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Research Article

Soliton Solutions for Quasilinear Schrödinger Equations

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By using a change of variables, we get new equations, whose respective associated functionals are well defined in $H^1(\mathbb{R}^N)$ and satisfy the geometric hypotheses of the mountain pass theorem. Using this fact, we obtain a nontrivial solution.

1. Introduction

We study the existence of solutions for the following quasilinear Schrödinger equations:

$$-\Delta u + V(x)u - \left[\Delta\left(1+u^2\right)^{\alpha/2}\right]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = u^q + u^p, \quad x \in \mathbb{R}^N, \tag{1}$$

where $V \in C(\mathbb{R}^N, \mathbb{R}^+)$ is bounded and periodic in each variable of x_i, $1 \le i \le N$, $N \ge 3$, $T(\alpha) < q+1 < p+1 < \alpha 2^* := 2\alpha N/(N-2)$, $\alpha \ge 1$, and here

$$T(\alpha) := \begin{cases} 2\alpha, & \alpha_0 \le \alpha, \\ 2\alpha_0, & 1 < \alpha < \alpha_0, \\ 12 - 4\sqrt{6}, & \alpha = 1, \end{cases} \tag{2}$$

where α_0 is defined in Lemma 2. These equations are related to existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + W(x)z - h\left(|z|^2\right)z - \Delta g\left(|z|^2\right)g'\left(|z|^2\right)z, \quad x \in \mathbb{R}^N, \tag{3}$$

where W is a given potential and g and h are real functions. Quasilinear equations such as (3) have been accepted as models of several physical phenomena corresponding to various types of g. The case of $g(s) = s^\alpha$ was used for the superfluid film equation in plasma physics [1]. Besides, (3) also appears in plasma physics and fluid mechanics [2], in dissipative quantum mechanics [3], and in the theory of Heisenberg ferromagnetism and magnons [4, 5]. See also [6, 7] for more physical backgrounds. Equations (3) with $\alpha = 1$ have been studied extensively recently; see [8, 9]. When $g(s) = (1+s)^{\alpha/2}$, then (3) turn into our equations (1) with $h(s) = s^q + s^p$. In particular if we let $\alpha = 1$, that is, $g(s) = (1+s)^{1/2}$, (3) models the self-channeling of a high-power ultrashort laser in matter [10]. In this case, few results are known. In [11], the authors proved global existence and uniqueness of small solutions in transverse space dimensions 2 and 3 and local existence without any smallness condition in transverse space dimension 1. In [12], the authors proved the existence of nontrivial solution. When $\alpha > 1$, although we do not know the physical background of (3), in a mathematical sense, we give the proof of the existence of nontrivial solution.

For (1), the main difficulty is that the energy functional associated to (1) is not well defined in $H^1(\mathbb{R}^N)$. To overcome this difficulty, enlightened by [8, 9], we give a new change of variables. Then we reduce the quasilinear problem (1) to a semilinear one, which we will prove has a nontrivial solutions.

Our main result is the following.

Theorem 1. *Assume that $\alpha \ge 1$ and $T(\alpha) < q+1 < p+1 < \alpha 2^*$. Then (1) has a nontrivial solution.*

In this paper, C denotes positive (possibly different) constant, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm

$|u|_p = (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$, $1 \le p < \infty$, and $H^1(\mathbb{R}^N)$ denotes the Sobolev space with norm $\|u\| = (\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx)^{1/2}$.

2. The Change of Variables

We note that the solutions of (1) are the critical points of the following functional:

$$\begin{aligned} I(u) = {} & \frac{1}{2}\int_{\mathbb{R}^N}\left[1 + \frac{\alpha^2 u^2}{2(1+u^2)^{2-\alpha}}\right]|\nabla u|^2 dx \\ & + \frac{1}{2}\int_{\mathbb{R}^N} V(x) u^2 dx \\ & - \frac{1}{q+1}\int_{\mathbb{R}^N} u^{q+1} dx - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1} dx. \end{aligned} \tag{4}$$

