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Research Article

Nonexistence Results for the Schrödinger-Poisson Equations with Spherical and Cylindrical Potentials in $\mathbb{R}^3$

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We study the following Schrödinger-Poisson system: $-\Delta u + V(x)u + \phi u = |u|^{p-1}u$, $-\Delta\phi = u^2$, $\lim_{|x|\to+\infty}\phi(x) = 0$, where $u, \phi : \mathbb{R}^3 \to \mathbb{R}$ are positive radial functions, $p \in (1, +\infty)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and $V(x)$ is allowed to take two different forms including $V(x) = 1/(x_1^2 + x_2^2 + x_3^2)^{\alpha/2}$ and $V(x) = 1/(x_1^2 + x_2^2)^{\alpha/2}$ with $\alpha > 0$. Two theorems for nonexistence of nontrivial solutions are established, giving two regions on the $\alpha - p$ plane where the system has no nontrivial solutions.

1. Introduction

Schrödinger-Poisson systems arise in quantum mechanics and have been studied by many researchers in the recent years. A number of researches have been focused on quantum transport in semiconductor devices using both mathematical analysis and numerical analysis. Mathematical analysis plays a very crucial role in any investigation. In this paper, we study the nonexistence of nontrivial solutions for the following system in $\mathbb{R}^3$:

$$\begin{gathered} -\Delta u + V(x)u + \phi(x)u = |u|^{p-1}u, \\ -\Delta\phi = u^2, \qquad \lim_{|x|\to+\infty}\phi = 0, \end{gathered} \tag{1}$$

where $u, \phi : \mathbb{R}^3 \to \mathbb{R}$ are positive radial functions, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $p \in (1, +\infty)$, and $V(x)$ is allowed to have two different forms including $V(x) = 1/(x_1^2 + x_2^2 + x_3^2)^{\alpha/2}$ and $V(x) = 1/(x_1^2 + x_2^2)^{\alpha/2}$ with $\alpha > 0$.

The above system was introduced in [1] in the study of an N-body quantum problem, that is, the Hartree-Fock system, Kohn-Sham system and, so forth [1–4]. For $V(x)$ in the form of a constant potential, the nonexistence of nontrivial solutions of (1) for $p \notin (1, 5)$ was proved in [5] by using a Pohožaev-type identity. For $V(x)$ in the form of the singular potentials as considered in this work, existence of positive solutions has been established under certain assumption [6]. However, the conditions under which nontrivial solutions do not exist have not yet been full established. Hence, in this paper, we study the nonexistence of solutions to the problem (1) with singular potential.

The main contribution of this work is the development of analytical results giving two regions on the $\alpha - p$ plane where the system (1) has no nontrivial solutions. The two $\alpha - p$ regions are shown in Figure 1. The rest of the paper is organized as follows. In Section 2, we first give some basic definitions and concepts and then, based on the method in Badiale et al. [7], establish a Pohožaev-type identity. In Section 3, we give two theorems summarizing the nonexistence results we obtained and then prove the theorems.

2. Preliminaries and a Pohožaev-Type Identity

Firstly, we briefly introduce some notation and definitions and recall some properties and known results of the second equations (Poisson equation) in (1). Throughout the paper, we

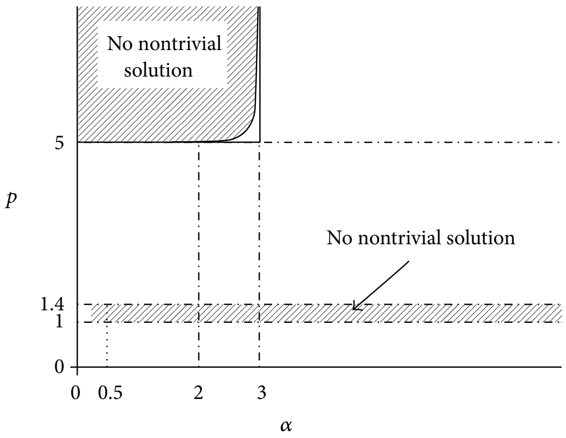

Figure 1: Diagram showing the two regions on the $\alpha - p$ plane where system (1) has no nontrivial solution.

