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Research Article

On Stability of a Third Order of Accuracy Difference Scheme for Hyperbolic Nonlocal BVP with Self-Adjoint Operator

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A third order of accuracy absolutely stable difference schemes is presented for nonlocal boundary value hyperbolic problem of the differential equations in a Hilbert space H with self-adjoint positive definite operator A. Stability estimates for solution of the difference scheme are established. In practice, one-dimensional hyperbolic equation with nonlocal boundary conditions is considered.

1. Introduction

In modeling several phenomena of physics, biology, and ecology mathematically, there often arise problems with nonlocal boundary conditions (see [1–5] and the references given therein). Nonlocal boundary value problems have been a major research area in the case when it is impossible to determine the boundary conditions of the unknown function. Over the last few decades, the study of nonlocal boundary value problems is of substantial contemporary interest (see, e.g., [6–14] and the references given therein).

We consider the nonlocal boundary value problem

$$\begin{gathered}\frac{d^2u(t)}{dt^2}+Au(t)=f(t),\quad 0<t<1,\\ u(0)=\alpha u(1)+\varphi,\\ u'(0)=\beta u'(1)+\psi,\end{gathered}\tag{1}$$

for hyperbolic equations in a Hilbert space H with self-adjoint positive definite linear operator A with domain $D(A)$.

A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied.

(i) $u(t)$ is twice continuously differentiable on the segment $[0,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t\in[0,1]$ and the function $Au(t)$ is continuous on the segment $[0,1]$.

(iii) $u(t)$ satisfies the equations and the nonlocal boundary conditions (1).

Here, $\varphi(x)$, $\psi(x)$ $(x\in[0,1])$ and $f(t,x)$ $(t,x\in[0,1])$ are smooth functions.

In the study of numerical methods for solving PDEs, stability is an important research area (see [6–27]). Many scientists work on difference schemes for hyperbolic partial differential equations, in which stability was established under the assumption that the magnitudes of the grid steps τ and h with respect to the time and space variables are connected. This particularly means that $\tau\|A_h\|\to 0$ when $\tau\to 0$.

We are interested in studying high order of accuracy unconditionally stable difference schemes for hyperbolic PDEs.

In the present paper, third order of accuracy difference scheme generated by integer power of A for approximately solving nonlocal boundary value problem (1) is presented.

The stability estimates for solution of the difference scheme are established.

In [8], some results of this paper, without proof, were presented.

The well posedness of nonlocal boundary value problems for parabolic equations, elliptic equations, and equations of mixed types have been studied extensively by many scientists (see, e.g., [11–14, 19–32] and the references therein).

2. Third Order of Accuracy Difference Scheme Subject to Nonlocal Conditions

In this section, we obtain stability estimates for the solution of third order of accuracy difference scheme

$$\begin{gathered}
\tau^{-2}\left(u_{k+1}-2u_k+u_{k-1}\right)+\frac{2}{3}Au_k+\frac{1}{6}A\left(u_{k+1}+u_{k-1}\right)+\frac{1}{12}\tau^2A^2u_{k+1}=f_k,\\
f_k=\frac{2}{3}f\left(t_k\right)+\frac{1}{6}\left(f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right)-\frac{1}{12}\tau^2\left(-Af\left(t_{k+1}\right)+f''\left(t_{k+1}\right)\right),\\
t_k=k\tau,\quad 1\le k\le N-1,\ N\tau=1,\\
u_0=\alpha u_N+\varphi,\\
\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\tau^{-1}\left(u_1-u_0\right)+\frac{\tau}{2}Au_0-\tau f_{1,1}
=\beta\left(I-\frac{\tau^2A}{12}\right)\times\left(\frac{7u_N-8u_{N-1}+u_{N-2}}{6\tau}+\frac{\tau}{3}\left(f_N-Au_N\right)\right)+\left(I-\frac{\tau^2A}{12}\right)\psi
\end{gathered}\tag{2}$$

for numerical solution of nonlocal boundary value problem (1). Here,

$$f_{1,1}=f\left(0\right)+\left(-f\left(0\right)+\tau f'\left(0\right)\right)\frac{1}{2}-2f'\left(0\right)\frac{\tau}{6}.\tag{3}$$

We study the stability of solutions of difference scheme (2) under the following assumption:

$$|\alpha|+2\left|\beta\right|+2\left|\alpha\right|\left|\beta\right|<1.\tag{4}$$

We give a lemma that will be needed in the sequel which was presented in [18]. First, let us present the following operators:

$$R=\left(I-\frac{1}{3}\tau^2A+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\times\left(I+\frac{1}{6}\tau^2A+\frac{1}{12}\tau^4A^2\right)^{-1},\tag{5}$$

and its conjugate $\widetilde{R}$,

$$\begin{aligned}
R_1=&\left(-\frac{5\tau^4}{144}A^2+\frac{7\tau^6}{216}A^3-i\tau A^{1/2}\right.\\
&\times\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\left.\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\\
&\times\left(-i\tau A^{1/2}\left(\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\right.\\
&\times\left.\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\right)^{-1},
\end{aligned}\tag{6}$$

and its conjugate $\widetilde{R}_1$,

$$\begin{aligned}
R_2=&\left(I-\frac{\tau^2}{12}A\right)\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\\
&\times\left(-iA^{1/2}\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\sqrt{I+\frac{1}{72}\tau^4A^2}\right)^{-1},\\
R_3=&\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\\
&\times\left(\left(I+\frac{\tau^2}{12}A+\frac{\tau^4}{144}A^2\right)\left(-i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\right)^{-1},\\
R_4=&\left(I+\frac{\tau^2}{3}A+\frac{\tau^4}{9}A^2+\frac{\tau^6}{72}A^3\right)\\
&\times\left(-iA^{1/2}\left(\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)\right.\\
&\times\left.\left(I+\frac{\tau^2}{6}A\right)\right)^{-1},\\
R_5=&\left(-\frac{\tau^2}{2}A-\frac{\tau^4}{12}A^2+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\\
&\times\left(I+\frac{\tau^2}{6}A+\frac{\tau^4}{12}A^2\right)^{-1},
\end{aligned}\tag{7}$$

and its conjugate $\widetilde{R}_5$, and

$$\begin{aligned}
R_6=&\left(I-\frac{1}{3}\tau^2A+i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)\\
&\times\left(\frac{\tau^2}{2}A+\frac{\tau^4}{12}A^2-i\tau A^{1/2}\sqrt{I+\frac{1}{72}\tau^4A^2}\right)^{-1},
\end{aligned}\tag{8}$$

and its conjugate $\widetilde{R}_6$.

We consider the following operators:

$$R_7 = \frac{(7R - I)}{6\tau} = \left(I - \frac{5}{12}\tau^2 A + \frac{1}{72}\tau^4 A^2 + \frac{7}{6} i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right) \times \tau^{-1}\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1}, \quad (9)$$

and its conjugate $\widetilde{R}_7$,

$$\widetilde{R}_7 = \frac{(7\widetilde{R} - I)}{6\tau} = \left(I - \frac{5}{12}\tau^2 A + \frac{1}{72}\tau^4 A^2 - \frac{7}{6} i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\right) \times \tau^{-1}\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-1},$$

$$R_8 = \left(\frac{7I - 2\tau^2 A}{6\tau}\right)\left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72}\right) \times \tau^{-1}\left(I + \frac{\tau^2 A}{6}\right)^{-1}\left(I + \frac{\tau^2}{6}A + \frac{\tau^4}{12}A^2\right)^{-2},$$

$$R_9 = \left(I - \frac{5}{3}\tau^2 A + \frac{\tau^4 A^2}{9}\right)\left(I + \frac{\tau^2 A}{3} + \frac{\tau^4 A^2}{9} + \frac{\tau^6 A^3}{72}\right) \times \tau^{-1}\left(I + \frac{\tau^2 A}{6}\right)^{-1}\left(I + \frac{1}{6}\tau^2 A + \frac{1}{12}\tau^4 A^2\right)^{-3},$$

$$R_{10} = I + \left(\frac{5}{144}\tau^4 A^2 - \frac{9}{288}\tau^6 A^3 + \frac{9}{1728}\tau^8 A^4\right) \times \left(i\tau A^{1/2}\sqrt{I + \frac{1}{72}\tau^4 A^2}\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)\right)^{-1}, \quad (10)$$

and its conjugate $\widetilde{R}_{10}$.