Since the functional $I(u)$ may not be well defined in the usual Sobolev spaces $H^1(\mathbb{R}^N)$, we make a change of variables as

$$v = G(u) = \int_0^u g(t)\,dt, \tag{5}$$

where $g(t) = \sqrt{1 + \alpha^2 t^2/2(1+t^2)^{2-\alpha}}$. Since $g(t)$ is monotonous with $|t|$, the inverse function $G^{-1}(t)$ of $G(t)$ exists. Then after the change of variables, $I(u)$ can be written by

$$\begin{aligned} J(v) = {} & \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left|G^{-1}(v)\right|^2 dx \\ & - \frac{1}{q+1}\int_{\mathbb{R}^N}\left|G^{-1}(v)\right|^{q+1} dx \\ & - \frac{1}{p+1}\int_{\mathbb{R}^N}\left|G^{-1}(v)\right|^{p+1} dx. \end{aligned} \tag{6}$$

By Lemma 2 listed below, we have $\lim_{t\to 0} G^{-1}(t)/t = 1$ and $\lim_{t\to\infty}|G^{-1}(t)|^\alpha/t = \sqrt{2}$ $(\alpha > 1)$ or $\sqrt{2/3}$ $(\alpha = 1)$, so $J(v)$ is well defined in $H^1(\mathbb{R}^N)$ and $J(v) \in C^1$.

If u is a nontrivial solution of (1), then for all $\phi \in C_0^\infty(\mathbb{R}^N)$ it should satisfy

$$\begin{aligned} \int_{\mathbb{R}^N} & \left[g^2(u)\nabla u \nabla\phi + g(u)g'(u)|\nabla u|^2\phi + V(x)u\phi \right. \\ & \left. - u^q\phi - u^p\phi\right] dx = 0. \end{aligned} \tag{7}$$

We show that (7) is equivalent to

$$\begin{aligned} J'(v)\psi = {} & \int_{\mathbb{R}^N}\left[\nabla v \nabla\psi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi \right. \\ & \left. - \frac{\left|G^{-1}(v)\right|^q}{g(G^{-1}(v))}\psi - \frac{\left|G^{-1}(v)\right|^p}{g(G^{-1}(v))}\psi\right] dx \\ = {} & 0, \quad \forall\psi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \tag{8}$$

Indeed, if we choose $\phi = (1/g(u))\psi$ in (7), then we get (8). On the other hand, since $u = G^{-1}(v)$, if we let $\psi = g(u)\phi$ in (8), we get (7). Therefore, in order to find the nontrivial solutions of (1), it suffices to study the existence of the nontrivial solutions of the following equations:

$$-\Delta v = -V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} + \frac{\left|G^{-1}(v)\right|^q}{g(G^{-1}(v))} + \frac{\left|G^{-1}(v)\right|^p}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{9}$$

Before we close this section, we give some properties of the change of variables.

Lemma 2. *For all $t > 0$, one has the following:*

(1) $\lim_{t\to 0}(G^{-1}(t)/t) = 1$,

(2) *(i) if $\alpha > 1$ then $\lim_{t\to\infty}(|G^{-1}(t)|^\alpha/t) = \sqrt{2}$, and (ii) if $\alpha = 1$ then $\lim_{t\to\infty}(|G^{-1}(t)|/t) = \sqrt{2/3}$,*

(3) $|G^{-1}(t)| \le t$,

(4) *(i) if $\alpha_0 \le \alpha$ then $tg'(t)/g(t) \le \alpha - 1$, (ii) if $1 < \alpha \le \alpha_0$ then $tg'(t)/g(t) \le \alpha_0 - 1$, and (iii) if $\alpha = 1$ then $tg'(t)/g(t) \le 5 - 2\sqrt{6}$, where $\alpha_0 \approx 1.36$ is a real root of the equation $\alpha^3 - 4\alpha^2 + 8\alpha - 6 = 0$.*

Proof. (1) We easily get $\lim_{t\to 0}(G^{-1}(t)/t) = (G^{-1}(t))'|_{t=0} = 1/g(G^{-1}(0)) = 1$.