let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $D^{1,2}(\mathbb{R}^3) = \{u(x) \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$, $r_1 = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, and $r_2 = (x_1^2 + x_2^2)^{1/2}$, and for $\alpha > 0$ we define

$$E_1 = \left\{ u \in D^{1,2}\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} \frac{1}{r_1^\alpha} u^2 dx < \infty \right\},$$

$$E_2 = \left\{ u \in D^{1,2}\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} \frac{1}{r_2^\alpha} u^2 dx < \infty; \quad u(x) = u(r_2, x_3) \right\}. \tag{2}$$

By Lemma 2.1 of [2], we know that $-\Delta\phi(x) = u^2$ has a unique solution in $D^{1,2}(\mathbb{R}^3)$ with the form of

$$\phi(x) := \phi_u(x) = \frac{\pi}{4} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \tag{3}$$

for any $u \in L^{12/5}(\mathbb{R}^3)$, and

$$\|\nabla \phi_u(x)\|_2 \leqslant C \|u\|_{12/5}^2, \qquad \int_{\mathbb{R}^3} \phi_u(x) u^2 dy \leqslant C \|u\|_{12/5}^4. \tag{4}$$

By the Hardy-Littlewood-Sobolev inequality, we know that $\int_{\mathbb{R}^3} \phi_u(x) uv\, dy$ is well defined for any $u, v \in L^2 \cap E$. So we can make the following definition.

Definition 1. For $i = 1$ or 2, if $(u, \phi) \in L^2 \cap L^{p+1} \cap E_i \cap C^2(\mathbb{R}^3 \setminus \{r_i = 0\}) \times D^{1,2} \cap C^2(\mathbb{R}^3 \setminus \{r_i = 0\})$ satisfies

$$\int_{\mathbb{R}^3} \left(\nabla u \nabla v + \frac{1}{r_i^\alpha} uv \right) dx + \int_{\mathbb{R}^3} \phi_u(x) uv\, dx = \int_{\mathbb{R}^3} |u|^{p-1} uv\, dx \tag{5}$$

for all $v \in L^2 \cap L^{p+1} \cap E$, we say that (u, ϕ) is a solution of (1).

Now we establish a Pohožaev-type identity based on the work by Badiale et al. [7]. For any $u \in C^2(\mathbb{R}^3 \setminus \{r_i = 0\})$, $x \in \mathbb{R}^3 \setminus \{r_i = 0\}$, where $i = 1, 2$, by a simple calculation, we have

$$\begin{gathered} (x \cdot \nabla u) \Delta u = \operatorname{div}\left[(x \cdot \nabla u) \nabla u - \frac{1}{2} |\nabla u|^2 x \right] + \frac{1}{2} |\nabla u|^2, \\ (x \cdot \nabla u) \frac{u}{r_i^\alpha} = \operatorname{div}\left[\frac{1}{2} \frac{u^2}{r_i^\alpha} x \right] - \frac{3-\alpha}{2} \frac{u^2}{r_i^\alpha}, \\ (x \cdot \nabla u) |u|^{p-1} u = \operatorname{div}\left[\frac{1}{p+1} |u|^{p+1} x \right] - \frac{3}{p+1} |u|^{p+1}. \end{gathered} \tag{6}$$

For any open subset $\Omega \subset \mathbb{R}^3 \setminus \{r_i = 0\}$, by using the divergence theorem and (6), we get

$$\begin{aligned} \int_\Omega (x \cdot \nabla u) \Delta u\, dx &= \int_{\partial\Omega} \left[(x \cdot \nabla u)(\nabla u \cdot \nu) - \frac{1}{2} |\nabla u|^2 x \cdot \nu d\sigma \right] + \int_\Omega \frac{1}{2} |\nabla u|^2 dx, \\ \int_\Omega (x \cdot \nabla u) \frac{u}{r_i^\alpha} dx &= \int_{\partial\Omega} \frac{1}{2} \frac{u^2}{r_i^\alpha} x \cdot \nu d\sigma - \frac{3-\alpha}{2} \int_\Omega \frac{u^2}{r_i^\alpha} dx, \\ \int_\Omega (x \cdot \nabla u) |u|^p dx &= \int_{\partial\Omega} \frac{1}{p+1} u^{p+1} x \cdot \nu d\sigma - \frac{3}{p+1} \int_\Omega |u|^{p+1} dx, \end{aligned}$$