Lemma 1. *The following estimates hold:*

$$\begin{aligned} &\|R\|_{H\to H} \le 1, && \|\widetilde{R}\|_{H\to H} \le 1, \\ &\|R_1\|_{H\to H} \le 1, && \|\widetilde{R}_1\|_{H\to H} \le 1, \\ &\|A^{1/2}R_2\|_{H\to H} \le 1, && \|\tau A^{1/2}R_3\|_{H\to H} \le 1, \\ &\|A^{1/2}R_4\|_{H\to H} \le 1, && \|A^{-1/2}R_5\|_{H\to H} \le \tau, \\ &\|A^{-1/2}\widetilde{R}_5\|_{H\to H} \le \tau, && \|\tau A^{1/2}R_6\|_{H\to H} \le 1, \\ &\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H} \le 1. \end{aligned} \quad (11)$$

Now let us give, without proof, the second lemma.

Lemma 2. *The following estimates hold:*

$$\begin{aligned} &\left\|\left(I + i\tau A^{1/2}\right)R\right\|_{H\to H} \le 2, \\ &\left\|\left(I + i\tau A^{1/2}\right)\widetilde{R}\right\|_{H\to H} \le 2, \\ &\|\tau R_7\|_{H\to H} \le 1, \quad \|\tau \widetilde{R}_7\|_{H\to H} \le 1, \\ &\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H} \le 1, \quad \left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H} \le 1, \\ &\|\tau R_8\|_{H\to H} \le \frac{7}{6}, \quad \|\tau R_9\|_{H\to H} \le 1, \\ &\left\|R_{10}\left(I + i\tau A^{1/2}\right)^{-1}\right\|_{H\to H} \le 2, \\ &\left\|\widetilde{R}_{10}\left(I + i\tau A^{1/2}\right)^{-1}\right\|_{H\to H} \le 2. \end{aligned} \quad (12)$$

Throughout the section, for simplicity, we denote

$$\begin{aligned} B_\tau = {} & \beta\frac{1}{2}R_2\left(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\right)\widetilde{R}^{N-2} \\ & + \beta\frac{1}{2}R_2\left(R_7 R_5 - \frac{\tau A}{3}R^2\right)R^{N-2} \\ & - \alpha\frac{1}{2}\left[\widetilde{R}_1 R^N - R_1\widetilde{R}^N\right] \\ & + \alpha\beta\frac{1}{4}\widetilde{R}_1 R_2\left(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\right)R^N\widetilde{R}^{N-2} \\ & + \alpha\beta\frac{1}{4}R_1 R_2\left(R_7 R_5 - \frac{\tau A}{3}R^2\right)\widetilde{R}^N R^{N-2} \\ & - \alpha\beta\frac{1}{4}\widetilde{R}_1 R_2\left(R_7 R_5 - \frac{\tau A}{3}R^2\right)\widetilde{R}^N R^{N-2} \\ & - \alpha\beta\frac{1}{4}R_1 R_2\left(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\right)R^N\widetilde{R}^{N-2}. \end{aligned} \quad (13)$$

Lemma 3. *Suppose that assumption* (4) *holds. Then, the operator $I - B_\tau$ has an inverse $T_\tau = (I - B_\tau)^{-1}$. From symmetry and positivity properties of operator A, the following estimate is satisfied:*

$$\|T_\tau\|_{H\mapsto H} \le \frac{1}{1 - |\alpha| - 2|\beta| - 2|\alpha||\beta|}. \quad (14)$$

Proof. Using the definitions of B_τ, R, $\widetilde{R}$, estimates (11), and the following simple estimates,

$$\begin{aligned} &\left\|\tau A^{1/2}\left(I - \frac{\tau^2}{12}A\right)\left(I + \frac{\tau^2}{12}A + \frac{\tau^4}{144}A^2\right)^{-1}\right\|_{H\to H} \le 12, \\ &\left\|\tau A^{1/2}\left(I + \frac{1}{12}\tau^2 A + \frac{1}{144}\tau^4 A^2\right)^{-1}\right\|_{H\to H} \le \frac{12\sqrt{11}}{12 + \sqrt{11}}, \end{aligned} \quad (15)$$

and the triangle inequality, we get

$$\begin{aligned}
B_\tau \le {} & |\beta| \frac{1}{2} \left\| A^{1/2} R_2 \right\|_{H\to H} \left\| \widetilde{R}^{N-2} \right\|_{H\to H} \\
& \times \left(\left\| \tau \widetilde{R}_7 \right\|_{H\to H} \left\| \tau^{-1} A^{-1/2} \widetilde{R}_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{1/2} \widetilde{R}^2 \right\|_{H\to H} \right) \\
& + |\beta| \frac{1}{2} \left\| A^{1/2} R_2 \right\|_{H\to H} \times \left\| R^{N-2} \right\|_{H\to H} \\
& \times \left(\left\| \tau R_7 \right\|_{H\to H} \left\| \tau^{-1} A^{-1/2} R_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H\to H} \right) \\
& + \alpha \frac{1}{2} \left[\left\| \widetilde{R}_1 \right\|_{H\to H} \left\| R^N \right\|_{H\to H} + \left\| R_1 \right\|_{H\to H} \left\| \widetilde{R}^N \right\|_{H\to H} \right] \\
& + |\alpha| \, |\beta| \frac{1}{4} \left\| \widetilde{R}_1 \right\|_{H\to H} \left\| A^{1/2} R_2 \right\|_{H\to H} \\
& \times \left\| R^N \right\|_{H\to H} \left\| \widetilde{R}^{N-2} \right\|_{H\to H} \\
& \times \left(\left\| \tau \widetilde{R}_7 \right\|_{H\to H} \left\| \tau^{-1} A^{-1/2} \widetilde{R}_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{1/2} \widetilde{R}^2 \right\|_{H\to H} \right) \\
& + |\alpha| \, |\beta| \frac{1}{4} \left\| R_1 \right\|_{H\to H} \left\| A^{1/2} R_2 \right\|_{H\to H} \\
& \times \left\| \widetilde{R}^N \right\|_{H\to H} \left\| R^{N-2} \right\|_{H\to H} \\
& \times \left(\left\| \tau R_7 \right\|_{H\to H} \left\| \tau^{-1} A^{-1/2} R_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H\to H} \right) \\
& + |\alpha| \, |\beta| \frac{1}{4} \left\| \widetilde{R}_1 \right\|_{H\to H} \left\| A^{1/2} R_2 \right\|_{H\to H} \\
& \times \left\| \widetilde{R}^N \right\|_{H\to H} \left\| R^{N-2} \right\|_{H\to H} \\
& \times \left(\left\| \tau R_7 \right\|_{H\to H} \left\| \tau^{-1} A^{-1/2} R_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{1/2} R^2 \right\|_{H\to H} \right) \\
& + |\alpha| \, |\beta| \frac{1}{4} \left\| R_1 \right\|_{H\to H} \left\| A^{1/2} R_2 \right\|_{H\to H} \\
& \times \left\| R^N \right\|_{H\to H} \left\| \widetilde{R}^{N-2} \right\|_{H\to H} \\
& \times \left(\left\| \tau \widetilde{R}_7 \right\|_{H\to H} \left\| \tau^{-1} A^{-1/2} \widetilde{R}_5 \right\|_{H\to H} + \left\| \frac{1}{3} \tau A^{1/2} \widetilde{R}^2 \right\|_{H\to H} \right) \\
\le {} & q,
\end{aligned} \tag{16}$$

where

$$q = |\alpha| + 2|\beta| + 2|\alpha||\beta|. \tag{17}$$

Since $q < 1$, the operator $I - B_\tau$ has a bounded inverse and

$$\left\| (I - B_\tau)^{-1} \right\|_{H\to H} \le \frac{1}{1-q} = \frac{1}{1 - |\alpha| - 2|\beta| - 2|\alpha||\beta|}. \tag{18}$$