For (2) if $\alpha > 1$, since $g(t) = \sqrt{1 + \alpha^2 t^2/2(1+t^2)^{2-\alpha}} = \sqrt{1 + (\alpha^2 t^2/2(1+t^2))(1+t^2)^{\alpha-1}}$, so $g(t) \sim \sqrt{(\alpha^2/2)t^{2(\alpha-1)}} = (\alpha/\sqrt{2})t^{\alpha-1}$ as $t \to \infty$, then $G(t) = \int_0^t g(s)ds \sim (1/\sqrt{2})t^\alpha$ as $t \to \infty$. Since $G^{-1}(t)$ is the inverse of $G(t)$, so $G^{-1}(t) \sim (\sqrt{2}t)^{1/\alpha}$ as $t \to \infty$, thus we have $\lim_{t\to\infty}(|G^{-1}(t)|^\alpha/t) = \sqrt{2}$. When $\alpha = 1$, the result is obvious since $g(t)$ is an increasing bounded function.

For (3), since $[G^{-1}(t) - (1/g(0))t]' = 1/g(G^{-1}(t)) - 1/g(0) \le 0$, so $G^{-1}(t) \le (1/g(0))t = t$, which proves (3).

Now we prove (4), since $(t/g(t))g'(t) = t^2/2(1+t^2)^2 g^2(t) = t^2/(2 + 5t^2 + 3t^4) = 1/(2/t^2 + 5 + 3t^2) \le 5 - 2\sqrt{6}$, which is (iii). To prove (i), that is,

$$\alpha^2(2-\alpha)t^2 \le 2(\alpha - 1)\left(1+t^2\right)^{3-\alpha}, \tag{10}$$

we set $j(t) = 2(\alpha-1)(1+t^2)^{3-\alpha} - \alpha^2(2-\alpha)t^2$, so $j'(t) = 2t[2(\alpha-1)(3-\alpha)(1+t^2)^{2-\alpha} - \alpha^2(2-\alpha)] := 2tk_\alpha(t)$, where $k_\alpha(t) = 2(\alpha-1)(3-\alpha)(1+t^2)^{2-\alpha} - \alpha^2(2-\alpha)$. Then $k'_\alpha(t) = 4(\alpha-1)(3-\alpha)(2-\alpha)t(1+t^2)^{1-\alpha}$. If $\alpha \le 2$ or $\alpha \ge 3$, we get $k'_\alpha(t) \ge 0$, so $k_\alpha(t) \ge k_\alpha(0)$. We notice that $k_\alpha(0) = \alpha^3 - 4\alpha^2 + 8\alpha - 6$ and $k_\alpha(0)$ is an increasing function with respect to α. By Cardano's formula for cubic equations, we know that $k_\alpha(0)$ has one real root and two complex roots. If we set $\alpha_0 \approx 1.36$ to be the real root of $k_\alpha(0)$, then $k_\alpha(t) \ge k_\alpha(0) \ge 0$ as $\alpha \ge \alpha_0$.

So $j'(t) = 2tk_\alpha(t) \geq 0$. That is $j(t)$ is a increasing function, so $j(t) \geq j(0) = 2(\alpha - 1) > 0$ as $\alpha > 1$. If $2 < \alpha < 3$, we get $k'_\alpha(t) < 0$, so $k_\alpha(t)$ is a decreasing function, but in this case $\lim_{t\to\infty} k_\alpha(t) = (\alpha - 2)\alpha^2 > 0$, so $k_\alpha(t) \geq 0$ for all $t > 0$. Thus we have the same result as $\alpha \leq 2$ or $\alpha \geq 3$, which proves (i). For (ii), by the definition of α_0, we have

$$\frac{\alpha_0^2 t^2 \left(1 + (\alpha_0 - 1) t^2\right)}{2(1+t^2)^{3-\alpha_0} + \alpha_0^2 t^2 \left(1+t^2\right)} \leq \alpha_0 - 1, \tag{11}$$

so

$$\begin{aligned} &\alpha_0^2 t^2 \left(1 + (\alpha_0 - 1) t^2\right) \\ &\quad \leq 2\left(\alpha_0 - 1\right)\left(1+t^2\right)^{3-\alpha_0} + \alpha_0^2\left(\alpha_0 - 1\right) t^2 \left(1+t^2\right). \end{aligned} \tag{12}$$

