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Research Article

Fractional Dynamics of Genetic Algorithms Using Hexagonal Space Tessellation

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The paper formulates a genetic algorithm that evolves two types of objects in a plane. The fitness function promotes a relationship between the objects that is optimal when some kind of interface between them occurs. Furthermore, the algorithm adopts an hexagonal tessellation of the two-dimensional space for promoting an efficient method of the neighbour modelling. The genetic algorithm produces special patterns with resemblances to those revealed in percolation phenomena or in the symbiosis found in lichens. Besides the analysis of the spacial layout, a modelling of the time evolution is performed by adopting a distance measure and the modelling in the Fourier domain in the perspective of fractional calculus. The results reveal a consistent, and easy to interpret, set of model parameters for distinct operating conditions.

1. Introduction

This paper analyzes the fractional order dynamics during the search for the optimal solution in a plane with an hexagonal tessellation by means of a genetic algorithm. These three distinct scientific topics are recognized to be efficient approaches in particular areas, namely, in the problems of modelling including long-range memory effects, space representation using geometric shapes with no overlaps and no gaps, and robust optimization in cases where standard techniques do not yield adequate solutions. This paper integrates the three methodologies in the analysis of a complex evolutionary optimization for producing solutions somehow resembling the percolation phenomenon, in the inorganic world, or, alternatively, the lichens, in the scope of living beings.

Fractional calculus (FC) is a branch of mathematical analysis that generalizes the operations of differentiation and integration from integer up to real or complex orders [1–5]. The concept emerged in September 30, 1695, when Guillaume de l'Hôpital wrote to Gottfried Leibniz a letter asking him about the meaning of $d^{1/2}y/dx^{1/2}$, to which Leibniz replied "an apparent paradox, from which one-day useful consequences will be drawn." Important minds such as Fourier, Euler, Laplace, Liouville, and Riemann devoted efforts to the development of the theory of FC, but the field remained primarily of pure mathematics. Things started to change in the beginning of the twentieth century with the work of Olivier Heaviside in operational calculus and electromagnetism [6]. Nevertheless, only in the last two decades FC witnessed a substantial progress in the application to physics, engineering, and biology [7–11].

Genetic algorithms (GAs) are a computer heuristic that mimics the process of evolution and belong to the class of the so-called evolutionary algorithms [12–16]. Genetic algorithms (GAs) were invented in the 60s by John Holland and developed by him and his colleagues. GAs are inspired in the genetic structure and behaviour of chromosomes within a population of individuals (the solutions) having in mind the ideas that (i) individuals compete between themselves, (ii) most successful individuals tend to produce more offspring, (iii) the genetic information of "good" individuals disseminates in the population and tends to produce offspring that are better than their parents, and (iv) successive generations become more suited to their living environment. GAs implement an intelligent exploration of the space of solutions by exploiting historical information to direct the search into the region of better performance. During the last decades a growing amount of successful application to real-world problems demonstrated that GAs are a powerful and robust optimisation technique.

The hexagonal tessellation is a regular tiling of the Euclidean plane, in which each vertex meets three hexagons [17, 18]. There are two other regular tessellations of the plane, namely, the triangular and the square tilings. Nevertheless, the hexagonal tessellation constitutes the best way to divide a given surface into regions of equal areas, while having the least total perimeter. This forms the so-called "honeycomb conjecture" that dates back to the ancient Greek mathematician Pappus of Alexandria (c. 290–c. 350) and was proven in 1999 by Hales [19]. We find this structure in nature, such as crystals or honeycombs, built by honey bees, and in man-made structures [20, 21], or even as art in the famous Maurits Escher woodcuts and lithographs [22]. Many other examples can be mentioned such as graphene and superbenzene, substances with atoms arranged in a regular hexagonal structure [23], or pineapples [24], a fruit with a rough skin having a hexagonal pattern of nodules.