Lemma 3 is proved. □

Now, let us obtain formula for the solution of problem (2). Using the results of [18], one can obtain the following formula:

$$\begin{aligned}
u_0 &= \mu, \\
u_1 &= \left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right)^{-1} \\
&\quad \times \left(\left(I - \frac{5}{12} \tau^2 A + \frac{\tau^4}{144} A^2 \right) \mu \right. \\
&\quad \left. + \tau \left(I - \frac{\tau^2}{12} A \right) \omega + \tau^2 f_{1,1} \right), \\
u_k &= \frac{1}{2} \left[\widetilde{R}_{10} R^k - R_{10} \widetilde{R}^k \right] \mu + \frac{1}{2} \left[\widetilde{R}^k - R^k \right] R_2 \omega \\
&\quad + \frac{1}{2} \left[\widetilde{R}^k - R^k \right] R_3 \tau^2 f_{1,1} + \frac{1}{2} R_4 \sum_{s=1}^{k-1} \left[\widetilde{R}^{k-s} - R^{k-s} \right] f_s \tau^2
\end{aligned} \tag{19}$$

for the solution of difference scheme

$$\begin{aligned}
& \tau^{-2} \left(u_{k+1} - 2u_k + u_{k-1} \right) + \frac{2}{3} A u_k + \frac{1}{6} A \left(u_{k+1} + u_{k-1} \right) \\
& \quad + \frac{1}{12} \tau^2 A^2 u_{k+1} = f_k, \\
& f_k = \frac{2}{3} f(t_k) + \frac{1}{6} \left(f(t_{k+1}) + f(t_{k-1}) \right) \\
& \quad - \frac{1}{12} \tau^2 \left(-A f(t_{k+1}) + f''(t_{k+1}) \right), \\
& t_k = k\tau, \quad 1 \le k \le N-1, \ N\tau = 1, \\
& u_0 = \mu, \\
& \left(I + \frac{\tau^2}{12} A + \frac{\tau^4}{144} A^2 \right) \tau^{-1} (u_1 - u_0) + \frac{\tau}{2} A u_0 - \tau f_{1,1} \\
& \quad = \left(I - \frac{\tau^2 A}{12} \right) \omega.
\end{aligned} \tag{20}$$

Applying formula (19) and nonlocal boundary conditions

$$\begin{aligned}
u_0 &= \alpha u_N + \varphi, \\
\omega &= \beta \left(\frac{7u_N - 8u_{N-1} + u_{N-2}}{6\tau} + \frac{\tau}{3} \left(f_N - A u_N \right) \right) + \psi,
\end{aligned} \tag{21}$$

one can write

$$\mu = \alpha\left\{\frac{1}{2}\left[\widetilde{R}_{10}R^N - R_{10}\widetilde{R}^N\right]\mu + \frac{1}{2}\left[\widetilde{R}^N - R^N\right]R_2\omega + \frac{1}{2}\left[\widetilde{R}^N - R^N\right]R_3\tau^2 f_{1,1} + \frac{1}{2}R_4\sum_{s=1}^{N-1}\left[\widetilde{R}^{N-s} - R^{N-s}\right]f_s\tau^2\right\} + \varphi,$$

$$\begin{aligned}\omega = \beta\Big\{&\frac{1}{2}\Big[\Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)\widetilde{R}_{10}R^{N-2} - \Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)R_{10}\widetilde{R}^{N-2}\Big]\mu\\ &+ \frac{1}{2}\Big[\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)R^{N-2}\Big]R_2\omega\\ &+ \frac{1}{2}\Big[\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)R^{N-2}\Big]R_3\tau^2 f_{1,1}\\ &+ \frac{\tau}{3}f_N + \frac{1}{2}R_8f_{N-1}\tau^2 + \frac{1}{2}R_9f_{N-2}\tau^2\\ &+ \frac{1}{2}R_4\tau\sum_{s=1}^{N-3}\Big[\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2-s} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)R^{N-2-s}\Big]f_s\tau\Big\} + \psi.\end{aligned} \tag{22}$$

Using formulas in (22), we obtain

$$\begin{aligned}\mu = T_\tau\Bigg\{&\Big[\alpha\Big(\frac{1}{2}\big(\widetilde{R}^N - R^N\big)R_3\tau^2 f_{1,1} + \frac{1}{2}R_4\tau\sum_{s=1}^{N-1}\big(\widetilde{R}^{N-s} - R^{N-s}\big)f_s\tau\Big) + \varphi\Big]\\ &\times\Big[I - \frac{1}{2}\Big(\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)R^{N-2}\Big)R_2\Big] + \Big[\alpha\frac{1}{2}\big(\widetilde{R}^N - R^N\big)R_2\Big]\\ &\times\Big[\beta\frac{1}{2}\Big\{\Big(\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)R^{N-2}\Big)\times R_3\tau^2 f_{1,1} + \frac{2\tau}{3}f_N + R_8f_{N-1}\tau^2\\ &+ R_9f_{N-2}\tau^2 + R_4\tau\sum_{s=1}^{N-3}\Big[\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2-s} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)\times R^{N-2-s}\Big]f_s\tau\Big\} + \psi\Big]\Bigg\},\end{aligned}$$

$$\begin{aligned}\omega = T_\tau\Bigg\{&\Big[I - \alpha\frac{1}{2}\big(\widetilde{R}_{10}R^N - R_{10}\widetilde{R}^N\big)\Big]\\ &\times\Big[\beta\frac{1}{2}\Big\{\Big(\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)R^{N-2}\Big)\times R_3\tau^2 f_{1,1} + \frac{2\tau}{3}f_N + R_8f_{N-1}\tau^2\\ &+ R_9f_{N-2}\tau^2 + R_4\tau\times\sum_{s=1}^{N-3}\Big(\Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2-s} - \Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)\times R^{N-2-s}\Big)f_s\tau\Big\} + \psi\Big]\\ &+ \frac{1}{2}\Big[\Big(R_7R_5 - \frac{\tau A}{3}R^2\Big)\widetilde{R}_{10}R^{N-2} + \Big(\widetilde{R}_7\widetilde{R}_5 - \frac{\tau A}{3}\widetilde{R}^2\Big)R_{10}\widetilde{R}^{N-2}\Big]\\ &\times\Big[\alpha\Big(\frac{1}{2}\big(\widetilde{R}^N - R^N\big)R_3\tau^2 f_{1,1} + \frac{1}{2}R_4\tau\sum_{s=1}^{N-1}\big(\widetilde{R}^{N-s} - R^{N-s}\big)f_s\tau\Big) + \varphi\Big]\Bigg\}.\end{aligned} \tag{23}$$

So, formulas (19) and (23) give a solution of problem (2).

Unfortunately, the estimates for $\max_{1\le k\le N}\|u_k\|_H$, $\max_{1\le k\le N}\|A^{1/2}u_k\|_H$, and $\max_{1\le k\le N}\|Au_k\|_H$ cannot be obtained under the conditions

$$\begin{aligned}&\max_{1\le k\le N}\|u_k\|_H \le M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau + \|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\},\\ &\max_{1\le k\le N}\left\|A^{1/2}u_k\right\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \tau\|f_{1,1}\|_H\right\},\\ &\max_{1\le k\le N}\|Au_k\|_H \le M\left\{\sum_{s=2}^{N-1}\|f_s - f_{s-1}\|_H + \|f_1\|_H + \|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}.\end{aligned} \tag{24}$$

Nevertheless, we have the following theorem.

Theorem 4. *Suppose that assumption* (4) *holds and* $\varphi \in D(A^{3/2})$, $\psi \in D(A^{1/2})$. *Then, for solution of difference scheme* (2), *the following stability estimates hold:*

$$\begin{aligned}
&\max_{1\le k\le N}\left\|u_k\right\|_H \\
&\quad\le M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau+\left\|\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right. \\
&\qquad\left.+\left\|A^{-1/2}\psi\right\|_H+\tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\}, \\
&\max_{1\le k\le N}\left\|A^{1/2}u_k\right\|_H \\
&\quad\le M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right. \\
&\qquad\left.+\left\|\psi\right\|_H+\tau\left\|f_{1,1}\right\|_H\right\}, \\
&\max_{1\le k\le N}\left\|Au_k\right\|_H \\
&\quad\le M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right. \\
&\qquad\left.+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\},
\end{aligned}\tag{25}$$

where M does not depend on τ, φ, ψ, $f_{1,1}(x)$, and $f_s(x)$, $1\le s\le N-1$.

