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Research Article

Extinction and Decay Estimates of Solutions for the p-Laplacian Equations with Nonlinear Absorptions and Nonlocal Sources

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We investigate the extinction and decay estimates of the p-Laplacian equations with nonlinear absorptions and nonlocal sources. By Gagliardo-Nirenberg inequality, we obtain the sufficient conditions of extinction solutions, and we also give the precise decay estimates of the extinction solutions.

1. Introduction

In this paper, we consider the following fast diffusive p-Laplacian equation:

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda \int_\Omega u^q(x,t)\,dx - ku^r, \quad x \in \Omega,\ t > 0, \tag{1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \qquad u(x,0) = u_0(x), \quad x \in \Omega, \tag{2}$$

where $1 < p < 2$, $k, q, \lambda > 0$, $0 < r < 1$, $\Omega \subset R^N$ $(N \geq 2)$ is a bounded domain with smooth boundary and $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a nonnegative function. Equation (1) is a class of nonlinear singular parabolic equations and appears to be relevant in the theory of non-Newtonian fluids perturbed by both nonlocal sources and nonlinear absorptions; see [1–4], for instance. Extinction is the phenomenon whereby the evolution of some nontrivial initial data $u_0(x)$ produces a nontrivial solution $u(x,t)$ in a time interval $0 < t < T$ and $u(x,t) \to 0$ as $t \to T$. As an important property of solutions of developing equations, the extinction recently has been studied intensively by several authors in [5–9]. In paper [10], the authors discussed the extinction behavior of solutions for Problem (1)-(2) when $r = 1$. In this paper, we investigated the extinction of solutions when $0 < r < 1$. Due to the nature of our problem, we would like to use the following lemmas by [11].

Lemma 1 (Gagliardo-Nirenberg inequality). *Suppose that $\beta \geq 0$, $N > p \geq 1$, $\beta + 1 \leq q \leq (\beta+1)Np/(N-p)$; then for u such that $|u|^\beta u \in W^{1,p}(\Omega)$, one has*

$$\|u\|_q \leq C^{1/(\beta+1)} \|u\|_r^{1-\theta} \left\|\nabla\left(|u|^\beta u\right)\right\|_p^{\theta/(\beta+1)} \tag{3}$$

with $\theta = ((\beta+1)r^{-1} - q^{-1})/(N^{-1} - p^{-1} + (\beta+1)r^{-1})$, where C is a constant depending only on N, p, and r.

2. Main Results and Proofs

Theorem 2. *Assume that $p - 1 = q$ with $r < 1$; then the non-negative nontrivial weak solution of Problem (1)-(2) vanishes in finite time for any non-negative initial data provided that $|\Omega|$ or λ is sufficiently small.*

(1) *For the case $2N/(N+2) \leq p < 2$, one has*

$$\|u(\cdot,t)\|_2 \leq \left(\|u_0\|_2^{2-k_1} - M_1(2-k_1)t\right)^{1/(2-k_1)}, \quad t \in [0, T_1), \qquad \|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_1, +\infty), \tag{4}$$

where k_1, M_1, and T_1 are given by (11), (16), *and* (17), *respectively.*

(2) *For the case* $1 < p < 2N/(N+2)$, *one has*

$$\|u(\cdot,t)\|_{1+s} \le \left(\|u_0\|_{1+s}^{1+s-k_2} - M_2(1+s-k_2)t\right)^{1/(1+s-k_2)}, \quad t \in [0,T_2), \tag{5}$$
$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_2,+\infty),$$

where s, k_2, M_2, *and* T_2 *are given by* (18), (22), (26), *and* (28), *respectively.*

Proof. (1) For the case $2N/(N+2) \le p < 2$, multiplying (1) by u and integrating over Ω, we deduce from the Hölder inequality that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_p^p + k\|u\|_{r+1}^{r+1} \le \lambda|\Omega|\,\|u\|_p^p. \tag{6}$$

inequality

$$\|u\|_p \le B\|\nabla u\|_p, \tag{7}$$

where B denotes the optimal embedding constant, combining (6) and (7) we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left(1-\lambda B^p|\Omega|\right)\|\nabla u\|_p^p + k\|u\|_{r+1}^{r+1} \le 0. \tag{8}$$

By Lemma 1, we have

$$\|u\|_2 \le C_1(N,p,r)\|\nabla u\|_p^{\theta}\|u\|_{1+r}^{1-\theta}, \tag{9}$$

where $\theta_1 = (1/(1+r)-1/2)(1/N-1/p+1/(1+r))^{-1}$.