The three scientific concepts are put together for simulating and modelling an evolutionary process in a two-dimensional space. First, it is considered a plane where some kind of process evolves. The plane is discretized by means of a regular hexagonal pattern, and the evolution consists of the optimization using a standard GA. Second, the evolution of the GA population is described using a fractional order model that approximates the numerical results. For that purpose, the best individual in each generation of the GA population is analysed in the viewpoint of fitness function, compared with the previous case, and the result is converted into the Fourier domain. An important aspect is also the fitness function that measures the "performance" of each GA individual. In the two-dimensional space are considered two distinct types of objects, and it is assumed that a "good performance" corresponds to a spacial arrangement exhibiting some type of interface between them. By other words, it is assumed that some kind of cooperation, or synergy, exists between the two objects, such that they should coexist close to each other in space. The resulting time-space population reveals fractal characteristics and patterns resembling those of percolation [25, 26], in the inanimate world, or of lichens, when thinking in living organisms [27, 28]. The possible examples correspond only to possible *interpretations* of the abstract algorithm implemented in the paper. Percolation is the phenomenon involved in the movement and filtering of fluids through porous materials. Nevertheless, in the last years percolation brought a new light into many topics such as material science, epidemiology, or geology. On the other hand, lichens are organisms consisting of two partners, namely, a fungus and a green alga growing in a symbiotic relationship. The body of a lichen consists of fungal filaments surrounding the cells of the algae. The basis of the symbiosis in lichens is that the fungus provides the algal protection and gains nutrients in return. Therefore, such examples are merely possible *interpretations* of the simulation results, but, in fact, an abstract formulation is the basis of the proposed study that primarily intends to model the GA evolution with FC tools.

Bearing these ideas in mind this paper is organized as follows. Section 2 formulates the main algorithms and methods. Section 3 presents the experiments and analyzes the results. Finally, Section 4 draws the main conclusions.

2. Main Algorithms and Methods

In this section are introduced briefly some aspects of FC and Laplace transform and the computational implementation of GA.

2.1. Fractional Calculus. The most used definitions of a fractional derivative of order α are the Riemann-Liouville, Grünwald-Letnikov, and Caputo formulations [29]. Fractional derivatives capture the history of past events, contrary to integer derivatives that are merely "local" operators. This property has been recognized both in natural and man-made phenomena, where modelling becomes simpler using FC rather than building complicated integer order expressions.

Using the Laplace transform, for zero initial conditions, we have the expression

$$\mathscr{L}\left\{D_t^{\alpha} f(t)\right\} = s^{\alpha}\mathscr{L}\left\{f(t)\right\}, \tag{1}$$

where s and $\mathscr{L}$ denote the Laplace variable and operator, respectively.

In the scope of FC it is also important to mention the Mittag-Leffler function $E_\alpha(t)$ defined as [30–33]

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathscr{C}, \quad \operatorname{Re}(\alpha) > 0. \tag{2}$$

The Mittag-Leffler function is a generalization of the exponential and the power laws. The first occurs in phenomena governed by integer dynamics, and the second emerges in fractional dynamics. In particular, when $\alpha = 1$ yields $E_1(t) = e^t$, while, for large values of t, the asymptotic behaviour of the ML leads to $E_\alpha(-t) \approx (1/\Gamma(1-\alpha))(1/t)$, $\alpha \neq 1$, $0 < \alpha < 2$. Applying the Laplace transform

$$\mathscr{L}\left\{E_\alpha\left(\pm a t^{\alpha}\right)\right\} = \frac{s^{\alpha-1}}{s^{\alpha} \mp a}, \tag{3}$$

we verify the generalization from the exponential up to the Mittag-Leffler function, that is, from integer up to fractional powers of s.

These results mean that standard methods in modelling and control, such as transfer functions and frequency response, can be directly applied as long as we allow the substitution of integer orders by their fractional counterparts.

2.2. Genetic Algorithms. GAs are a computer method to find approximate solutions in optimization problems. GAs are implemented such that a population of N possible solutions evolves with successive iterations towards better approximations. In the GA formulation it is necessary to define the genetic representation of the problem and the fitness function that measures how successfully a given individual approximates the solution. In the GA execution the population is initialized randomly and after it is improved applying iteratively the operations of mutation, crossover, and selection that mimic Darwin's theory. During the evolution a given part of the population is selected to breed the new generation. Solutions are selected by means of the fitness function. Therefore, those individuals that have the best fitness values are preferred. The GA execution is ended when some predefined condition is obeyed, such as when the maximum number of generations $t_{\max}$ is reached or when a satisfactory fitness value is obtained. The technique of "elitism" is often adopted that allows the better individuals to carry over, unaltered, to the next generation.