We add $(\alpha_0 - 1)(\alpha^2 - \alpha_0^2)t^2(1+t^2)$ to both sides of (12), where $\alpha < \alpha_0$. Then

$$\begin{aligned} &\alpha_0^2 t^2 + \alpha_0^2\left(\alpha_0 - 1\right) t^4 + \left(\alpha_0 - 1\right)\left(\alpha^2 - \alpha_0^2\right)\left(t^2 + t^4\right) \\ &\quad \leq 2\left(\alpha_0 - 1\right)\left(1+t^2\right)^{3-\alpha_0} + \left(\alpha_0 - 1\right)\alpha^2 t^2\left(1+t^2\right), \\ &\therefore \left(\alpha_0^2 + \left(\alpha_0 - 1\right)\left(\alpha^2 - \alpha_0^2\right)\right) t^2 + \left(\alpha_0 - 1\right)\alpha^2 t^4 \\ &\quad \leq \left(\alpha_0 - 1\right)\left[2\left(1+t^2\right)^{3-\alpha} + \alpha^2 t^2\left(1+t^2\right)\right]. \end{aligned} \tag{13}$$

We notice that $\alpha_0^2 + (\alpha_0 - 1)(\alpha^2 - \alpha_0^2) \geq \alpha^2$. In fact, $\alpha_0^2 + \alpha_0\alpha^2 - \alpha_0^3 - \alpha^2 + \alpha_0^2 \geq \alpha^2 \Leftrightarrow 2(\alpha_0^2 - \alpha^2) + \alpha_0(\alpha^2 - \alpha_0^2) \geq 0 \Leftrightarrow (\alpha_0^2 - \alpha^2)(2 - \alpha_0) > 0$, and the last inequality is obvious. So

$$\begin{aligned} &\alpha^2 t^2 + \left(\alpha_0 - 1\right)\alpha^2 t^4 \\ &\quad \leq \left(\alpha_0 - 1\right)\left[2\left(1+t^2\right)^{3-\alpha} + \alpha^2 t^2\left(1+t^2\right)\right] \end{aligned} \tag{14}$$

$$\begin{aligned} &\therefore \alpha^2 t^2 + \left(\alpha - 1\right)\alpha^2 t^4 \\ &\quad \leq \left(\alpha_0 - 1\right)\left[2\left(1+t^2\right)^{3-\alpha} + \alpha^2 t^2\left(1+t^2\right)\right], \end{aligned} \tag{15}$$

which implies that $tg'(t)/g(t) \leq \alpha_0 - 1$. □

3. Mountain Pass Geometry

In this section, we establish the geometric hypotheses of the mountain pass theorem.

Lemma 3. *There exist $\rho_0, a_0 > 0$ such that $J(v) \geq a_0$ for all $\|v\| = \rho_0$.*

Proof. Let

$$\begin{aligned} Q(x,t) :=\; & -\frac{1}{2}V(x)\left|G^{-1}(t)\right|^2 + \frac{1}{q+1}\left|G^{-1}(t)\right|^{q+1} \\ & + \frac{1}{p+1}\left|G^{-1}(t)\right|^{p+1}. \end{aligned} \tag{16}$$

Then, by Lemma 2 and $p + 1 < \alpha 2^*$, we have

$$\begin{aligned} \lim_{t\to 0}\frac{Q(x,t)}{t^2} &= \lim_{t\to 0}\left[-\frac{1}{2}V(x)\left(\frac{G^{-1}(t)}{t}\right)^2 \right. \\ &\qquad + \frac{1}{q+1}\left(\frac{G^{-1}(t)}{t}\right)^2\left|G^{-1}(t)\right|^{q-1} \\ &\qquad \left. + \frac{1}{p+1}\left(\frac{G^{-1}(t)}{t}\right)^2\left|G^{-1}(t)\right|^{p-1}\right] \\ &= -\frac{1}{2}V(x), \\ \lim_{t\to\infty}\frac{Q(x,t)}{t^{2^*}} &= \lim_{t\to\infty}\left[\frac{1}{2}V(x)\left(\frac{\left|G^{-1}(t)\right|^\alpha}{t}\right)^{2/\alpha}\frac{1}{t^{2^*-2/\alpha}}\right. \\ &\qquad + \frac{1}{q+1}\left(\frac{\left|G^{-1}(t)\right|^\alpha}{t}\right)^{(q+1)/\alpha}\frac{1}{t^{2^*-(q+1)/\alpha}} \\ &\qquad \left. + \frac{1}{p+1}\left(\frac{\left|G^{-1}(t)\right|^\alpha}{t}\right)^{(p+1)/\alpha}\frac{1}{t^{2^*-(p+1)/\alpha}}\right] \\ &= 0. \end{aligned} \tag{17}$$