Proof. Using formulas in (23) and estimates (11), (12), and (14), we obtain

$$\begin{aligned}
&\left\|\left(I+i\tau A^{1/2}\right)\mu\right\|_H\le\left\|T_\tau\right\|_{H\to H} \\
&\quad\times\left\{\left[|\alpha|\left(\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^N\right\|_{H\to H}\right.\right.\right.\right. \\
&\qquad\left.+\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right) \\
&\qquad\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|A^{-1/2}f_{1,1}\right\|_H \\
&\qquad+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H} \\
&\qquad\times\sum_{s=1}^{N-1}\left(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^{N-s}\right\|_{H\to H}\right. \\
&\qquad\left.+\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}\right) \\
&\qquad\left.\left.\times\left\|A^{-1/2}f_s\right\|_{H\to H}\tau\right)+\left\|\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right] \\
&\quad\times\left[1+\frac{1}{2}\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right.\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\qquad\times\left\|\widetilde{R}^{N-2}\right\|_{H\to H} \\
&\qquad+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right) \\
&\qquad\left.\times\left\|R^{N-2}\right\|_{H\to H}\right) \\
&\qquad\left.\times\left\|A^{1/2}R_2\right\|_{H\to H}\right] \\
&\quad+|\alpha|\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^N\right\|_{H\to H}\right. \\
&\qquad\left.+\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right) \\
&\quad\times\left\|A^{1/2}R_2\right\|_{H\to H} \\
&\quad\times\left[|\beta|\frac{1}{2}\left\{\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right.\right.\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\qquad\times\left\|\widetilde{R}^{N-2}\right\|_{H\to H} \\
&\qquad+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right) \\
&\qquad\left.\times\left\|R^{N-2}\right\|_{H\to H}\right) \\
&\qquad\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|A^{-1/2}f_{1,1}\right\|_H \\
&\qquad+\frac{2\tau}{3}\left\|A^{-1/2}f_N\right\|_H \\
&\qquad+\left\|\tau R_8\right\|_{H\to H}\left\|A^{-1/2}f_{N-1}\right\|_H\tau \\
&\qquad+\left\|\tau R_9\right\|_{H\to H}\left\|A^{-1/2}f_{N-2}\right\|_H\tau \\
&\qquad+\left\|\tau A^{1/2}R_4\right\|_{H\to H} \\
&\qquad\times\sum_{s=1}^{N-3}\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\right.\right. \\
&\qquad\quad\times\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H} \\
&\qquad\quad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\qquad\quad\times\left\|\widetilde{R}^{N-2-s}\right\|_{H\to H}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \Big(\|\tau R_7\|_{H\to H} \\
&\quad \times \left\|\tau^{-1} A^{-1/2} R_5\right\|_{H\to H} + \left\|\frac{1}{3}\tau A^{1/2} R^2\right\|_{H\to H} \Big) \\
&\quad \times \left\|R^{N-2-s}\right\|_{H\to H} \Big) \\
&\quad \times \left\|A^{-1/2} f_s\right\|_H \tau \Big\} + \left\|A^{-1/2}\psi\right\|_H \Big] \Big\} \\
&\le M \left\{ \sum_{s=1}^{k-1} \left\|A^{-1/2} f_s\right\|_H \tau + \left\|\left(I + i\tau A^{1/2}\right)\varphi\right\|_H \right. \\
&\quad \left. + \left\|A^{-1/2}\psi\right\|_H + \tau \left\|A^{-1/2} f_{1,1}\right\|_H \right\}, \\
&\left\|A^{-1/2}\omega\right\|_H \le \|T_\tau\|_{H\to H} \\
&\quad \times \Big\{ \Big[1 + |\alpha| \frac{1}{2} \Big(\left\|\left(I + i\tau A^{1/2}\right)^{-1} \widetilde{R}_{10}\right\|_{H\to H} \\
&\quad \times \left\|\left(I + i\tau A^{1/2}\right) R^N\right\|_{H\to H} \\
&\quad + \left\|\left(I + i\tau A^{1/2}\right)^{-1} R_{10}\right\|_{H\to H} \\
&\quad \times \left\|\left(I + i\tau A^{1/2}\right) \widetilde{R}^N\right\|_{H\to H} \Big) \Big] \\
&\quad \times \Big[|\beta| \frac{1}{2} \Big\{ \Big(\Big(\|\tau \widetilde{R}_7\|_{H\to H} \left\|\tau^{-1} A^{-1/2} \widetilde{R}_5\right\|_{H\to H} \\
&\quad + \left\|\frac{1}{3}\tau A^{1/2} \widetilde{R}^2\right\|_{H\to H} \Big) \\
&\quad \times \left\|\widetilde{R}^{N-2}\right\|_{H\to H} \\
&\quad + \Big(\|\tau R_7\|_{H\to H} \left\|\tau^{-1} A^{-1/2} R_5\right\|_{H\to H} \\
&\quad + \left\|\frac{1}{3}\tau A^{1/2} R^2\right\|_{H\to H} \Big) \\
&\quad \times \left\|R^{N-2}\right\|_{H\to H} \Big) \\
&\quad \times \left\|\tau A^{1/2} R_3\right\|_{H\to H} \left\|A^{-1/2} f_{1,1}\right\|_H \tau \\
&\quad + \frac{2}{3} \left\|A^{-1/2} f_N\right\|_H \tau \\
&\quad + \|\tau R_8\|_{H\to H} \left\|A^{-1/2} f_{N-1}\right\|_H \tau \\
&\quad + \|\tau R_9\|_{H\to H} \left\|A^{-1/2} f_{N-2}\right\|_H \tau \\
&\quad + \left\|\tau A^{1/2} R_4\right\|_{H\to H} \\
&\quad \times \sum_{s=1}^{N-3} \Big(\Big(\|\tau \widetilde{R}_7\|_{H\to H} \\
&\quad \times \left\|\tau^{-1} A^{-1/2} \widetilde{R}_5\right\|_{H\to H} + \left\|\frac{1}{3}\tau A^{1/2} \widetilde{R}^2\right\|_{H\to H} \Big) \\
&\quad \times \left\|\widetilde{R}^{N-2-s}\right\|_{H\to H} + \Big(\|\tau R_7\|_{H\to H} \\
&\quad \times \left\|\tau^{-1} A^{-1/2} R_5\right\|_{H\to H} + \left\|\frac{1}{3}\tau A^{1/2} R^2\right\|_{H\to H} \Big) \\
&\quad \times \left\|R^{N-2-s}\right\|_{H\to H} \Big) \\
&\quad \times \left\|A^{-1/2} f_s\right\|_H \tau \Big\} + \left\|A^{-1/2}\psi\right\|_H \Big] \\
&\quad + \frac{1}{2} \Big[\Big(\|\tau R_7\|_{H\to H} \left\|\tau^{-1} A^{-1/2} R_5\right\|_{H\to H} \\
&\quad + \left\|\frac{1}{3}\tau A^{1/2} R^2\right\|_{H\to H} \Big) \\
&\quad \times \left\|\left(I + i\tau A^{1/2}\right)^{-1} \widetilde{R}_{10}\right\|_{H\to H} \left\|R^{N-2}\right\|_{H\to H} \\
&\quad + \Big(\|\tau \widetilde{R}_7\|_{H\to H} \left\|\tau^{-1} A^{-1/2} \widetilde{R}_5\right\|_{H\to H} \\
&\quad + \left\|\frac{1}{3}\tau A^{1/2} \widetilde{R}^2\right\|_{H\to H} \Big) \\
&\quad \times \left\|\left(I + i\tau A^{1/2}\right)^{-1} R_{10}\right\|_{H\to H} \left\|\widetilde{R}^{N-2}\right\|_{H\to H} \Big] \\
&\quad \times \Big[|\alpha| \frac{1}{2} \Big(\Big(\left\|\left(I + i\tau A^{1/2}\right) \widetilde{R}^N\right\|_{H\to H} \\
&\quad + \left\|\left(I + i\tau A^{1/2}\right) R^N\right\|_{H\to H} \Big) \\
&\quad \times \left\|\tau A^{1/2} R_3\right\|_{H\to H} \tau \left\|A^{-1/2} f_{1,1}\right\|_H \\
&\quad + \left\|\tau A^{1/2} R_4\right\|_{H\to H} \\
&\quad \times \sum_{s=1}^{N-1} \Big(\left\|\left(I + i\tau A^{1/2}\right) \widetilde{R}^{N-s}\right\|_{H\to H} \\
&\quad + \left\|\left(I + i\tau A^{1/2}\right) R^{N-s}\right\|_{H\to H} \Big) \\
&\quad \times \left\|A^{-1/2} f_s\right\|_H \tau \Big) + \left\|\left(I + i\tau A^{1/2}\right)\varphi\right\|_H \Big] \Big\} \\
&\le M \left\{ \sum_{s=1}^{k-1} \left\|A^{-1/2} f_s\right\|_H \tau + \left\|\left(I + i\tau A^{1/2}\right)\varphi\right\|_H \right. \\
&\quad \left. + \left\|A^{-1/2}\psi\right\|_H + \tau \left\|A^{-1/2} f_{1,1}\right\|_H \right\}.
\end{aligned}
\tag{26}
$$

