilon^{(N-2)/2}\right),\end{aligned} \tag{69}$$

where $U(x) = (1+|x|^2)^{-(N-2)/2}\in W^{1,2}(R^N)$ is a minimizer of

$$\left\{\frac{\int_\Omega\left(|\nabla u|^2 + a(x)|u|^2\right)dx}{\|u\|^2_{L^\alpha(R^N)}}\right\}_{u\in W^{1,2}(R^N)\setminus\{0\}}; \tag{70}$$

that is,

$$\begin{aligned}&\frac{\int_\Omega\left(|\nabla U|^2 + a(x)|U|^2\right)dx}{\|U\|^2_{L^\alpha(R^N)}}\\ &\quad = S = \inf_{u\in W^{1,2}(R^N)\setminus\{0\}}\frac{\int_\Omega\left(|\nabla u|^2 + a(x)|u|^2\right)dx}{\|u\|^2_{L^\alpha(R^N)}}.\end{aligned} \tag{71}$$

Set $u_0 = e_1 u_\varepsilon$, $v_0 = e_2 u_\varepsilon$ and $(u_0, v_0) \in E$, where $(e_1, e_2) \in (R^+)^2$ and $\inf_{x\in\overline{\Omega}} F(x, e_1, e_2) \geq K$. Then, by (F1), (9), the definition of S_F, and (69), we obtain that

$$\begin{aligned}
&\sup_{t\geq 0} I\left(te_1u_\varepsilon, te_2u_\varepsilon\right)\\
&\leq \frac{N+2}{4N}\left(\frac{a_1\left\|e_1u_\varepsilon\right\|^2 + a_2\left\|e_2u_\varepsilon\right\|^2}{\left(\int_\Omega F\left(x, e_1u_\varepsilon, e_2u_\varepsilon\right)dx\right)^{2/\alpha}}\right)^{2N/(N+2)}\\
&\leq \frac{N+2}{4N}\left(\frac{\left(a_1e_1^2 + a_2e_2^2\right)\left\|u_\varepsilon\right\|^2}{\left(\int_\Omega \left|u_\varepsilon\right|^\alpha F\left(x, e_1, e_2\right)dx\right)^{2/\alpha}}\right)^{2N/(N+2)}\\
&\leq \frac{N+2}{4N}\left(\frac{\left(a_1e_1^2 + a_2e_2^2\right)}{K^{2/\alpha}}\right)^{2N/(N+2)}\\
&\quad\times\left(\frac{\left\|u_\varepsilon\right\|^2}{\left(\int_\Omega \left|u_\varepsilon\right|^\alpha dx\right)^{2/\alpha}}\right)^{2N/(N+2)}\\
&\leq \frac{N+2}{4N}\left(\frac{\left(a_1e_1^2 + a_2e_2^2\right)}{K^{2/\alpha}}\right)^{2N/(N+2)}\\
&\quad\times\left(S + O\left(\varepsilon^{(N-2)/2}\right)\right)^{2N/(N+2)}\\
&= \frac{N+2}{4N}\left(\frac{\left(a_1e_1^2 + a_2e_2^2\right)}{K^{2/\alpha}}\right)^{2N/(N+2)}\\
&\quad\times\left(S^{2N/(N+2)} + O\left(\varepsilon^{(N-2)/2}\right)\right)\\
&\leq \frac{N+2}{4N}S_F^{2N/(N-2)} + O\left(\varepsilon^{(N-2)/2}\right),
\end{aligned} \tag{72}$$

where the following fact has been used:

$$\sup_{t\geq 0}\left(\frac{t^2}{2}A - \frac{t^\alpha}{\alpha}B\right) = \frac{N+2}{4N}\left(\frac{A}{B^{2/\alpha}}\right)^{(N-2)/2} \qquad A, B > 0. \tag{73}$$

We can choose $\delta_1 > 0$ such that, for all $\varrho_{\lambda,\mu} \in (0, \delta_1)$, we have

$$c_\infty = \left(\frac{1}{4} - \frac{1}{\alpha}\right)S_F^{2N/(N+2)} - \Lambda\varrho_{\lambda,\mu} > 0. \tag{74}$$

Using the definitions of $J_{\lambda,\mu}$ and (u_0, v_0), we get

$$J_{\lambda,\mu}\left(tu_0, tv_0\right) \leq \frac{t^2\max\left\{a_1, a_2\right\}}{2}\left\|\left(u_0, v_0\right)\right\|_E^2 \quad \forall t \geq 0,\ \lambda, \mu > 0, \tag{75}$$

which implies that there exists $t_0 \in (0, 1)$ satisfying

$$\sup_{0\leq t\leq t_0} J_{\lambda,\mu}\left(t_0u_0, t_0v_0\right) < c_\infty, \quad \forall\varrho_{\lambda,\mu} \in \left(0, \delta_1\right). \tag{76}$$