The pseudo-code of a GA is as follow:

(1) generate randomly the initial population of individuals (solutions);

(2) evaluate the fitness function for each individual in the population;

(3) repeat:

   (a) select the individuals with best fitness value for reproducing;

   (b) treat the population by means of the crossover and mutation operators and produce offspring;

   (c) evaluate the fitness value of each individual in the offspring;

   (d) replace the worst ranked part of previous population by the best individuals of the produced offspring;

   (e) until termination.

3. Numerical Experiments

In this section we describe the experiments with the GA and we analyse the results in the perspective of fractional dynamics.

3.1. Genetic Algorithm Using Hexagonal Tessellation. We consider a two-dimensional space subdivided in discrete cells and where three types of situation may occur. The cells consists of a tessellation using regular hexagons as represented in Figure 1, where (i, j) denotes the cell indexing and the six gray cells define the set $\mathscr{A}$ of neighbours.

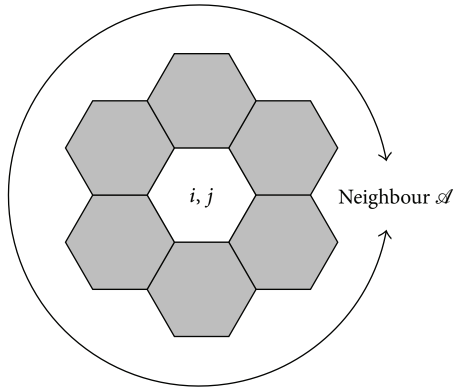

Figure 1: Hexagonal tessellation of the plane and definition of neighbour set $\mathscr{A}$ of cell (i, j).

Each cell has three possible situations, namely, "*empty*," "*occupied with object type 1,*" and "*occupied with object type 2.*"

In the GA these objects interact by means of a fitness function J defined as

$$J = \sum_{i=1}^{i_{\max}} \sum_{j=1}^{j_{\max}} \left\{ (i, j) \perp \mathscr{A} \right\}, \tag{4}$$

where $i_{\max}$ and $j_{\max}$ denote the maximum values for indices i and j, respectively. The notation $(i, j) \perp \mathscr{A}$ describes the logical operation of comparing the object present in cell (i, j) with the set of six neighbours $\mathscr{A}$ and incrementing the value of J by "1" (or "0") each time the cells have different (or identical) types of objects. Therefore, the logical operation $(i, j) \perp \mathscr{A}$ yields a value between 0 and 6, with each term "1" (or "0") being the result of verifying if cell (i, j) is occupied with "object type 1" (or "object type 2") and a given cell in $\mathscr{A}$ is occupied with "object type 2" (or "object type 1"). For "empty" cells, or for cells near the boundary of the tessellation space, no logical action is performed.

In the numerical experiments neighbours with more cells were tested, namely, with a second and a third ring having 12 and 18 hexagons and having different weights. Nevertheless, the results were qualitatively of the same type, and, therefore, these experiments are not reported in this paper. Furthermore, for testing the surrounding cells different fitness functions were also evaluated. Again, while leading to different plots, the results were not significantly distinct and are not described in the sequel.

We must note that fitness (4) is straightforward to calculate, leading to a fast computational implementation, while accessing as "good" a genetic species that includes a high number of variations between objects in neighbour cells. This abstract notion of discontinuity can be *interpreted* according to the type of application. As mentioned in Introduction we can *interpret* for percolation as some type of interface between two distinct materials or for lichens as the interface between two symbiotic species.

For the crossover operation a simple one-point scheme is considered. First, the indices i and j, for the one-point crossover, are randomly generated. Second, the individual 1 of the offspring is generated by selecting the upper left and lower right corners of parent 1 and the upper right and lower left corners of parent 2. The individual 2 of the offspring is obtained using the complementary selection of the corners in parents 1 and 2.