Thus, for $\epsilon > 0$ sufficiently small, there exists a constant $C_\epsilon > 0$ such that

$$Q(x,t) \leq \left(-\frac{1}{2}V(x) + \epsilon\right)t^2 + C_\epsilon |t|^{2^*}. \tag{18}$$

Then, we have

$$\begin{aligned} J(v) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)\left|G^{-1}(v)\right|^2 dx \\ &\quad - \frac{1}{q+1}\int_{\mathbb{R}^N}\left|G^{-1}(v)\right|^{q+1}dx - \frac{1}{p+1} \\ &\quad \times \int_{\mathbb{R}^N}\left|G^{-1}(v)\right|^{p+1}dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)v^2 dx - \epsilon \\ &\quad \times \int_{\mathbb{R}^N}v^2 dx - C_\epsilon\int_{\mathbb{R}^N}v^{2^*}dx \\ &\geq C\|v\|^2 - C\|v\|^{2^*}. \end{aligned} \tag{19}$$

Thus, by choosing ρ_0 small, we get the result when $\|v\| = \rho_0$. □

Lemma 4. *There exists $v \in H^1(\mathbb{R}^N)$ such that $J(v) < 0$.*

Proof. Given $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\operatorname{supp}\phi := \overline{B}_1$, we will prove that $J(s\phi) \to -\infty$ as $s \to \infty$, which will prove the result if we take $v = s\phi$ with s large enough. By the proof of Lemma 2, we have $G^{-1}(t) \geq Ct^{1/\alpha}$ as $t \geq 1$, so

$$J(s\phi) \leq \frac{1}{2}s^2 \int_{\mathbb{R}^N} |\nabla\phi|^2 dx + \frac{1}{2}s^2 \times \int_{\mathbb{R}^N} V(x)\phi^2 dx - s^{(p+1)/\alpha} \times \int_{\{|s\phi|\geq 1\}} \phi^{(p+1)/\alpha} dx \longrightarrow -\infty, \tag{20}$$

as $s \to \infty$. Thus, we get the result. □

4. Existence

In consequence of Lemmas 3 and 4 of the Ambrosetti-Rabinowitz mountain pass Theorem [13], see also [14–16], for the constant

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)) > 0, \tag{21}$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0,\ \gamma(1) \neq 0,\ J(\gamma(1)) < 0\}$, and there exists a Palais-Smale seq-uence at level c; that is, $J(v_n) \to c$ and $J'(v_n) \to 0$ as $n \to \infty$.

Lemma 5. *The Palais-Smale sequence $\{v_n\}$ for J is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Since $\{v_n\} \subset H^1(\mathbb{R}^N)$ satisfies

$$J(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left|G^{-1}(v_n)\right|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} \left|G^{-1}(v_n)\right|^{p+1} dx - \frac{1}{q+1}\int_{\mathbb{R}^N} \left|G^{-1}(v_n)\right|^{q+1} dx = c + o(1) \tag{22}$$

and for any $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$J'(v_n)\psi = \int_{\mathbb{R}^N} \left[\nabla v_n \nabla\psi + V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi - \frac{\left|G^{-1}(v_n)\right|^q}{g(G^{-1}(v_n))}\psi - \frac{\left|G^{-1}(v_n)\right|^p}{g(G^{-1}(v_n))}\psi\right] dx = o(1)\|\psi\|. \tag{23}$$