$$\begin{aligned} &\int_\Omega \phi(x) u (x \cdot \nabla u)\, dx \\ &= \Sigma_{k=1}^3 \int_\Omega \phi(x) u u_k x_k dx \\ &= \frac{1}{2} \Sigma_{k=1}^3 \int_\Omega \phi(x) \left(u^2\right)_k x_k dx \\ &= \frac{1}{2} \Sigma_{k=1}^3 \left[-\int_\Omega u^2 (x_k \phi(x)_k)\, dx - \int_\Omega \phi(x) u^2 dx + \int_{\partial\Omega} \phi u^2 (x_k \cdot \nu_k)\, d\sigma \right] \\ &= -\frac{1}{2} \int_\Omega u^2 (\nabla\phi \cdot x)\, dx - \frac{3}{2} \int_\Omega \phi(x) u^2 dx + \frac{1}{2} \int_{\partial\Omega} \phi u^2 (x \cdot \nu)\, d\sigma \\ &= \frac{1}{2} \int_\Omega \Delta\phi (\nabla\phi \cdot x)\, dx - \frac{3}{2} \int_\Omega \phi(x) u^2 dx + \frac{1}{2} \int_{\partial\Omega} \phi u^2 (x \cdot \nu)\, d\sigma \\ &= \frac{1}{2} \int_{\partial\Omega} \phi u^2 (x \cdot \nu)\, d\sigma + \frac{1}{2} \int_{\partial\Omega} (x \cdot \nabla\phi)(\nabla\phi \cdot \nu)\, d\sigma - \frac{1}{2} \int_{\partial\Omega} |\nabla\phi|^2 (x \cdot \nu)\, d\sigma \\ &\quad - \frac{3}{2} \int_\Omega \phi u^2 dx + \frac{1}{4} \int_\Omega |\nabla\phi|^2 dx. \end{aligned} \tag{7}$$

So, by multiplying (1) by $(x \cdot \nabla u)$ and using (7), we get

$$\begin{aligned} &\frac{1}{2}\int_{\Omega}|\nabla u|^2 dx + \frac{3-\alpha}{2}\int_{\Omega}\frac{u^2}{r_i^{\alpha}} - \frac{3}{p+1}\int_{\Omega}|u|^{p+1}dx \\ &\quad + \frac{3}{2}\int_{\Omega}\phi u^2 dx - \frac{1}{4}\int_{\Omega}|\nabla\phi|^2 dx \\ &= \int_{\partial\Omega}\frac{1}{2}(x\cdot\nabla\phi)(\nabla\phi\cdot\nu) - (x\cdot\nabla u)(\nabla u\cdot\nu)\, d\sigma \\ &\quad + \int_{\partial\Omega}\left\{\frac{1}{2}\left(|\nabla u|^2 - |\nabla\phi|^2 + \frac{u^2}{r_i^{\alpha}} + \phi u^2\right)\right. \\ &\qquad \left. - \frac{1}{p+1}|u|^{p+1}\right\}(x\cdot\nu)\, d\sigma. \end{aligned} \tag{8}$$

3. Nonexistence Results for the System of Pohožaev-Type Identity Equations

The nonexistence results we obtained for system (1) are summarized in the following two theorems.

Theorem 2. *For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $V(x) = 1/(x_1^2 + x_2^2 + x_3^2)^{\alpha/2}$, if $\alpha \in (0,3)$ and $p \in (1, \min\{7/5, (3+\alpha)/(3-\alpha)\}) \cup [\max\{5, (3+\alpha)/(3-\alpha)\}, +\infty)$, or $\alpha \in [3,\infty)$ and $p \in (1, 7/5]$, any solution (u,ϕ) of problem (1) is trivial.*

Proof of Theorem 2. Let $\infty > R_2 > R_1 > 0$, $B_R = \{x \in \mathbb{R}^3, |x| < R\}$, $\overline{B}_R = \{x \in \mathbb{R}^3, |x| \leqslant R\}$, and $\Omega = B_{R_2} \setminus \overline{B}_{R_1}$; we then have $\partial\Omega = \partial B_{R_1} \cup \partial B_{R_2}$. Since $u \in E_1 \cap L^{p+1}$, $\phi \in D^{1,2}$, we have

$$\begin{aligned} &\int_0^{\infty} dr \int_{\partial B_r} |\nabla u|^2 + |\nabla\phi|^2 \\ &\quad + \frac{u^2}{|x|^{\alpha}} + \phi u^2 + |u|^{p+1} d\sigma \\ &= \int_{\mathbb{R}^3} |\nabla u|^2 + |\nabla\phi|^2 + \frac{u^2}{|x|^{\alpha}} + \phi u^2 \\ &\quad + |u|^{p+1} dx < +\infty. \end{aligned} \tag{9}$$