It is easy to check that $\theta_1 \in (0,1]$; using Young's inequality with ε, it follows from (9) that

$$\|u\|_2^{k_1} \le C_1^{k_1}(N,p,r)\left(\varepsilon_1\|\nabla u\|_p^p + C(\varepsilon_1)\|u\|_{1+r}^{pk_1(1-\theta_1)/(p-k_1\theta_1)}\right), \tag{10}$$

where $\varepsilon_1 > 0$ and $k_1 > 0$ will be determined later. We choose

$$k_1 = \frac{2[(1+r)p + N(p-1-r)]}{2p + N(p-1-r)}. \tag{11}$$

Then we can conclude that $k_1 \in (1,2)$ and $pk_1(1-\theta_1)/(p-k_1\theta_1) = 1+r$. Therefore, it follows from (10) that

$$\|u\|_{1+r}^{1+r} \ge \left(C_1^{-k_1}(N,p,r)\|u\|_2^{k_1} - \varepsilon_1\|\nabla u\|_p^p\right)\frac{1}{C(\varepsilon_1)}. \tag{12}$$

By combining (8) and (12), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left(1-\lambda B^p|\Omega| - \frac{k\varepsilon_1}{C(\varepsilon_1)}\right)\|\nabla u\|_p^p + \frac{kC_1^{-k_1}(N,p,r)}{C(\varepsilon_1)}\|u\|_2^{k_1} \le 0. \tag{13}$$

Choosing ε_1 small enough such that $1-k\varepsilon_1/C(\varepsilon_1) > 0$ and $|\Omega| \le (1-k\varepsilon_1/C(\varepsilon_1))/\lambda B^p$, then we have $1-k\varepsilon_1/C(\varepsilon_1) - B^p\lambda|\Omega| > 0$. Therefore, we deduce from $k_1 \in (1,2)$ that

$$\frac{d}{dt}\|u\|_2 + M_1\|u\|_2^{k_1-1} \le 0, \tag{14}$$

which implies that

$$\|u(\cdot,t)\|_2 \le \left(\|u_0\|_2^{2-k_1} - M_1(2-k_1)t\right)^{1/(2-k_1)}, \quad t \in [0,T_1), \tag{15}$$
$$\|u(\cdot,t)\|_2 \equiv 0, \quad t \in [T_1,+\infty),$$

where

$$M_1 = \frac{kC_1^{-k_1}(N,p,r)}{C(\varepsilon_1)}, \tag{16}$$

$$T_1 = \frac{\|u_0\|_2^{2-k_1}}{M_1(2-k_1)}. \tag{17}$$

(2) For the case $1 < p < 2N/(N+2)$, multiplying (1) by u^s, where

$$s > l = \frac{2N - p(1+N)}{p} > 1, \tag{18}$$

integrating over Ω, we deduce from the Hölder inequality that

$$\frac{1}{1+s}\frac{d}{dt}\|u\|_{1+s}^{1+s} + \frac{sp^p}{(p+s-1)^p}\left\|\nabla u^{(p+s-1)/p}\right\|_p^p + k\|u\|_{s+r}^{s+r} \le \lambda|\Omega|\,\|u\|_{p+s-1}^{p+s-1}. \tag{19}$$

By Lemma 1 and $s > 1$, we have

$$\|u\|_{s+1} \le C_2(N,p,r)\left\|\nabla u^{(p+s-1)/p}\right\|_p^{p\theta_2/(p+s-1)}\|u\|_{s+r}^{1-\theta_2}, \tag{20}$$

where $\theta_2 = N(1-r)(p+s-1)/(s+1)[p(s+r)+N(p-1-r)]$. By (18) and $r < 1$, it is easy to check that $\theta_2 \in (0,1)$. By Young's inequality with ε, it follows from (19) that

$$\|u\|_{s+1}^{k_2} \le C_2^{k_2}(N,p,r,s)\left(\varepsilon_2\left\|\nabla u^{(p+s-1)/p}\right\|_p^p + C(\varepsilon_2) \times\|u\|_{s+r}^{(1-\theta_2)k_2(p+s-1)/(p+s-1-k_2\theta_2)}\right), \tag{21}$$

where $\varepsilon_2 > 0$ and $k_2 > 0$ will be determined later. We choose

$$k_2 = \frac{(s+1)[(s+r)p + N(p-1-r)]}{(s+1)p + N(p-1-r)}; \tag{22}$$

then it follows that $k_2 \in (s, s+1)$ and $(p+s-1)k_2(1-\theta_2)/(p+s-1-k_2\theta_2) = s+r$. Therefore, it follows from (21) that

$$\|u\|_{s+r}^{s+r} \ge \frac{C_2^{-k_2}(N,p,r,s)}{C(\varepsilon_2)}\|u\|_{s+1}^{k_2} - \frac{\varepsilon_2}{C(\varepsilon_2)}\left\|\nabla u^{(p+s-1)/p}\right\|_p^p. \tag{23}$$