Applying $A^{1/2}$ to formulas in (23), we get

$$
\begin{aligned}
&\left\|A^{1/2}\left(I + i\tau A^{1/2}\right)\mu\right\|_H \le \|T_\tau\|_{H\to H} \\
&\quad \times \Big\{ \Big[|\alpha| \Big(\frac{1}{2} \Big(\left\|\left(I + i\tau A^{1/2}\right) \widetilde{R}^N\right\|_{H\to H} \\
&\quad + \left\|\left(I + i\tau A^{1/2}\right) R^N\right\|_{H\to H} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|f_{1,1}\right\|_H \\
&+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H} \\
&\times\sum_{s=1}^{N-1}\left(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^{N-s}\right\|_{H\to H}\right. \\
&\qquad\left.+\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}\right) \\
&\left.\times\left\|f_s\right\|_{H\to H}\tau\right)+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\Bigg] \\
&\times\Bigg[1+\frac{1}{2}\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\times\left\|\widetilde{R}^{N-2}\right\|_{H\to H}+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right. \\
&\left.\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\left\|R^{N-2}\right\|_{H\to H}\right) \\
&\times\left\|A^{1/2}R_2\right\|_{H\to H}\Bigg] \\
&+|\alpha|\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^{N}\right\|_{H\to H}\right. \\
&\left.+\left\|\left(I+i\tau A^{1/2}\right)R^{N}\right\|_{H\to H}\right)\times\left\|A^{1/2}R_2\right\|_{H\to H} \\
&\times\Bigg[|\beta|\frac{1}{2}\left\{\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right.\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\times\left\|\widetilde{R}^{N-2}\right\|_{H\to H}+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right. \\
&\qquad\left.\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\left\|R^{N-2}\right\|_{H\to H}\right) \\
&\qquad\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|f_{1,1}\right\|_H+\frac{2\tau}{3}\left\|f_N\right\|_H \\
&\qquad+\left\|\tau R_8\right\|_{H\to H}\left\|f_{N-1}\right\|_H\tau+\left\|\tau R_9\right\|_{H\to H} \\
&\qquad\times\left\|f_{N-2}\right\|_H\tau+\left\|\tau A^{1/2}R_4\right\|_{H\to H} \\
&\qquad\times\sum_{s=1}^{N-3}\left(\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right.\right. \\
&\qquad\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\quad\times\left\|\widetilde{R}^{N-2-s}\right\|_{H\to H} \\
&\quad+\left(\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right.\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right) \\
&\qquad\left.\times\left\|R^{N-2-s}\right\|_{H\to H}\right) \\
&\qquad\left.\left.\times\left\|f_s\right\|_H\tau\right\}+\left\|\psi\right\|_H\right]\Bigg\} \\
&\le M\left\{\sum_{s=1}^{k-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right. \\
&\qquad\left.+\left\|\psi\right\|_H+\tau\left\|f_{1,1}\right\|_H\right\},
\end{aligned}
$$

$$
\begin{aligned}
\|\omega\|_H\le{}&\left\|T_\tau\right\|_{H\to H} \\
&\times\Bigg\{\Bigg[1+|\alpha|\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)^{-1}\widetilde{R}_{10}\right\|_{H\to H}\right. \\
&\qquad\times\left\|\left(I+i\tau A^{1/2}\right)R^{N}\right\|_{H\to H} \\
&\qquad+\left\|\left(I+i\tau A^{1/2}\right)^{-1}R_{10}\right\|_{H\to H} \\
&\qquad\left.\times\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^{N}\right\|_{H\to H}\right)\Bigg] \\
&\times\Bigg[|\beta|\frac{1}{2}\left\{\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right.\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\qquad\times\left\|\widetilde{R}^{N-2}\right\|_{H\to H} \\
&\qquad+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right. \\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right) \\
&\qquad\left.\times\left\|R^{N-2}\right\|_{H\to H}\right) \\
&\qquad\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|f_{1,1}\right\|_H\tau+\frac{2}{3}\left\|f_N\right\|_H\tau \\
&\qquad+\left\|\tau R_8\right\|_{H\to H}\left\|f_{N-1}\right\|_H\tau \\
&\qquad+\left\|\tau R_9\right\|_{H\to H}\left\|f_{N-2}\right\|_H\tau+\left\|A^{1/2}R_4\right\|_{H\to H} \\
&\qquad\times\sum_{s=1}^{N-3}\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right. \\
&\qquad\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right) \\
&\times\left\|\widetilde{R}^{N-2-s}\right\|_{H\to H}+\left(\left\|\tau R_7\right\|_{H\to H}\right. \\
&\times\left.\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)
\end{aligned}
$$

$$\begin{aligned}
&\times\left\|R^{N-2-s}\right\|_{H\to H}\Big)\left\|f_s\right\|_H\tau\Big\}+\|\psi\|_H\Big]\\
&+\frac{1}{2}\Big[\Big(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\\
&\qquad+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\Big)\\
&\qquad\times\left\|\left(I+i\tau A^{1/2}\right)^{-1}\widetilde{R}_{10}\right\|_{H\to H}\left\|R^{N-2}\right\|_{H\to H}\\
&\qquad+\Big(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\\
&\qquad+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\Big)\\
&\qquad\times\left\|\left(I+i\tau A^{1/2}\right)^{-1}R_{10}\right\|_{H\to H}\left\|\widetilde{R}^{N-2}\right\|_{H\to H}\Big]\\
&\times\Big[|\alpha|\Big(\frac{1}{2}\Big(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^N\right\|_{H\to H}\\
&\qquad+\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\Big)\\
&\qquad\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|f_{1,1}\right\|_H\\
&\qquad+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\\
&\qquad\times\sum_{s=1}^{N-1}\Big(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^{N-s}\right\|_{H\to H}\\
&\qquad\qquad+\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}\Big)\\
&\times\left\|f_s\right\|_H\tau\Big)+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\Big]\Big\}\\
&\le M\left\{\sum_{s=1}^{k-1}\left\|f_s\right\|_H\tau+\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right.\\
&\qquad\left.+\|\psi\|_H+\tau\left\|f_{1,1}\right\|_H\right\}.
\end{aligned}\tag{27}$$

Now, applying Abel's formula to (23), we obtain the following formulas:

$$\begin{aligned}
\mu=T_\tau&\Big\{\Big[\alpha\Big(\frac{1}{2}\left(\widetilde{R}^N-R^N\right)R_3\tau^2f_{1,1}\\
&+\frac{1}{2}R_4\tau^2\Big(\sum_{s=2}^{N-1}\left(R_6R^{N-s}-\widetilde{R}_6\widetilde{R}^{N-s}\right)\\
&\times\left(f_s-f_{s-1}\right)+\left(\widetilde{R}_6-R_6\right)f_{N-1}\\
&-\left(\widetilde{R}_6\widetilde{R}^{N-1}-R_6R^{N-1}\right)f_1\Big)\Big)+\varphi\Big]\\
&\times\Big[I-\frac{1}{2}\Big(\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2}\\
&\qquad-\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)R^{N-2}\Big)R_2\Big]\\
&+\alpha\frac{1}{2}\left(\widetilde{R}^N-R^N\right)R_2\\
&\times\Big[\beta\Big\{\frac{1}{2}\Big(\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2}\\
&\qquad-\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)R^{N-2}\Big)\\
&\qquad\times R_3\tau^2f_{1,1}+\frac{\tau}{3}f_N+\frac{1}{2}R_8f_{N-1}\tau^2+\frac{1}{2}R_9f_{N-2}\tau^2\\
&\qquad+R_4\frac{1}{2}\tau^2\Big(\sum_{s=2}^{N-3}\Big(R_6\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)R^{N-2-s}\\
&\qquad\qquad-\widetilde{R}_6\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2-s}\Big)\\
&\qquad\qquad\times\left(f_s-f_{s-1}\right)\\
&\qquad\qquad+\Big(\widetilde{R}_6\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\\
&\qquad\qquad-R_6\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)\Big)f_{N-3}\\
&\qquad\qquad-\Big(\widetilde{R}_6\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-3}\\
&\qquad\qquad-R_6\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)R^{N-3}\Big)\\
&\qquad\qquad\times f_1\Big)\Big\}+\psi\Big]\Big\},
\end{aligned}\tag{28}$$