On the other hand,

$$\begin{aligned}
&\sup_{t\geq t_0} J_{\lambda,\mu}\left(tu_0, tv_0\right)\\
&\quad= \sup_{t\geq t_0}\left(I\left(tu_0, tv_0\right) + \frac{b_1}{4}\left\|tu_0\right\|^4 + \frac{b_2}{4}\left\|tv_0\right\|^4\right.\\
&\qquad\left. - \frac{t^q}{q}K_{\lambda,\mu}\left(u_0, v_0\right)\right)\\
&\quad\leq \frac{N+2}{4N}S_F^{2N/(N-2)} + O\left(\varepsilon^{(N-2)/2}\right)\\
&\qquad+ \frac{\max\left\{b_1, b_2\right\}t^4\left(e_1^4 + e_2^4\right)}{2}\left\|u_\varepsilon\right\|^4\\
&\qquad- \frac{t_0^q}{q}\left(e_1^q\lambda b_0 + e_2^q\mu c_0\right)\int_{B^N(0,\rho_0)}\left|u_\varepsilon\right|^q dx\\
&\quad\leq \frac{N+2}{4N}S_F^{2N/(N-2)} + O\left(\varepsilon^{(N-2)/2}\right)\\
&\qquad+ \frac{\max\left\{b_1, b_2\right\}t^4\left(e_1^4 + e_2^4\right)}{2}\left\|u_\varepsilon\right\|^4\\
&\qquad- \frac{t_0^q}{q}m\left(\lambda b_0 + \mu c_0\right)\int_{B^N(0,\rho_0)}\left|u_\varepsilon\right|^q dx,
\end{aligned} \tag{77}$$

where $m = \min\{e_1^q, e_2^q\}$, $b_0 = \min b(x)$, and $c_0 = \min c(x)$ on $B(0, \rho_0)$.

Let $0 < \varepsilon \leq \rho_0^2$; we get

$$\begin{aligned}
\int_{B^N(0,\rho_0)}\left|u_\varepsilon\right|^q dx &= \int_{B^N(0,\rho_0)}\frac{1}{\left(\varepsilon + |x|^2\right)^{((N-2)/2)q}}dx\\
&\geq \int_{B^N(0,\rho_0)}\frac{1}{\left(2\rho_0^2\right)^{((N-2)/2)q}}dx\\
&= C_2\left(N, q, \rho_0\right).
\end{aligned} \tag{78}$$

Combining with (77) and the above inequality, for all $\varepsilon = \varrho_{\lambda,\mu}^{2/(N-2p)} \in (0, \rho_0^2)$, we have the following.

According to properties of u_ε and $F((F_1)$–$(F_3), 4.4)$, we can conclude that there exists the positive constant C_3 such that $\|u_\varepsilon\|^4 \leq C_3$, so we have

$$\begin{aligned}
&\sup_{t\geq t_0} J_{\lambda,\mu}\left(tu_0, tv_0\right)\\
&\quad\leq \frac{N+2}{4N}S_F^{2N/(N+2)} + O\left(\varrho_{\lambda,\mu}\right)\\
&\qquad+ \frac{\max\left\{b_1, b_2\right\}t^4\left(e_1^4 + e_2^4\right)}{2}C_3 - \frac{t_0^q}{q}\varrho_{\lambda,\mu}mC_2.
\end{aligned} \tag{79}$$

There exists a constant Λ_1, such that $((N+2)/4N)S_F^{2N/(N+2)} - \Lambda_1\varrho_{\lambda,\mu} \leq c_\infty$.

Hence, we can choose $\delta_2 > 0$ such that, for all $\varrho_{\lambda,\mu} \in (0, \delta_2)$, we obtain

$$O\left(\varrho_{\lambda,\mu}\right) + \frac{\max\{b_1, b_2\} t^4 \left(e_1^4 + e_2^4\right)}{2} C_3 - \frac{t_0^q}{q} \varrho_{\lambda,\mu} m C_2 < -\Lambda_1 \varrho_{\lambda,\mu}. \tag{80}$$

If we see $C^* = \min\{\delta_1, \rho_0^{N-2}, \delta_2\}$ and $\varepsilon = \varrho_{\lambda,\mu}^{2/(N-2p)}$, then for $\lambda \in (0, C^*)$ we have

$$\sup_{t \geq t_0} J_{\lambda,\mu}\left(t u_0, t v_0\right) < c_\infty. \tag{81}$$

Finally, we prove that $\theta_{\lambda,\mu}^- < c_\infty$ for all $\varrho_{\lambda,\mu} \in (0, C^*)$. Recall that $(u_0, v_0) = (e_1 u_\varepsilon, e_2 u_\varepsilon)$. It is easy to see that

$$\int F\left(x, u_0, v_0\right) dx > 0. \tag{82}$$

Combining this with Lemma 9, from the definition of $\theta_{\lambda,\mu}^-$ and (81), we get that there exists $t_0 > 0$ such that $(t_0 u_0, t_0 v_0) \in N_{\lambda,\mu}^-$ and

$$\theta_{\lambda,\mu}^- \leq J_{\lambda,\mu}\left(t_0 u_0, t_0 v_0\right) \leq \sup_{t \geq 0} J_{\lambda,\mu}\left(t_0 u_0, t_0 v_0\right) < c_\infty \tag{83}$$

for all $\varrho_{\lambda,\mu} \in (0, C^*)$. □

Theorem 17. *If $\varrho_{\lambda,\mu} \in (0, C_0^*)$ and (F1)–(F3) hold, then $J_{\lambda,\mu}$ has a minimizer (u_0^-, v_0^-) in $N_{\lambda,\mu}^-$ and it satisfies the following:*

(i) $J_{\lambda,\mu}(u_0^-, v_0^-) = \theta_{\lambda,\mu}^-$,

(ii) (u_0^-, v_0^-) *is a positive solution of problem (1),*

where $C_0^* = \min\{C^*, C_0\}$.