3.2. Numerical Experiments and Fractional Dynamics. In this subsection experiments with GA populations of $N = \{100, 200, 500, 1000\}$ individuals and a two-dimensional space are developed such that $i_{\max} = 30$ and $j_{\max} = 60$. The GA terminates for $t_{\max} = 10^3$ generations. In the plots are adopted the colours white, red, and green for the cases of cells *"empty," "occupied with object type 1,"* and *"occupied with*

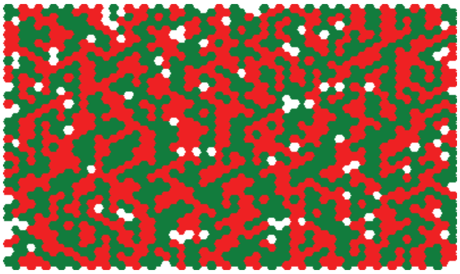

FIGURE 2: GA result for $N = 100$, $P_0 = \{0.2, 0.4, 0.4\}$.

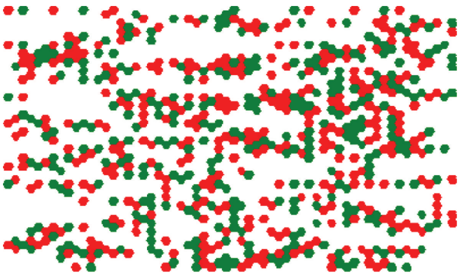

FIGURE 3: GA result for $N = 100$, $P_0 = \{0.8, 0.2, 0.2\}$.

object type 2," respectively. Moreover, different initializations of the GA population, such that the three types of cells have distinct probabilities, are also tested. Let us represent the initialization probabilities $P_0 = \{white, red, green\}$. In the sequel the cases $P_0 = \{0.2, 0.4, 0.4\}$, $P_0 = \{0.33, 0.33, 0.33\}$, $P_0 = \{0.5, 0.25, 0.25\}$, and $P_0 = \{0.8, 0.1, 0.1\}$ are considered, corresponding to environmental conditions varying from "fertile" up to "arid." For the GA parameters the one-point, crossover with tournament selection is adopted, 100% crossover rate and elitism, a mutation probability of 0.05.

Figures 2 and 3 show the plots resulting for $N = 100$ with $P_0 = \{0.2, 0.4, 0.4\}$ and $P_0 = \{0.8, 0.2, 0.2\}$, respectively. We observe clearly the fractal structure in space and the interlacing between the two distinct types of objects. Moreover, the effect of the initial conditions is also clear, that is, the outcome of having rich or poor populating conditions of the tessellated plane.

The analysis of the GA dynamics [34, 35] in space time requires the definition and clarification of several concepts. In this line of thought, the term "space" represents the plane where the objects are laid. Since we consider an hexagonal tessellation, the space points are represented by cells (i, j), $i = 1, \ldots, i_{\max}$, and $j = 1, \ldots, j_{\max}$. The term "time" denotes the *iteration time* consisting of the successive GA generations $t = 1, \ldots, t_{\max}$. For the characterization of the GA population the individual with best fitness value J is considered, since usually in GA applications only the best solution is selected. Furthermore, for describing the *dynamics, d*, an index measuring the "distance" between two consecutive best individuals is considered, defined as

$$d(t) = \sum_{i=1}^{i_{\max}} \sum_{j=1}^{j_{\max}} \left\{ (i, j)_t \neg (i, j)_{t-1} \right\}, \qquad (5)$$

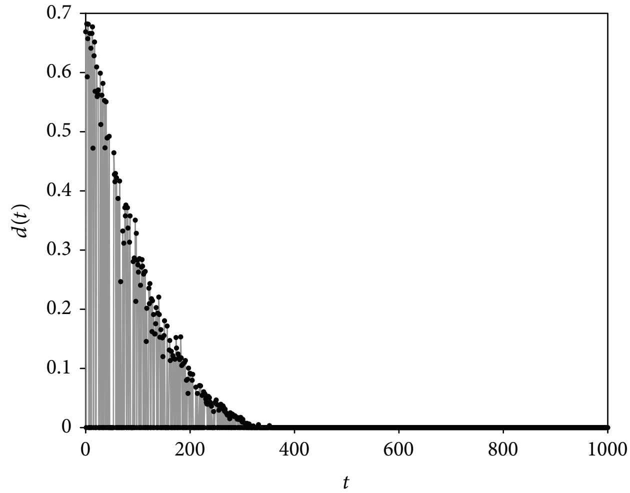

FIGURE 4: Plot of $d(t)$ for $N = 100$, $P_0 = \{0.33, 0.33, 0.33\}$.