Now, we consider the function $G^{-1}(v_n)g(G^{-1}(v_n))$. Note by Lemma 2 that

$$\left|\nabla\left(G^{-1}(v_n)\,g\left(G^{-1}(v_n)\right)\right)\right| = \left[1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}g'\left(G^{-1}(v_n)\right)\right]|\nabla v_n| \leq \frac{1}{2}T(\alpha)|\nabla v_n|. \tag{24}$$

Combining Lemma 2, we have $G^{-1}(v_n)g(G^{-1}(v_n)) \in H^1(\mathbb{R}^N)$. Thus, since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, by choosing $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ in (23), we deduce that

$$\begin{aligned} o(1)\|v_n\| &= J'(v_n)G^{-1}(v_n)\,g\left(G^{-1}(v_n)\right) \\ &= \int_{\mathbb{R}^N} \left[\left(1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}g'\left(G^{-1}(v_n)\right)\right)|\nabla v_n|^2 + V(x)\left|G^{-1}(v_n)\right|^2 - \left|G^{-1}(v_n)\right|^{q+1} - \left|G^{-1}(v_n)\right|^{p+1}\right] dx \\ &\leq \int_{\mathbb{R}^N} \left[\frac{1}{2}T(\alpha)|\nabla v_n|^2 + V(x)\left|G^{-1}(v_n)\right|^2 - \left|G^{-1}(v_n)\right|^{q+1} - \left|G^{-1}(v_n)\right|^{p+1}\right] dx. \end{aligned} \tag{25}$$

Therefore, by (22) and (25), we have

$$\begin{aligned} &(q+1)J(v_n) - J'(v_n)G^{-1}(v_n)\,g\left(G^{-1}(v_n)\right) \\ &\geq \left(\frac{q+1}{2} - \frac{1}{2}T(\alpha)\right) \times \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{q-1}{2}\int_{\mathbb{R}^N} V(x)\left|G^{-1}(v_n)\right|^2 dx + \left(1 - \frac{q+1}{p+1}\right)\int_{\mathbb{R}^N} \left|G^{-1}(v_n)\right|^{p+1} dx \\ &\geq \left(\frac{q+1}{2} - \frac{1}{2}T(\alpha)\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{q-1}{2} \times \int_{\mathbb{R}^N} V(x)\left|G^{-1}(v_n)\right|^2 dx. \end{aligned} \tag{26}$$

Combining (22) and (26), we get $\int_{\mathbb{R}^N} |G^{-1}(v_n)|^{p+1} dx$ is bounded. To verify that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ we start splitting

$$\int_{\mathbb{R}^N} V(x)v_n^2 dx = \int_{\{x:|v_n(x)|>1\}} V(x)v_n^2 dx + \int_{\{x:|v_n(x)|\leq 1\}} V(x)v_n^2 dx. \tag{27}$$

By the proof of Lemma 2, we have $G^{-1}(t) \geq Ct^{1/\alpha}$, for all $t > 1$ and $G(1) = \int_0^1 g(t)dt \geq \int_0^1 dt = 1$. Therefore

$$\begin{aligned} &\int_{\{x:|v_n(x)|>1\}} V(x)v_n^2 dx \\ &\leq C\int_{\{x:|v_n(x)|>1\}} \left|G^{-1}(v_n)\right|^{2\alpha} dx \\ &\leq C\int_{\{x:|v_n(x)|>1\}} \left|G^{-1}(v_n)\right|^{p+1} dx \leq C. \end{aligned} \tag{28}$$

Since $g(t)$ is increasing and $G(t) = \int_0^t g(s)ds \le g(t)t$, we have

$$\int_{\{x:|v_n(x)|\le 1\}} V(x)\left|G^{-1}(v_n)\right|^2 dx \ge \frac{1}{g^2(G^{-1}(1))}\int_{\{x:|v_n(x)|\le 1\}} V(x)v_n^2 dx. \tag{29}$$

Hence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and this proves Lemma 5. □

Now we give the completion of the proof of Theorem 1.