$$\begin{aligned}
\omega=T_\tau&\Big\{\Big[I-\alpha\frac{1}{2}\left(\widetilde{R}_1R^N-R_1\widetilde{R}^N\right)\Big]\\
&\times\Big[\beta\Big\{\frac{1}{2}\Big(\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2}\\
&\qquad-\Big(R_7R_5-\frac{\tau A}{3}R^2\Big)R^{N-2}\Big)\\
&\qquad\times R_3\tau^2f_{1,1}+\frac{\tau}{3}f_N+\frac{1}{2}R_8f_{N-1}\tau^2\\
&\qquad+\frac{1}{2}R_9f_{N-2}\tau^2\\
&+R_4\frac{1}{2}\tau^2\Big(\sum_{s=2}^{N-3}\Big(R_6\Big(\frac{R_7R_5}{\tau}-\frac{\tau A}{3}R^2\Big)R^{N-2-s}\\
&\qquad-\widetilde{R}_6\Big(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\Big)\widetilde{R}^{N-2-s}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times\left(f_s-f_{s-1}\right)\\
&+\left(\widetilde{R}_6\left(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\right)\right.\\
&\left.-R_6\left(R_7R_5-\frac{\tau A}{3}R^2\right)\right)f_{N-3}\\
&-\left(\widetilde{R}_6\left(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\right)\widetilde{R}^{N-3}\right.\\
&\quad-R_6\left(R_7R_5-\frac{\tau A}{3}R^2\right)\\
&\left.\left.\left.\quad\times R^{N-3}\right)\ f_1\right)\right\}+\psi\Big]\\
&+\frac{1}{2}\left[\left(R_7R_5-\frac{\tau A}{3}R^2\right)\widetilde{R}_1R^{N-2}\right.\\
&\quad\left.-\left(\widetilde{R}_7\widetilde{R}_5-\frac{\tau A}{3}\widetilde{R}^2\right)R_1\widetilde{R}^{N-2}\right]\\
&\times\left[\alpha\left(\frac{1}{2}\left(\widetilde{R}^N-R^N\right)R_3\tau^2f_{1,1}\right.\right.\\
&\quad+\frac{1}{2}R_4\tau^2\left(\sum_{s=2}^{N-1}\left(R_6R^{N-s}-\widetilde{R}_6\widetilde{R}^{N-s}\right)\right.\\
&\qquad\times\left(f_s-f_{s-1}\right)\\
&\qquad+\left(\widetilde{R}_6-R_6\right)f_{N-1}\\
&\qquad-\left(\widetilde{R}_6\widetilde{R}^{N-1}-R_6R^{N-1}\right)\\
&\qquad\left.\left.\left.\left.\times f_1\right)\right)+\varphi\right]\right\}.
\end{aligned}
\tag{29}
$$

Next, let us obtain the estimates for $\|A(I+i\tau A^{1/2})\mu\|_H$ and $\|A^{1/2}\omega\|_H$. First, applying A to formula (28) and using estimates (11), (12), and (14) and the triangle inequality, one can obtain

$$
\begin{aligned}
&\left\|A\left(I+i\tau A^{1/2}\right)\mu\right\|_H\leq\|T_\tau\|_{H\to H}\\
&\quad\times\left\{\left[|\alpha|\left(\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^N\right\|_{H\to H}\right.\right.\right.\right.\\
&\qquad\left.+\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right)\\
&\qquad\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|A^{1/2}f_{1,1}\right\|_H\tau\\
&\qquad+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\\
&\qquad\times\left(\sum_{s=2}^{N-1}\left(\left\|\tau A^{1/2}R_6\right\|_{H\to H}\right.\right.\\
&\qquad\quad\times\left\|\left(I+i\tau A^{1/2}\right)R^{N-s}\right\|_{H\to H}\\
&\qquad\quad+\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\\
&\qquad\quad\left.\times\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^{N-s}\right\|_{H\to H}\right)\\
&\qquad\quad\times\left\|f_s-f_{s-1}\right\|_H\\
&\qquad+\left(\left\|\tau A^{1/2}\left(I+i\tau A^{1/2}\right)\widetilde{R}_6\right\|_{H\to H}\right.\\
&\qquad\quad\left.+\left\|\tau A^{1/2}\left(I+i\tau A^{1/2}\right)R_6\right\|_{H\to H}\right)\times\left\|f_{N-1}\right\|_H\\
&\qquad+\left(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\right.\\
&\qquad\quad\times\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^{N-1}\right\|_{H\to H}\\
&\qquad\quad+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\\
&\qquad\quad\left.\times\left\|\left(I+i\tau A^{1/2}\right)R^{N-1}\right\|_{H\to H}\right)\\
&\qquad\quad\left.\left.\left.\times\left\|f_1\right\|_H\right)\right)+\left\|A\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right]\\
&\quad\times\left[1+\frac{1}{2}\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right.\right.\\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right)\left\|\widetilde{R}^{N-2}\right\|_{H\to H}\\
&\qquad+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right.\\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\\
&\qquad\left.\left.\times\left\|R^{N-2}\right\|_{H\to H}\right)\left\|A^{1/2}R_2\right\|_{H\to H}\right]\\
&\quad+|\alpha|\frac{1}{2}\left(\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^N\right\|_{H\to H}\right.\\
&\qquad\left.+\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\right)\left\|A^{1/2}R_2\right\|_{H\to H}\\
&\quad\times\left[|\beta|\frac{1}{2}\left\{\left(\left(\left(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\right.\right.\right.\right.\right.\\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\right)\left\|\widetilde{R}^{N-2}\right\|_{H\to H}\\
&\qquad+\left(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\right.\\
&\qquad\left.+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\right)\\
&\qquad\left.\times\left\|R^{N-2}\right\|_{H\to H}\right)\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\\
&\qquad\times\tau\left\|A^{1/2}f_{1,1}\right\|_H+\frac{2\tau}{3}\left\|A^{1/2}f_N\right\|_H\\
&\qquad+\left\|\tau R_8\right\|_{H\to H}\left\|A^{1/2}f_{N-1}\right\|_H\tau\\
&\qquad+\left\|\tau R_9\right\|_{H\to H}\left\|A^{1/2}f_{N-2}\right\|_H\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\tau A^{1/2}R_4\right\|_{H\to H}\\
&\times\Bigg(\sum_{s=2}^{N-3}\Big(\left\|\tau A^{1/2}R_6\right\|_{H\to H}\\
&\qquad\times\Big(\left\|\tau R_7\right\|_{H\to H}\\
&\qquad\qquad\times\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\\
&\qquad\qquad+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\Big)\\
&\qquad\times\left\|R^{N-2-s}\right\|_{H\to H}\\
&\qquad+\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\\
&\qquad\times\Big(\left\|\tau \widetilde{R}_7\right\|_{H\to H}\\
&\qquad\qquad\times\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\\
&\qquad\qquad+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\Big)\\
&\qquad\times\left\|\widetilde{R}^{N-2-s}\right\|_{H\to H}\Big)\left\|f_s-f_{s-1}\right\|_H\\
&\quad+\Big(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\\
&\qquad\times\Big(\left\|\tau \widetilde{R}_7\right\|_{H\to H}\\
&\qquad\qquad\times\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\\
&\qquad\qquad+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\Big)\\
&\qquad+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\\
&\qquad\times\Big(\left\|\tau R_7\right\|_{H\to H}\\
&\qquad\qquad\times\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\\
&\qquad\qquad+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\Big)\Big)\\
&\times\left\|f_{N-3}\right\|_H\\
&+\Bigg(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\\
&\quad\times\Big(\left\|\tau \widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\\
&\qquad+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\Big)\\
&\quad\times\left\|\widetilde{R}^{N-3}\right\|_{H\to H}+\left\|\tau A^{1/2}R_6\right\|_{H\to H}\\
&\quad\times\Big(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\\
&\qquad+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\Big)\\
&\quad\times\left\|R^{N-3}\right\|_{H\to H}\Bigg)\left\|f_1\right\|_H\Bigg)\Bigg\}\\
&+\left\|A^{1/2}\psi\right\|_H\Bigg]\Bigg\}\\
&\leq M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right.\\
&\qquad\left.+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\}.
\end{aligned}
\tag{30}
$$