So, (9) shows that there exist sequences $R_{1,n} \xrightarrow{n} 0$ and $R_{2,n} \xrightarrow{n} +\infty$ such that

$$\begin{aligned} &R_{1,n}\int_{\partial B_{R_{1,n}}} |\nabla u|^2 + |\nabla\phi|^2 + \frac{u^2}{|x|^{\alpha}} + \phi u^2 + |u|^{p+1} d\sigma \xrightarrow{n} 0, \\ &R_{2,n}\int_{\partial B_{R_{2,n}}} |\nabla u|^2 + |\nabla\phi|^2 + \frac{u^2}{|x|^{\alpha}} + \phi u^2 + |u|^{p+1} d\sigma \xrightarrow{n} 0. \end{aligned} \tag{10}$$

On $\partial B_{R_{1,n}}$ we have $\nu(x) = -x/R_{1,n}$. By using Cauchy inequality and (10), we get

$$\begin{aligned} &\left|\int_{\partial B_{R_{1,n}}}\frac{1}{2}(x\cdot\nabla\phi)(\nabla\phi\cdot\nu) - (x\cdot\nabla u)(\nabla u\cdot\nu)\, d\sigma\right| \\ &\quad \leqslant R_{1,n}\int_{\partial B_{R_{1,n}}} |\nabla\phi|^2 + |\nabla u|^2 d\sigma \xrightarrow{n} 0, \\ &\left|\int_{\partial B_{R_{1,n}}}\left\{\frac{1}{2}\left(|\nabla u|^2 - |\nabla\phi|^2 + \frac{u^2}{|x|^{\alpha}} + \phi u^2\right) - \frac{1}{p+1}|u|^{p+1}\right\}\right. \\ &\qquad \left. \times (x\cdot\nu)\, d\sigma\right| \\ &\quad \leqslant R_{1,n}\int_{\partial B_{R_{1,n}}}\frac{1}{2}\left(|\nabla u|^2 - |\nabla\phi|^2 + \frac{u^2}{|x|^{\alpha}} + \phi u^2\right) \\ &\qquad + \frac{1}{p+1}|u|^{p+1} d\sigma \xrightarrow{n} 0. \end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned} &\left|\int_{\partial B_{R_{2,n}}}\frac{1}{2}(x\cdot\nabla\phi)(\nabla\phi\cdot\nu) - (x\cdot\nabla u)(\nabla u\cdot\nu)\, d\sigma\right| \\ &\quad \leqslant R_{2,n}\int_{\partial B_{R_{2,n}}} |\nabla\phi|^2 + |\nabla u|^2 d\sigma \xrightarrow{n} 0. \\ &\left|\int_{\partial B_{R_{2,n}}}\left\{\frac{1}{2}\left(|\nabla u|^2 - |\nabla\phi|^2 + \frac{u^2}{|x|^{\alpha}} + \phi u^2\right)\right.\right. \\ &\qquad \left.\left. - \frac{1}{p+1}|u|^{p+1}\right\}(x\cdot\nu)\, d\sigma\right| \\ &\quad \leqslant R_{2,n}\int_{\partial B_{R_{2,n}}}\frac{1}{2}\left(|\nabla u|^2 - |\nabla\phi|^2 + \frac{u^2}{|x|^{\alpha}} + \phi u^2\right) \\ &\qquad + \frac{1}{p+1}|u|^{p+1} d\sigma \xrightarrow{n} 0. \end{aligned} \tag{12}$$

Hence in (8), by setting $\Omega = B_{R_{2,n}} \setminus \overline{B}_{R_{1,n}}$, as $n \to \infty$, from (11) and (12), we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx + \frac{3-\alpha}{2}\int_{\mathbb{R}^3}\frac{u^2}{|x|^{\alpha}} - \frac{3}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx \\ &\quad + \frac{3}{2}\int_{\mathbb{R}^3}\phi u^2 dx - \frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi|^2 dx = 0. \end{aligned} \tag{13}$$