Proof. First, we will prove that $J'(v) = 0$. That is, v is a weak solution of (9). To prove this, it suffices to show that

$$J'(v)\psi = \int_{\mathbb{R}^N}\left[\nabla v\nabla\psi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi - \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))}\psi - \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))}\psi\right]dx = 0, \quad \forall\psi\in C_0^\infty(\mathbb{R}^N). \tag{30}$$

From Lemma 5, $\{v_n\}$ is a bounded Palais-Smale sequence, and there exists $v\in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. By the Lebesgue dominated theorem, we have

$$\begin{aligned} J'(v_n)\psi - J'(v)\psi &= \int_{\mathbb{R}^N}(\nabla v_n - \nabla v)\nabla\psi\, dx \\ &+ \int_{\mathbb{R}^N} V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right]\psi\, dx \\ &- \int_{\mathbb{R}^N}\left[\frac{|G^{-1}(v_n)|^q}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^q}{g(G^{-1}(v))}\right]\psi\, dx \\ &- \int_{\mathbb{R}^N}\left[\frac{|G^{-1}(v_n)|^p}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))}\right]\psi\, dx \longrightarrow 0. \end{aligned} \tag{31}$$

Hence, $J'(v) = 0$. That is, v is a weak solution of (1).

Next, in order to complete the proof of Theorem 1, we must show that v is nontrivial. By contradiction, we assume $v = 0$. To prove this, we claim that, for all $R > 0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} v_n^2 dx = 0 \tag{32}$$

cannot occur. Suppose by contradiction that (32) occurs; that is, $\{v_n\}$ vanishes. Then by the Lions compactness lemma [16], $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for any $r\in(2,2^*)$. By the proof of Lemma 2, we get $G^{-1}(t) : (\sqrt{2}t)^{1/\alpha}$ as $t\to\infty$, so there exists a suitable constant C such that $G^{-1}(t) \le Ct^{1/\alpha}$. In addition, since $G(t)\le g(t)t$, we have

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|G^{-1}(v_n)|^p}{g(G^{-1}(v_n))}v_n dx &\le \lim_{n\to\infty}\int_{\mathbb{R}^N}\left|G^{-1}(v_n)\right|^{p+1}dx \\ &\le \lim_{n\to\infty} C\int_{\mathbb{R}^N} v_n^{(p+1)/\alpha}dx = 0, \\ \lim_{n\to\infty}\int_{\mathbb{R}^N}\left|G^{-1}(v_n)\right|^{q+1}dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left|G^{-1}(v_n)\right|^{p+1}dx = 0, \end{aligned} \tag{33}$$

and $\lim_{n\to\infty}\int_{\mathbb{R}^N}(|G^{-1}(v_n)|^q/g(G^{-1}(v_n)))v_n dx = 0$ is obvious since $q<p$, which implies that

$$\begin{aligned} 0 &= \lim_{n\to\infty} J'(v_n)v_n \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left[|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n - \frac{|G^{-1}(v_n)|^q}{g(G^{-1}(v_n))}v_n - \frac{|G^{-1}(v_n)|^p}{g(G^{-1}(v_n))}v_n\right]dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left[|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right]dx. \end{aligned} \tag{34}$$

Then,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2 dx = 0, \tag{35}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n dx = 0. \tag{36}$$

On the other hand, by (25), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)\left|G^{-1}(v_n)\right|^2 dx = 0. \tag{37}$$

Combining (35) and (37), we get a contradiction since $J(v_n)\to c>0$. Thus, $\{v_n\}$ does not vanish and there exist $k,\ R>0$ and $\{y_n\}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{B_R(y_n)} v_n^2 dx \ge k > 0. \tag{38}$$

Define $\tilde{v}_n(x) = v_n(x+y_n)$. We may assume that the components of $\{y_n\}$ are integer multiples of the periods of $V(x)$. Since $\{v_n\}$ is a Palais-Smale sequence for J and $V(x)$ is periodic in x_i, $1\le i\le N$, $\{\tilde{v}_n\}$ is also a Palais-Smale sequence for J with $J'(\tilde{v}) = 0$ if $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^1(\mathbb{R}^N)$. Since $\{\tilde{v}_n\}$ does not vanish, we have that $\tilde{v}\ne 0$ is a nontrivial solution of (9). □

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