Second, applying $A^{1/2}$ to formula (29) and using estimates (11), (12), and (14) and the triangle inequality, we get

$$
\begin{aligned}
\left\|A^{1/2}\omega\right\|_H &\leq \left\|T_\tau\right\|_{H\to H}\\
&\times\Bigg\{\Bigg[1+|\alpha|\frac{1}{2}\Big(\left\|\left(I+i\tau A^{1/2}\right)^{-1}\widetilde{R}_{10}\right\|_{H\to H}\\
&\qquad\times\left\|\left(I+i\tau A^{1/2}\right)R^N\right\|_{H\to H}\\
&\qquad+\left\|\left(I+i\tau A^{1/2}\right)^{-1}R_{10}\right\|_{H\to H}\\
&\qquad\times\left\|\left(I+i\tau A^{1/2}\right)\widetilde{R}^N\right\|_{H\to H}\Big)\Bigg]\\
&\times\Bigg[|\beta|\frac{1}{2}\Bigg\{\Bigg(\Big(\Big(\left\|\tau\widetilde{R}_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}\widetilde{R}_5\right\|_{H\to H}\\
&\qquad+\left\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\right\|_{H\to H}\Big)\left\|\widetilde{R}^{N-2}\right\|_{H\to H}\\
&\qquad+\Big(\left\|\tau R_7\right\|_{H\to H}\left\|\tau^{-1}A^{-1/2}R_5\right\|_{H\to H}\\
&\qquad\qquad+\left\|\frac{1}{3}\tau A^{1/2}R^2\right\|_{H\to H}\Big)\\
&\qquad\times\left\|R^{N-2}\right\|_{H\to H}\Big)\\
&\qquad\times\left\|\tau A^{1/2}R_3\right\|_{H\to H}\tau\left\|A^{1/2}f_{1,1}\right\|_H\\
&\qquad+\frac{2}{3}\tau\left\|A^{1/2}f_N\right\|_H\\
&\qquad+\left\|\tau R_8\right\|_{H\to H}\left\|A^{1/2}f_{N-1}\right\|_H\tau\\
&\qquad+\left\|\tau R_9\right\|_{H\to H}\left\|A^{1/2}f_{N-2}\right\|_H\tau\\
&\qquad+\left\|\tau A^{1/2}R_4\right\|_{H\to H}\tau\\
&\qquad\times\Bigg(\sum_{s=2}^{N-3}\Big(\left\|\tau A^{1/2}R_6\right\|_{H\to H}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}R^2\Big\|_{H\to H}\Big)\\
&\times\|R^{N-2-s}\|_{H\to H}+\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\\
&\times\Big(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\Big\|_{H\to H}\Big)\\
&\times\|\widetilde{R}^{N-2-s}\|_{H\to H}\Big)\|f_s-f_{s-1}\|_H\\
&+\Big(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\times\Big(\|\tau\widetilde{R}_7\|_{H\to H}\times\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\Big\|_{H\to H}\Big)\\
&+\|\tau A^{1/2}R_6\|_{H\to H}\times\Big(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}R^2\Big\|_{H\to H}\Big)\Big)\\
&\times\|f_{N-3}\|_H\\
&+\Big(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\times\Big(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\Big\|_{H\to H}\Big)\\
&\times\|\widetilde{R}^{N-3}\|_{H\to H}+\|\tau A^{1/2}R_6\|_{H\to H}\\
&\times\Big(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}R^2\Big\|_{H\to H}\Big)\\
&\times\|R^{N-3}\|_{H\to H}\Big)\|f_1\|_H\Big)\Big\}\\
&+\|A^{1/2}\psi\|_H\Big]\\
&+\frac{1}{2}\Big[\Big(\Big(\|\tau R_7\|_{H\to H}\|\tau^{-1}A^{-1/2}R_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}R^2\Big\|_{H\to H}\Big)\\
&\times\Big\|\widetilde{R}_{10}\big(I+i\tau A^{1/2}\big)^{-1}\Big\|_{H\to H}\times\Big\|\big(I+i\tau A^{1/2}\big)R^{N-2}\Big\|_{H\to H}\\
&+\Big(\|\tau\widetilde{R}_7\|_{H\to H}\|\tau^{-1}A^{-1/2}\widetilde{R}_5\|_{H\to H}+\Big\|\frac{1}{3}\tau A^{1/2}\widetilde{R}^2\Big\|_{H\to H}\Big)\\
&\times\Big\|R_{10}\big(I+i\tau A^{1/2}\big)^{-1}\Big\|_{H\to H}\times\Big\|\big(I+i\tau A^{1/2}\big)\widetilde{R}^{N-2}\Big\|_{H\to H}\Big]\\
&\times\Big[|\alpha|\frac{1}{2}\big(\|\widetilde{R}^N\|_{H\to H}+\|R^N\|_{H\to H}\big)\times\|\tau A^{1/2}R_3\|_{H\to H}\tau\|A^{1/2}f_{1,1}\|_H\\
&+\|\tau A^{1/2}R_4\|_{H\to H}\times\Big(\sum_{s=2}^{N-1}\big(\|\tau A^{1/2}R_6\|_{H\to H}\|R^{N-s}\|_{H\to H}+\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\widetilde{R}^{N-s}\|_{H\to H}\big)\\
&\times\|f_s-f_{s-1}\|_H+\big(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}+\|\tau A^{1/2}R_6\|_{H\to H}\big)\times\|f_{N-1}\|_{H\to H}\\
&+\big(\|\tau A^{1/2}\widetilde{R}_6\|_{H\to H}\|\widetilde{R}^{N-1}\|_{H\to H}+\|\tau A^{1/2}R_6\|_{H\to H}\|R^{N-1}\|_{H\to H}\big)\times\|f_1\|_H\Big)\\
&+\Big\|\big(I+i\tau A^{1/2}\big)^{-1}\Big\|_{H\to H}\Big\|A\big(I+i\tau A^{1/2}\big)\varphi\Big\|_H\Big]\Big\}\\
&\le M\Big\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\Big\|A\big(I+i\tau A^{1/2}\big)\varphi\Big\|_H+\|A^{1/2}\psi\|_H+\tau\|A^{1/2}f_{1,1}\|_H\Big\}.
\end{aligned}
\tag{31}
$$

Now, we will prove estimates (25). Using formula (19), estimates (11), (12), (26), and (27), and the triangle inequality, we obtain

$$
\begin{aligned}
\|u_k\|_H\le\frac{1}{2}&\Big(\Big\|\widetilde{R}_{10}\big(I+i\tau A^{1/2}\big)^{-1}\Big\|_{H\to H}\|R^k\|_{H\to H}+\Big\|R_{10}\big(I+i\tau A^{1/2}\big)^{-1}\Big\|_{H\to H}\|\widetilde{R}^k\|_{H\to H}\Big)\\
&\times\Big\|\big(I+i\tau A^{1/2}\big)\mu\Big\|_H
\end{aligned}
$$

$$+\frac{1}{2}\left(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right) \times \left\|A^{-1/2}\omega\right\|_H$$
$$+\frac{1}{2}\left(\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\tau\left\|A^{-1/2}f_{1,1}\right\|_H$$
$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\sum_{s=1}^{k-1}\left[\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right] \times \left\|A^{-1/2}f_s\right\|_H\tau$$
$$\leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1/2}f_s\right\|_H\tau + \left\|\left(I+i\tau A^{1/2}\right)\varphi\right\|_H + \left\|A^{-1/2}\psi\right\|_H + \tau\left\|A^{-1/2}f_{1,1}\right\|_H\right\} \tag{32}$$

for any $k \geq 2$. Applying $A^{1/2}$ to (19), we get

$$\left\|A^{1/2}u_k\right\|_H \leq \frac{1}{2}\left(\left\|\widetilde{R}_{10}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^k\right\|_{H\to H} + \left\|R_{10}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right)$$
$$\times\left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\mu\right\|_H$$
$$+\frac{1}{2}\left(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\times\|\omega\|_H$$
$$+\frac{1}{2}\left(\left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|\tau A^{1/2}R_3\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)\times\tau\left\|f_{1,1}\right\|_H$$
$$+\frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}\sum_{s=1}^{k-1}\left(\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\left\|R^{k-s}\right\|_{H\to H}\right)$$
$$\times\left\|f_s\right\|_H\tau \leq M\left\{\sum_{s=1}^{N-1}\left\|f_s\right\|_H\tau + \left\|A^{1/2}\left(I+i\tau A^{1/2}\right)\varphi\right\|_H + \|\psi\|_H + \tau\left\|f_{1,1}\right\|_H\right\} \tag{33}$$

for $k \geq 2$. Now, applying Abel's formula to (19), we have

$$u_k = \frac{1}{2}\left[\widetilde{R}_1R^k - R_1\widetilde{R}^k\right]\mu + \frac{1}{2}\left[\widetilde{R}^k - R^k\right]R_2\omega + \frac{1}{2}\left[\widetilde{R}^k - R^k\right]R_3\tau^2 f_{1,1}$$
$$+\tau^2 R_4\frac{1}{2}\left(\sum_{s=2}^{k-1}\left[R_6R^{k-s}-\widetilde{R}_6\widetilde{R}^{k-s}\right]\left(f_s - f_{s-1}\right) + \left(\widetilde{R}_6 - R_6\right)f_{k-1} - \left[\widetilde{R}_6\widetilde{R}^{k-1} - R_6R^{k-1}\right]f_1\right), \quad 2 \leq k \leq N. \tag{34}$$