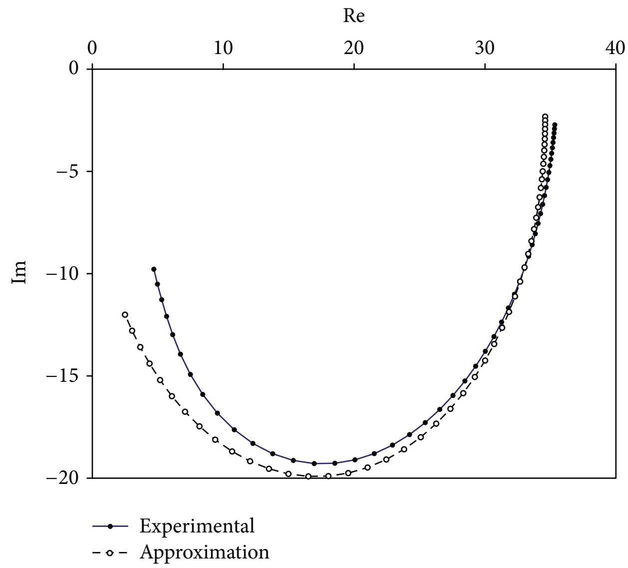

FIGURE 5: Polar diagram of $d(t)$ and its approximation (7) for $N = 100$, $P_0 = \{0.33, 0.33, 0.33\}$ and $0.001 \leq \omega \leq 0.03$.

where the notation $(i, j)_t \neg (i, j)_{t-1}$ describes the logical operation of comparing the objects present in cell (i, j) at iterations t and $t-1$ and incrementing the value of d by "1" (or "0") if the cells have different (or identical) types of objects. Therefore, d yields the value 0 ($i_{\max} \times j_{\max}$) for two completely identical (distinct) consecutive best individuals.

Figure 4 depicts $d(t)$ for $N = 100$, $P_0 = \{0.33, 0.33, 0.33\}$. We observe a vanishing transient with severe discontinuities. The vanishing value means the convergence towards the final value while the discontinuities reveal that often successive generations change only slightly or even do not evolve at all. These transients vary with the initial conditions, that is, with the probabilities P_0 and the GA parameters, namely, the population size N. Therefore, in the sequel several combinations of values of P_0 and N are tested.

It was decided to identify a parametric model in the Fourier domain since usually it leads to a simple and robust

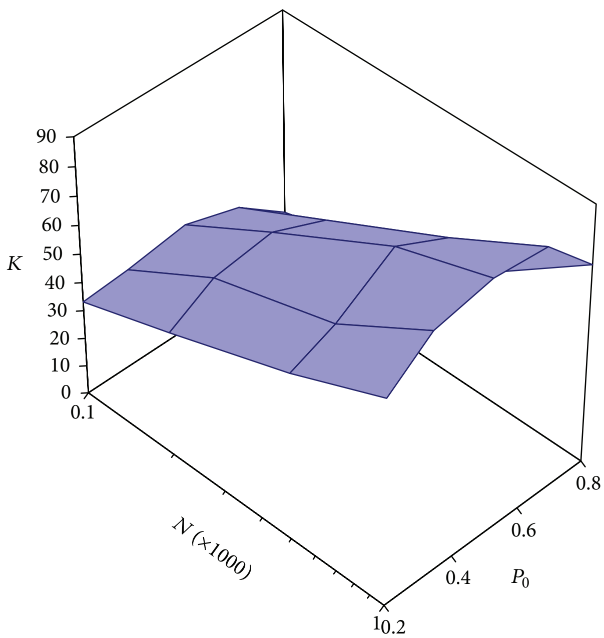

Figure 6: Variation of the transfer function parameter K versus N and P_0.