By combining (19) and (23), we have by poincare inequality

$$\frac{1}{1+s}\frac{d}{dt}\|u\|_{1+s}^{1+s} + \left(\frac{sp^p}{(p+s-1)^p} - \frac{k\varepsilon_2}{C(\varepsilon_2)} - \lambda|\Omega|B^p\right) \times\left\|\nabla u^{(p+s-1)/p}\right\|_p^p + \frac{kC_2^{k_2}(N,p,r,s)}{C(\varepsilon_2)}\|u\|_{s+1}^{k_2} \le 0. \tag{24}$$

Choosing $\varepsilon_2 > 0$ small enough such that $sp^p/(p+s-1)^p - k\varepsilon_2/C(\varepsilon_2) > 0$ and $|\Omega| \le (sp^p/(p+s-1)^p - k\varepsilon_2/C(\varepsilon_2))/\lambda|\Omega|B^p$, then we have $sp^p/(p+s-1)^p - k\varepsilon_2/C(\varepsilon_2) - \lambda|\Omega|B^p > 0$. Therefore, we deduce from $k_2 \in (s, s+1)$ that

$$\frac{d}{dt}\|u\|_{1+s} + M_2\|u\|_{1+s}^{k_2-s} \le 0, \tag{25}$$

where

$$M_2 = \frac{kC_2^{-k_2}(N,p,r,s)}{C(\varepsilon_2)}, \tag{26}$$

which implies that

$$\begin{aligned}\|u(\cdot,t)\|_{1+s} &\le \left[\|u_0\|_{1+s}^{1+s-k_2} - M_2(1+s-k_2)t\right]^{1/(1+s-k_2)}, \\ &\qquad t \in [0, T_2), \\ \|u(\cdot,t)\|_{1+s} &\equiv 0, \quad t \in [T_2, +\infty),\end{aligned} \tag{27}$$

where

$$T_2 = \frac{\|u_0\|_{1+s}^{1+s-k_2}}{M_2(1+s-k_2)}. \tag{28}$$

The proof of Theorem 2 is complete. □

Theorem 3. *Assume that $r < 1$.*

(1) *If $2N/(N+2) \le p < 2$ with $q > k_1 - 1 = (2rp + N(p-1-r))/(2p+N(p-1-r))$, then the non-negative nontrivial weak solution of Problem* (1)-(2) *vanishes in finite time provided that u_0 (or $|\Omega|$ or λ) is sufficiently small and*

$$\begin{aligned}\|u(\cdot,t)\|_2 &\le \left(\|u_0\|_2^{2-k_1} - (2-k_1)M_3t\right)^{1/(2-k_1)}, \\ &\qquad t \in [0, T_3), \\ \|u(\cdot,t)\|_2 &\equiv 0, \quad t \in [T_3, +\infty),\end{aligned} \tag{29}$$

where k_1, M_3, and T_3 are given by (11), (35), *and* (33), *respectively.*

(2) *If $1 < p < 2N/(N+2)$ with $q > k_2 - s = ((s+1)rp + N(p-1-r))/((s+1)p + N(p-1-r))$, then the non-negative nontrivial weak solution of Problem* (1)-(2) *vanishes if finite time provided that u_0 (or $|\Omega|$ or λ) is sufficiently small and*

$$\begin{aligned}\|u(\cdot,t)\|_{s+1} &\le \left(\|u_0\|_{s+1}^{s+1-k_2} - (s+1-k_2M_4)t\right)^{1/(s+1-k_2)}, \\ &\qquad t \in [0, T_4), \\ \|u(\cdot,t)\|_{s+1} &\equiv 0, \quad t \in [T_4, +\infty),\end{aligned} \tag{30}$$

where s, k_2, M_4, and T_4 are given by (18), (22), (39), *and* (41), *respectively.*

Proof. (1) If $2N/(N+2) \le p < 2$, multiplying (1) by u and integrating over Ω, we deduce from (12) and the Hölder inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \left(1 - \frac{k\varepsilon_1}{C(\varepsilon_1)}\right)\|\nabla u\|_p^p + \frac{kC_1^{-k_1}(N,p,r)}{C(\varepsilon_1)} \\ &\quad \times \|u\|_2^{k_1} - \lambda|\Omega|^{(3-q)/2}\|u\|_2^{q+1} \le 0.\end{aligned} \tag{31}$$