Proof. By Lemma 10(ii), there is a $(PS)_{\theta_{\lambda,\mu}^-}$-sequence $\{(u_n, v_n)\} \subseteq N_{\lambda,\mu}^-$ in E for $J_{\lambda,\mu}$ for all $\varrho_{\lambda,\mu} \in (0, C_0)$. From Lemmas 15, 16, and 8(ii), for $\varrho_{\lambda,\mu} \in (0, C^*)$, $J_{\lambda,\mu}$ satisfies $(PS)_{\theta_{\lambda,\mu}^-}$ condition and $\theta_{\lambda,\mu}^- > 0$. Since $J_{\lambda,\mu}$ is coercive on $N_{\lambda,\mu}$, we get that (u_n, v_n) is bounded in E. Therefor, there exists a subsequence still denoted by (u_n, v_n) and $(u_0^-, v_0^-) \in N_{\lambda,\mu}^-$ such that $(u_n, v_n) \to (u_0^-, v_0^-)$ strongly in E and $J_{\lambda,\mu}(u_0^-, v_0^-) = \theta_{\lambda,\mu}^- > 0$ for all $\varrho_{\lambda,\mu} \in (0, C_0^*)$. Finally, by the same arguments as in the proof of Theorem 11, for all $\varrho_{\lambda\mu} \in (0, C_0^*)$, we have that u_0^- is a positive solution of problem (1). □

Now, we complete the proof of Theorems 1 and 2. By Theorem 11, we obtain that, for all $\varrho_{\lambda,\mu} \in C(q, N, K, S, |\Omega|, a_1, a_2)$, problem (1) has a positive solution $(u_0^+, v_0^+) \in N_{\lambda,\mu}^+$. On the other hand, from Theorem 17, we get the second positive solution $\left(u_0^-, v_0^-\right) \in N_{\lambda,\mu}^-$ for all $0 < \varrho_{\lambda,\mu} < C_0^* < C(q, N, K, S, |\Omega|, a_1, a_2)$. Since $N_{\lambda,\mu}^+ \cap N_{\lambda,\mu}^- = \emptyset$, this implies that (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct. This completes the proof of Theorems 1 and 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] G. R. Kirchhoff, *Vorlesungen über mathematische Physik: Mechanik*, B.G. Teubner, Leipzig, Germany, 1883.

[2] F. Julio, S. A. Correa, and G. M. Figueiredo, "On an elliptic equation of p-Kirchhoff type via variational methods," *Bulletin of the Australian Mathematical Society*, vol. 74, pp. 246–277, 2006.

[3] C. Y. Chen, Y. C. Kuo, and T. F. Wu, "The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions," *Journal of Differential Equations*, vol. 250, no. 4, pp. 1876–1908, 2011.

[4] J. H. Jin and X. Wu, "Infinitly many radial solutions for Kirchhoff type problem in $\mathbb{R}^N$," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 2, pp. 564–574, 2010.

[5] X. M. He and W. M. Zou, "Infinitely many positive solutions for Kirchhoff-type problems," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 70, no. 3, pp. 1407–1414, 2009.

[6] K. Perera and Z. T. Zhang, "Nontrivial solutions of Kirchhoff-type problems via the Yang index," *Journal of Differential Equations*, vol. 221, no. 1, pp. 246–255, 2006.

[7] M. Ferrara, S. Khademloo, and S. Heidarkhani, "Multiplicity results for perturbed fourth-order Kirchhoff type elliptic problems," *Applied Mathematics and Computation*, vol. 234, pp. 316–325, 2014.

[8] F. Julio, S. A. Correa, and R. G. Nascimento, "On a nonlocal elliptic system of *p*-Kirchhoff-type under Neumann boundary condition," *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 598–604, 2009.

[9] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, Mass, USA, 1996.

[10] Y. Shen and J. Zhang, "Multiplicity of positive solutions for a semilinear *p*-Laplacian system with Sobolev critical exponent," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 4, pp. 1019–1030, 2011.

[11] K. J. Brown and T.-F. Wu, "A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1326–1336, 2008.

[12] T. F. Wu, "A semilinear elliptic equations involving nonlinear bou ndary codition and signchanging potential," *Electronic Journal of Differential Equations*, vol. 131, pp. 1–15, 2006.

[13] H. Brézis and A. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, pp. 486–490, 1983.

[14] T.-S. Hsu, "Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2688–2698, 2009.