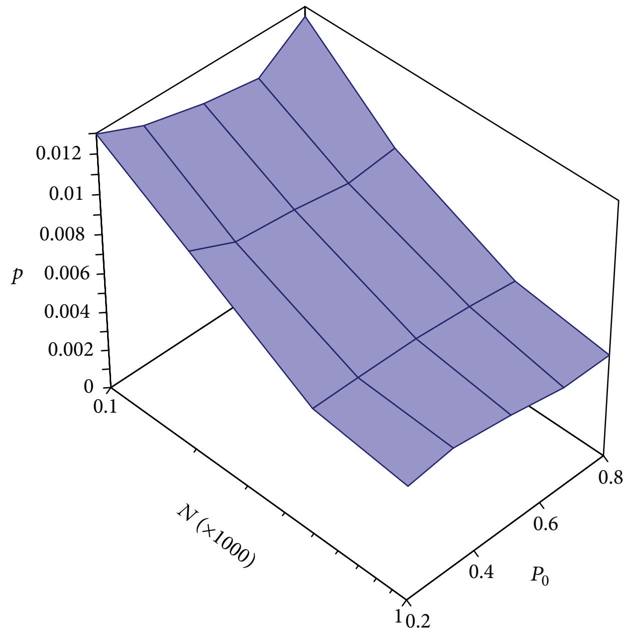

Figure 7: Variation of the transfer function parameter p versus N and P_0.

procedure. Therefore, the Fourier transform of each time response, $D(\imath\omega) = \mathcal{F}[d(t)]$, was determined. For the identification several transfer functions were tested trying to establish a compromise between accuracy and complexity, while trying to preserve the same type of expression for all cases under study. The final choice fell on a function $G(s)$ with 3 parameters given by the expression

$$G(s) = \frac{K}{1 + (s/p)^{\alpha}} e^{-s\tau}, \tag{6}$$

where K denotes the gain, p represents a pole of fractional order α, and τ stands for a time delay.

For example, Figure 5 depicts the polar diagram of the experimental result and approximation (6), that is, $\mathrm{Re} = \Re\{D(\imath\omega)\}$ versus $\mathrm{Im} = \Im\{D(\imath\omega)\}$, for $N = 100$ and $P_0 = \{0.33, 0.33, 0.33\}$. Several experiments demonstrated that the low frequency content of $D(\imath\omega)$ is invariant with different GA seeds, in opposition with the high frequency behaviour that reflects the stochastic nature of the algorithm and reveals noisy characteristics in the Fourier domain. Therefore, in the sequel a bandwidth limitation is considered such that $\omega \leq 0.03$.

Given the large number of combinations of values for N and P_0, the identification was performed automatically by means of a second GA having a population of 500 individuals and terminating after calculating 500 iterations. Several tests demonstrated that good identification results were produced by the fitness function:

$$\sum_{\omega=0.001}^{\omega=0.03} \{\Re[D(\imath\omega)] - \Re[G(\imath\omega)]\}^2 + \{\Im[D(\imath\omega)] - \Im[G(\imath\omega)]\}. \tag{7}$$

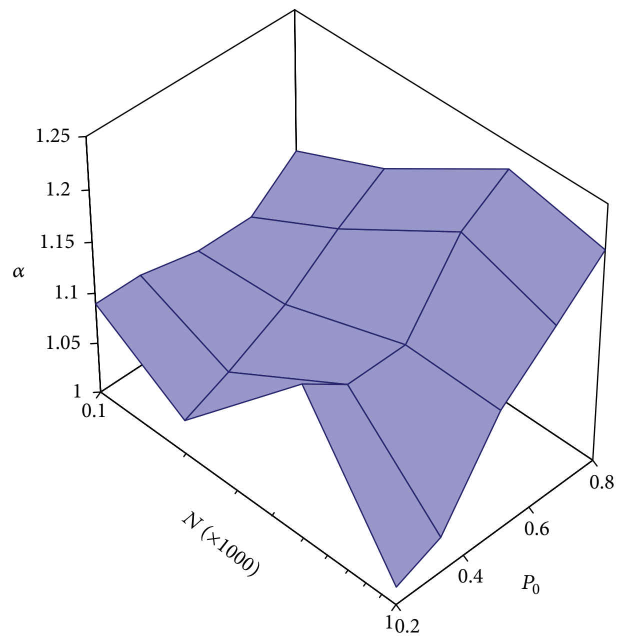

Figure 8: Variation of the transfer function parameter α versus N and P_0.