By the second equation of (1), we have

$$\int_{\mathbb{R}^3}|\nabla\phi|^2 dx = \int_{\mathbb{R}^3}\phi u^2 dx. \tag{14}$$

From (13) and (14), we get

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{3-\alpha}{2}\int_{\mathbb{R}^3}\frac{u^2}{|x|^\alpha}+\frac{5}{4}\int_{\mathbb{R}^3}\phi u^2dx -\frac{3}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx=0. \tag{15}$$

On the other hand, multiplying (1) by u and integrating the result over Ω, where $\Omega\subset\mathbb{R}^3\setminus\{0\}$, we have

$$\int_\Omega -\Delta uu+\frac{u^2}{|x|^\alpha}+\phi(x)u^2dx=\int_\Omega |u|^{p+1}dx. \tag{16}$$

Using the divergence theorem to the first term of (16) yields that

$$\int_\Omega -\Delta uu\,dx=\int_\Omega \nabla u\nabla u\,dx-\int_{\partial\Omega}u\frac{\partial u}{\partial\nu}d\sigma, \tag{17}$$

while the Hölder inequality gives

$$\left|\int_{\partial\Omega}u\frac{\partial u}{\partial\nu}d\sigma\right|\leqslant\left\{\int_{\partial\Omega}u^6d\sigma\right\}^{1/6}\left\{\int_{\partial\Omega}|\nabla u|^2d\sigma\right\}^{1/2}|\partial\Omega|^{1/3}. \tag{18}$$

Setting $\Omega=B_{R_{2,n}}\setminus\overline{B}_{R_{1,n}}$, we have

$$\begin{aligned}&\left|\int_{\partial B_{R_{1,n}}}u\frac{\partial u}{\partial\nu}d\sigma\right|\\&\quad\leqslant C\left\{\int_{\partial B_{R_{1,n}}}u^6d\sigma\right\}^{1/6}\left\{\int_{\partial B_{R_{1,n}}}|\nabla u|^2d\sigma\right\}^{1/2}\\&\quad\times|R_{1,n}|^{2/3}\xrightarrow{n}0,\\&\left|\int_{\partial B_{R_{2,n}}}u\frac{\partial u}{\partial\nu}d\sigma\right|\\&\quad\leqslant C\left\{\int_{\partial B_{R_{2,n}}}u^6d\sigma\right\}^{1/6}\left\{\int_{\partial B_{R_{2,n}}}|\nabla u|^2d\sigma\right\}^{1/2}\\&\quad\times|R_{2,n}|^{2/3}\xrightarrow{n}0.\end{aligned} \tag{19}$$

From (16)-(17) and (19), we have

$$\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}\frac{u^2}{|x|^\alpha}dx+\int_{\mathbb{R}^3}\phi u^2dx-\int_{\mathbb{R}^3}|u|^{p+1}dx=0. \tag{20}$$

By combining (15) and (20), we have

$$\left(\frac{1}{2}-\frac{3}{p+1}\right)\int_{\mathbb{R}^3}|\nabla u|^2dx+\left(\frac{3-\alpha}{2}-\frac{3}{p+1}\right)\int_{\mathbb{R}^3}\frac{u^2}{|x|^\alpha}dx+\left(\frac{5}{4}-\frac{3}{p+1}\right)\int_{\mathbb{R}^3}\phi u^2dx=0. \tag{21}$$

For $1<p\leqslant\min\{(3+\alpha)/(3-\alpha),7/5\}$ or $p\geqslant\max\{5,(3+\alpha)/(3-\alpha)\}$, we have

$$\begin{aligned}&\frac{3}{p+1}\leqslant\min\left\{\frac{1}{2},\frac{3-\alpha}{2},\frac{5}{4}\right\}\\ \text{or}\quad&\frac{3}{p+1}\geqslant\max\left\{\frac{1}{2},\frac{3-\alpha}{2},\frac{5}{4}\right\}.\end{aligned} \tag{22}$$

Then (21) gives that the solution $(u,\phi)\in L^{p+1}(\mathbb{R}^3)\cap E_1(\mathbb{R}^3)\cap C^2(\mathbb{R}^3\setminus\{0\})\times D^{1,2}\cap C^2(\mathbb{R}^3)$ must be trivial. □

Let $L^4_{\mathrm{loc}}(\mathbb{R}^3)=\{u(x):$ for any open domain $\Omega\subset\mathbb{R}^3$, $u(x)\in L^4(\Omega)\}$. Similar to Theorem 2, we get another nonexistence result to the system (1) with potential function $V(x)=1/(x_1^2+x_2^2)^{\alpha/2}$.