By choosing $\varepsilon_1 > 0$ small enough such that $1 - k\varepsilon_1/C(\varepsilon_1) \ge 0$, we obtain that

$$\frac{d}{dt}\|u\|_2 + M_3\|u\|_2^{k_1-1} \le 0, \tag{32}$$

provided that $\|u_0\|_2 \le (kC_1^{-k_1}(N,p,r)/C(\varepsilon_1)\lambda|\Omega|^{(3-q)/2})^{1/(q-k_1+1)}$ and $q > k_1 - 1 = (2rp + N(p-1-r))/(2p+N(p-1-r))$, where

$$M_3 = \frac{kC_1^{-k_1}(N,p,r)}{C(\varepsilon_1)} - \lambda|\Omega|^{(3-q)/2}\|u_0\|_2^{q-k_1+1} > 0. \tag{33}$$

From (32) and $k_1 \in (1,2)$, we can derive that

$$\begin{aligned}\|u(\cdot,t)\|_2 &\le \left(\|u_0\|_2^{2-k_1} - (2-k_1)M_3t\right)^{1/(2-k_1)}, \\ &\qquad t \in [0, T_3), \\ \|u(\cdot,t)\|_2 &\equiv 0, \quad t \in [T_3, +\infty),\end{aligned} \tag{34}$$

where

$$T_3 = \frac{\|u_0\|_2^{2-k_1}}{(2-k_1)M_3}. \tag{35}$$

(2) If $1 < p < 2N/(N+2)$, multiplying (1) by u^s, where s is given by (18) and integrating over Ω, we deduce from the Hölder inequality and (23) that

$$\begin{aligned}&\frac{1}{1+s}\frac{d}{dt}\|u\|_{1+s}^{1+s} + \left(\frac{sp^p}{(p+s-1)^p} - \frac{k\varepsilon_2}{C(\varepsilon_2)}\right)\left\|\nabla u^{(p+s-1)/p}\right\|_p^p \\ &\quad + \frac{kC_2^{-k_2}(N,p,r,s)}{C(\varepsilon_2)}\|u\|_{s+1}^{k_2} \le \lambda\|u\|_{1+s}^{q+s}|\Omega|^{(2+s-q)/(1+s)}.\end{aligned} \tag{36}$$

Choosing $\varepsilon_2 > 0$ small enough such that $sp^p/(p+s-1)^p - k\varepsilon_2/C(\varepsilon_2) > 0$, we have

$$\begin{aligned}&\frac{d}{dt}\|u\|_{1+s} + \|u\|_{1+s}^{k_2-s}\left(\frac{kC_2^{k_2}(N,p,r,s)}{C(\varepsilon_2)}\right. \\ &\qquad \left. - \lambda|\Omega|^{(2+s-q)/(1+s)}\|u\|_{1+s}^{q+s-k_2}\right) \le 0.\end{aligned} \tag{37}$$

Therefore, we have

$$\frac{d}{dt}\|u\|_{1+s} + M_4\|u\|_{1+s}^{k_2-s} \le 0, \tag{38}$$

provided that $\|u_0\|_{1+s} \le (kC_2^{-k_2}(N,p,r,s)/\ C(\varepsilon_2)\lambda |\Omega|^{(2+s-q)/(1+s)})^{1/(q+s-k_2)}$ and $q > k_2 - s = ((s+1)rp + N(p-1-r))/((s+1)p + N(p-1-r))$, where

$$M_4 = \frac{kC_2^{-k_2}(N,p,r,s)}{C(\varepsilon_2)} - \lambda|\Omega|^{(2+s-q)/(1+s)} \|u_0\|_{1+s}^{q+s-k_2} > 0. \tag{39}$$

It follows from (38) and $k_2 \in (s, s+1)$ that

$$\begin{gathered} \|u(\cdot,t)\|_{1+s} \le \left(\|u_0\|_{s+1}^{s+1-k_2} - M_4(s+1-k_2)t\right)^{1/(s+1-k_2)}, \\ t \in [0, T_4), \\ \|u(\cdot,t)\|_{s+1} \equiv 0, \quad t \in [T_4, +\infty), \end{gathered} \tag{40}$$

where

$$T_4 = \frac{\|u_0\|_{1+s}^{1+s-k_2}}{M_4(s+1-k_2)}. \tag{41}$$

The proof of Theorem 3 is complete. □

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