Figures 6, 7, 8, and 9 show the variation of the transfer function parameters $\{K, p, \alpha, \tau\}$ versus N and P_0.

We observe that the parameters of the transfer function (6) have the following behaviour:

(i) K grows with N and has a maximum for $P_0 = \{0.5, 0.25, 0.25\}$;

(ii) p decreases with N and is independent of P_0;

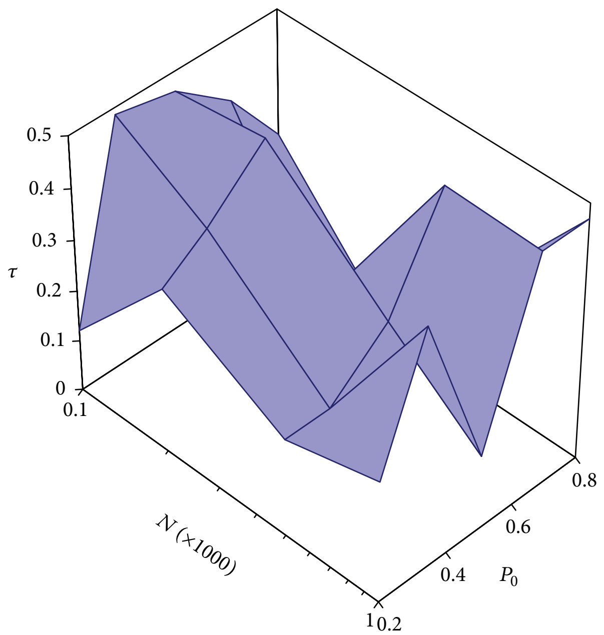

Figure 9: Variation of the transfer function parameter τ versus N and P_0.

(iii) α has a fractional value and seems to be almost independent of N and grows slightly with P_0;

(iv) τ has not a clear relationship with N or P_0. It has a small value and its average is $\tau_{av} = 0.3$.

We verify that we can model the GA dynamical behaviour in terms of a simple fractional order model. In fact, stochastic results of the GA seem to be of minor influence in the proposed modelling scheme, not only due to the comprehensive variation of the parameters, but also because several numerical experiments with distinct seeds lead to similar results. Nevertheless, we have a phenomenal modelling perspective based on the analytical approximation and supported by the outcome results. Therefore, a new challenge is the inverse problem. By other words, the problem of defining the fractional order model remains open and, as a consequence, designing the GA rules that produce such dynamical behaviour. In the scope of this problematic it can not be forgotten that the GA optimizes a symbiotic behaviour using a regular hexagonal space tessellation. Therefore, at a higher level several problems remain open such as the design of other fitness functions, the effect of other tessellation methods, or the phenomena produced by a larger number of object types in the GA population.

4. Conclusions

This paper presented a GA dynamical evolution and its description by means of a fractional model. The GA adopts a tessellation of the space using regular hexagons in order to provide an efficient scheme to handle the neighbour cells in the numerical discretization. Furthermore, the GA includes a fitness function that evaluates positively the interface between distinct objects in the population. This scheme has an abstract nature but can be interpreted as representing a simplified version of some kind of interaction between distinct materials or, even, as symbiotic relation between two different species. During the experiments several conditions for the GA evolution were tested, namely, with initial populations having different number of objects and distinct percentages of each type. For describing the space-time GA dynamics a simple measure was defined. The index, inspired in the notion of distance, captures the differences between two consecutive best individuals. It is verified, in all cases, that the proposed description leads to simple models capable of being traduced by analytical expressions of fractional order.

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