Theorem 3. *For $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and $V(x)=1/(x_1^2+x_2^2)^{\alpha/2}$, if $\alpha\in(0,3)$ and $p\in(1,\min\{7/5,(3+\alpha)/(3-\alpha)\})\cup[\max\{5,(3+\alpha)/(3-\alpha)\},+\infty)$, or $\alpha\in[3,\infty)$ and $p\in(1,7/5]$, any solution (u,ϕ) of problem (1) with $(\nabla u,\nabla\phi)\in L^4_{loc}(\mathbb{R}^3)\times L^4_{loc}(\mathbb{R}^3)$ is trivial.*

Proof of Theorem 3. For any $R_2>R_1>0$, setting $\Omega=\Omega_{R_1,R_2}:=\{x\in B_{R_2}:r_2>R_1\}$, then $\partial\Omega_{R_1,R_2}=\{x\in\partial B_{R_2}:r_2\geqslant R_1\}\cup\{x\in B_{R_2}:r_2=R_1\}:=\Sigma_{R_1,R_2}\cup\Gamma_{R_1,R_2}$, where $\Gamma_{R_1,R_2}=\{x\in\mathbb{R}^3:r_2=R_1,\ |x_3|<\sqrt{R_2^2-R_1^2}\}$ and $\nu(x)=(-x_1/R_1,-x_2/R_1,0)$ on Γ_{R_1,R_2}. Note that

$$\begin{aligned}&\int_0^{R_2}dR_1\int_{\tau_{R_1,R_2}}\left(|\nabla u|^2+|\nabla u|^4+|\nabla\phi|^2+|\nabla\phi|^4+\frac{u^2}{|y|^2}+\phi u^2+|u|^{p+1}\right)d\sigma\\&\quad=\int_{B_{R_2}}\left(|\nabla u|^2+|\nabla u|^4+|\nabla\phi|^2+|\nabla\phi|^4+\frac{u^2}{r_2^\alpha}+\phi u^2+|u|^{p+1}\right)dx<\infty.\end{aligned} \tag{23}$$

Let

$$f(R_1)=R_1\int_{T_{R_1,R_2}}\left(|\nabla u|^2+|\nabla u|^4+|\nabla\phi|^2+|\nabla\phi|^4+\phi u^2+\frac{u^2}{r_2^\alpha}+|u|^{p+1}\right)d\sigma\geqslant0. \tag{24}$$