Applying A to formula (34) and using estimates (11) and (12) and the triangle inequality, we obtain

$$\left\|Au_k\right\|_H \leq \frac{1}{2}\left(\left\|\widetilde{R}_{10}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|R^k\right\|_{H\to H} + \left\|R_{10}\left(I+i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H}\right)$$
$$\times\left\|A\left(I+i\tau A^{1/2}\right)\mu\right\|_H$$
$$+\frac{1}{2}\left(\left\|A^{1/2}R_2\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|A^{1/2}R_2\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)$$
$$\times\left\|A^{1/2}\omega\right\|_H + \frac{1}{2}\left(\left\|\tau A^{-1/2}R_3\right\|_{H\to H}\left\|\widetilde{R}^k\right\|_{H\to H} + \left\|\tau A^{-1/2}R_3\right\|_{H\to H}\left\|R^k\right\|_{H\to H}\right)$$
$$\times\tau\left\|A^{1/2}f_{1,1}\right\|_H + \frac{1}{2}\left\|\tau A^{1/2}R_4\right\|_{H\to H}$$
$$\times\left(\sum_{s=2}^{k-1}\left[\left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-s}\right\|_{H\to H} + \left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\left\|\widetilde{R}^{k-s}\right\|_{H\to H}\right]\right.$$
$$\times\left\|f_s - f_{s-1}\right\|_H$$
$$+\left(\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H} + \left\|\tau A^{1/2}R_6\right\|_{H\to H}\right)\left\|f_{k-1}\right\|_H$$
$$+\left[\left\|\tau A^{1/2}\widetilde{R}_6\right\|_{H\to H}\left\|\widetilde{R}^{k-1}\right\|_{H\to H}\right.$$
$$\left.\left. + \left\|\tau A^{1/2}R_6\right\|_{H\to H}\left\|R^{k-1}\right\|_{H\to H}\right]\left\|f_1\right\|_H\right)$$

$$\le M\left\{\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H+\left\|A\left(I+i\tau A^{1/2}\right)\varphi\right\|_H\right.$$
$$\left.+\left\|A^{1/2}\psi\right\|_H+\tau\left\|A^{1/2}f_{1,1}\right\|_H\right\} \tag{35}$$

for $k\ge 2$. Theorem 4 is proved. □

Note that the stability estimates obtained previously permit us to get the convergence estimate of difference scheme (2) under the smoothness property of solution (1). Actually, under the condition $u(t)\in C([0,1],H)$, we can obtain the third order of accuracy for the error of difference scheme (2). Since $u^{(6)}(t)=-A^3u(t)+A^2f(t)-Af''(t)+f^{(4)}(t)$, this condition is satisfied under the given data $\varphi\in D(A^3)$, $\psi\in D(A^{5/2})$, $f'(t)\in D(A^2)$, and $f(0)\in D(A^3)$.

Now, let us give application of this abstract result for nonlocal boundary value problem

$$\begin{gathered} u_{tt}-\left(a(x)u_x\right)_x+\delta u=f(t,x),\quad 0<t<1,\ 0<x<1,\\ u(0,x)=\alpha u(1,x)+\varphi(x),\quad 0\le x\le 1,\\ u_t(0,x)=\beta u_t(1,x)+\psi(x),\quad 0\le x\le 1,\\ u(t,0)=u(t,1),\quad u_x(t,0)=u_x(t,1),\quad 0\le t\le 1 \end{gathered}\tag{36}$$

for hyperbolic equation. Problem (36) has a unique smooth solution $u(t,x)$, $\delta>0$ and the smooth functions $a(x)\ge a>0$ $(a(0)=a(1), x\in(0,1))$, $\varphi(x)$, $\psi(x)$ $(x\in[0,1])$, and $f(t,x)\,(t,x\in[0,1])$. This allows us to reduce mixed problem (36) to nonlocal boundary value problem (1) in a Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (36).

The discretization of problem (36) is carried out in two steps. In the first step, let us define the grid space

$$[0,1]_h=\left\{x:x_r=rh,0\le r\le K,Kh=1\right\}. \tag{37}$$

We introduce Hilbert space $L_{2h}=L_2([0,1]_h)$, $W_{2h}^1=W_{2h}^1([0,1]_h)$, and $W_{2h}^2=W_{2h}^2([0,1]_h)$ of the grid functions $\varphi^h(x)=\{\varphi_r\}_1^{K-1}$ defined on $[0,1]_h$, and we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\{-(a(x)\varphi_{\bar{x}})_{x,r}+\delta\varphi_r\}_1^{K-1}, \tag{38}$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi_r\}_0^K$ satisfying the conditions $\varphi_0=\varphi_K$, $\varphi_1-\varphi_0=\varphi_K-\varphi_{K-1}$.

With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{gathered} \frac{d^2v^h(t,x)}{dt^2}+A_h^xv^h(t,x)=f^h(t,x),\\ 0<t<1,\ x\in[0,1]_h,\\ v^h(0,x)=\alpha v^h(1,x)+\varphi^h(x),\quad x\in[0,1]_h,\\ v_t^h(0,x)=\beta v_t^h(1,x)+\psi^h(x),\quad x\in[0,1]_h \end{gathered}\tag{39}$$

for a system of ordinary differential equations.

In the second step, we replace problem (2) with difference scheme (40)

$$\begin{aligned} &\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{2}{3}A_h^xu_k^h(x)\\ &\quad+\frac{1}{6}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)\\ &\quad+\frac{1}{12}\tau^2\left(A_h^x\right)^2u_{k+1}^h(x)=f_k^h(x),\\ &f_k^h(x)=\frac{2}{3}f^h(t_k,x)\\ &\quad+\frac{1}{6}\left(f^h(t_{k+1},x)+f^h(t_{k-1},x)\right)\\ &\quad-\frac{1}{12}\tau^2\left(-Af^h(t_{k+1},x)+f_{tt}^h(t_{k+1},x)\right),\quad x\in[0,1]_h,\\ &t_k=k\tau,\quad N\tau=1,\quad 1\le k\le N-1,\\ &u_0^h(x)=\alpha u_N^h(x)+\varphi^h(x),\quad x\in[0,1]_h,\\ &\left(I+\frac{\tau^2}{12}\left(A_h^x\right)+\frac{\tau^4}{144}\left(A_h^x\right)^2\right)\tau^{-1}\left(u_1^h(x)-u_0^h(x)\right)\\ &\quad+\frac{\tau}{2}\left(A_h^x\right)\varphi^h(x)-\tau f_{1,1}^h(x)\\ &\quad=\beta\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\\ &\quad\times\left(\frac{1}{6\tau}\left(7u_N^h(x)-8u_{N-1}^h(x)+u_{N-2}^h(x)\right)\right.\\ &\quad\left.+\frac{\tau}{3}\left(f_N^h(x)-Au_N^h(x)\right)\right)\\ &\quad+\left(I-\frac{\tau^2}{12}\left(A_h^x\right)\right)\psi^h(x),\quad x\in[0,1]_h,\\ &f_{1,1}^h(x)=\frac{1}{2}f^h(0,x)+\frac{\tau}{6}f_t^h(0,x). \end{aligned}\tag{40}$$

Theorem 5. *Let τ and h be sufficiently small numbers. Then, the solution of difference scheme* (40) *satisfies the following stability estimates:*

$$\begin{aligned} &\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^1}\\ &\quad\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}\right.\\ &\quad\left.+\left\|\varphi^h\right\|_{W_{2h}^1}+\tau\left\|\varphi^h\right\|_{W_{2h}^2}+\tau\left\|f_{1,1}^h\right\|_{L_{2h}}\right],\\ &\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|u_k^h\right\|_{W_{2h}^2}\\ &\quad\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.\\ &\quad+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\varphi^h\right\|_{W_{2h}^2}\\ &\quad\left.+\tau\left\|\varphi^h\right\|_{W_{2h}^3}+\tau\left\|f_{1,1}^h\right\|_{W_{2h}^1}\right]. \end{aligned}\tag{41}$$

Here, M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$, $f_{1,1}^h(x)$, *and* $f_k^h(x)$, $1 \leq k < N$.

The proof of Theorem 5 is based on the proof of abstract Theorem 4 and the symmetry property of operator A_h^x defined by (38).

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