Then

$$\int_0^{R_2}\frac{f(R_1)}{R_1}dR_1<\infty. \tag{25}$$

So we must have $R_{1,n}\xrightarrow{n}0$ such that

$$f(R_{1,n})\xrightarrow{n}0. \tag{26}$$

By using Cauchy inequality and (24)–(26), we have

$$\begin{aligned}
&\left|\int_{\tau_{R_{1,n},R_2}}\left\{\frac{1}{2}\left(|\nabla u|^2-|\nabla\phi|^2+\frac{u^2}{r_2^\alpha}+\phi u^2\right)\right.\right.\\
&\qquad\left.\left.-\frac{1}{p+1}|u|^{p+1}\right\}(x\cdot\nu)\,d\sigma\right|\\
&\qquad\leqslant R_{1,n}\int_{\tau_{R_{1,n},R_2}}\left(|\nabla u|^2+|\nabla\phi|^2+\frac{u^2}{r_2^\alpha}+|u|^{p+1}\right)d\sigma\xrightarrow{n}0,\\
&\left|\int_{\tau_{R_{1,n},R_2}}\left[\frac{1}{2}(x\cdot\nabla\phi)(\nabla\phi\cdot\nu)-(x\cdot\nabla u)(\nabla u\cdot\nu)\right]d\sigma\right|\\
&\quad\leqslant\left|\int_{\tau_{R_{1,n},R_2}}(x\cdot\nabla\phi)(\nabla\phi\cdot\nu)\,d\sigma\right|\\
&\quad\quad+\left|\int_{\tau_{R_{1,n},R_2}}(x\cdot\nabla u)(\nabla u\cdot\nu)\,d\sigma\right|\\
&\quad\leqslant R_2\int_{\tau_{R_{1,n},R_2}}\left(|\nabla u|^2+|\nabla\phi|^2\right)d\sigma\\
&\quad\leqslant R_2\left[\left\{\int_{\tau_{R_{1,n},R_2}}d\sigma\right\}^{1/2}\left\{\int_{\tau_{R_{1,n},R_2}}|\nabla\phi|^4d\sigma\right\}^{1/2}\right.\\
&\quad\quad\left.+\left\{\int_{\tau_{R_{1,n},R_2}}d\sigma\right\}^{1/2}\left\{\int_{\tau_{R_{1,n},R_2}}|\nabla u|^4d\sigma\right\}^{1/2}\right]\\
&\quad=\sqrt{2\pi}R_2^{3/2}\left[\left\{R_{1,n}\int_{\tau_{R_{1,n},R_2}}|\nabla\phi|^4d\sigma\right\}^{1/2}\right.\\
&\quad\quad\left.+\left\{R_{1,n}\int_{\tau_{R_{1,n},R_2}}|\nabla u|^4d\sigma\right\}^{1/2}\right]\xrightarrow{n}0.
\end{aligned}\tag{27}$$

It is easy to see that $\Sigma_{R_{1,n},R_2}\subset\Sigma_{R_{1,n+1},R_2}$ and $\cup_n\Sigma_{R_{1,n},R_2}=\{x\in\partial B_{R_2}: r_2\neq 0\}$. Let $\Omega=\Omega_{R_1,R_2}$ in (18) by using the definition of $\partial\Omega_{R_1,R_n}$ and (27), we get

$$\begin{aligned}
&\frac{1}{2}\int_{B_{R_2}}|\nabla u|^2dx+\frac{3-\alpha}{2}\int_{B_{R_2}}\frac{u^2}{r_2^\alpha}-\frac{3}{p+1}\int_{B_{R_2}}|u|^{p+1}dx\\
&\qquad+\frac{3}{2}\int_{B_{R_2}}\phi u^2dx-\frac{1}{4}\int_{B_{R_2}}|\nabla\phi|^2dx\\
&\quad=\int_{\partial B_{R_2}}\frac{1}{2}(x\cdot\nabla\phi)(\nabla\phi\cdot\nu)-(x\cdot\nabla u)(\nabla u\cdot\nu)\,d\sigma\\
&\qquad+\int_{\partial B_{R_2}}\left\{\frac{1}{2}\left(|\nabla u|^2-|\nabla\phi|^2+\frac{u^2}{r_2^\alpha}+\phi u^2\right)\right.\\
&\qquad\qquad\left.-\frac{1}{p+1}|u|^{p+1}\right\}(x\cdot\nu)\,d\sigma.
\end{aligned}\tag{28}$$

Similar to (12), we have $R_{2,n}\xrightarrow{n}+\infty$ such that

$$\begin{aligned}
&\left|\int_{\partial B_{R_{2,n}}}\left\{\frac{1}{2}\left(|\nabla u|^2-|\nabla\phi|^2+\frac{u^2}{r_2^\alpha}+\phi u^2\right)\right.\right.\\
&\qquad\left.\left.-\frac{1}{p+1}|u|^{p+1}\right\}(x\cdot\nu)\,d\sigma\right|\\
&\quad\leqslant R_{2,n}\int_{\partial B_{R_{2,n}}}|\nabla u|^2+|\nabla\phi|^2+\frac{u^2}{r_2^\alpha}\\
&\qquad+\phi u^2+|u|^{p+1}d\sigma\xrightarrow{n}0,\\
&\left|\int_{\partial B_{R_{2,n}}}\frac{1}{2}(x\cdot\nabla\phi)(\nabla\phi\cdot\nu)-(x\cdot\nabla u)(\nabla u\cdot\nu)\,d\sigma\right|\\
&\quad\leqslant R_{2,n}\int_{\partial B_{R_{2,n}}}|\nabla u|^2+|\nabla\phi|^2d\sigma\xrightarrow{n}0.
\end{aligned}\tag{29}$$

As $n\to+\infty$, (28)–(29) imply that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{3-\alpha}{2}\int_{\mathbb{R}^3}\frac{u^2}{r_i^\alpha}-\frac{3}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx\\
&\qquad+\frac{3}{2}\int_{\mathbb{R}^3}\phi u^2dx-\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi|^2dx=0.
\end{aligned}\tag{30}$$

Since $\int_{\mathbb{R}^3}|\nabla\phi|^2dx=\int_{\mathbb{R}^3}\phi u^2dx$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx+\frac{3-\alpha}{2}\int_{\mathbb{R}^3}\frac{u^2}{r_2^\alpha}+\frac{5}{4}\int_{\mathbb{R}^3}\phi u^2dx\\
&\qquad-\frac{3}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx=0.
\end{aligned}\tag{31}$$

On the other hand, we have

$$\begin{aligned}
&\left|\int_{\tau_{R_{1,n},R_2}}u(\nabla u\cdot\nu)\,d\sigma\right|\\
&\quad\leqslant\left\{\int_{\tau_{R_{1,n},R_2}}|u|^6d\sigma\right\}^{1/6}\left\{\int_{\tau_{R_{1,n},R_2}}|\nabla u|^3d\sigma\right\}^{1/3}\left|\tau_{R_{1,n},R_2}\right|^{1/2}\\
&\quad\leq C\cdot\left\{\int_{\tau_{R_{1,n},R_2}}R_{1,n}|u|^6d\sigma\right\}^{1/6}\\
&\qquad\times\left\{\int_{\tau_{R_{1,n},R_2}}R_{1,n}|\nabla u|^3d\sigma\right\}^{1/3}\xrightarrow{n}0.
\end{aligned}\tag{32}$$

So if we multiply (1) by u and then integrate over the domain $\Omega_{R_{1,n},R_2}$ and let $n\to+\infty$, we have

$$\begin{aligned}
&\int_{B_{R_2}}|\nabla u|^2dx+\int_{B_{R_2}}\frac{u^2}{r_2^\alpha}dx+\int_{B_{R_2}}\phi u^2dx-\int_{B_{R_2}}|u|^{p+1}dx\\
&\quad=\int_{\partial B_{R_2}}u(\nabla u\cdot\nu)\,d\sigma.
\end{aligned}\tag{33}$$

As for (20), we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \frac{u^2}{r_2^{\alpha}} dx + \int_{\mathbb{R}^3} \phi u^2 dx - \int_{\mathbb{R}^3} |u|^{p+1} dx = 0. \tag{34}$$

From (31) and (34), we have

$$\left(\frac{1}{2} - \frac{3}{p+1}\right) \int_{\mathbb{R}^3} |\nabla u|^2 dx + \left(\frac{3-\alpha}{2} - \frac{3}{p+1}\right) \int_{\mathbb{R}^3} \frac{u^2}{r_2^{\alpha}} dx + \left(\frac{5}{4} - \frac{3}{p+1}\right) \int_{\mathbb{R}^3} \phi u^2 dx = 0. \tag{35}$$

For $1 < p \leqslant \min\{(3+\alpha)/(3-\alpha), 7/5\}$ or $p \geqslant \max\{5, (3+\alpha)/(3-\alpha)\}$, (35) implies that the solution of problem (1) with $i = 2, (u, \phi) \in L^{p+1}(\mathbb{R}^3) \cap E_2(\mathbb{R}^3) \cap C^2(\mathbb{R}^3 \setminus \{0\}) \times D^{1,2} \cap C^2(\mathbb{R}^3)$, which satisfies $(\nabla u, \nabla \phi) \in L^4_{\text{loc}}(\mathbb{R}^3) \times L^4_{\text{loc}}(\mathbb{R}^3)$, must be trivial. □

4. Conclusion

We mainly study the nonexistence of nontrivial solutions to system (1) in this paper, giving two regions on the $\alpha - p$ plane where the system (1) has no nontrivial solutions; see Figure 1. In another paper, we will study the existence of nontrivial solutions to system (1).

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