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Research Article

Hopf Bifurcation Control in a Delayed Predator-Prey System with Prey Infection and Modified Leslie-Gower Scheme

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Hopf bifurcation of a delayed predator-prey system with prey infection and the modified Leslie-Gower scheme is investigated. The conditions for the stability and existence of Hopf bifurcation of the system are obtained. The state feedback and parameter perturbation are used for controlling Hopf bifurcation in the system. In addition, direction of Hopf bifurcation and stability of the bifurcated periodic solutions of the controlled system are obtained by using normal form and center manifold theory. Finally, numerical simulation results are presented to show that the hybrid controller is efficient in controlling Hopf bifurcation.

1. Introduction

The dynamics of epidemiological models have been investigated by many scholars [1–7] since Kermack and McKendrick [8] proposed the classical SIR model. Based on the classical SIR model, Chattopadhyay and Arino [9] proposed a predator-prey epidemiological model with disease spreading in the prey, and they studied the boundedness of the solutions and the existence of Hopf bifurcation for the model. In order to study the influence of disease on an environment where two or more interacting species are present, Zhou et al. [10] proposed the following ecoepidemiological system consisting of three species:

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1-\frac{S+I}{K}\right)-\beta SI, \\ \frac{dI}{dt} &= \beta SI - cI - \frac{c_1 I y}{I+K_1}, \\ \frac{dy}{dt} &= \left(a_2-\frac{c_2 y}{I+K_2}\right)y, \end{aligned} \tag{1}$$

where S, I, and y denote, respectively, the population density of the susceptible prey, the infected prey, and the predator. It is assumed that the predator eats only the infected prey with the modified Leslie-Gower scheme [11–14]. The coefficients r, K, β, c, c_1, K_1, a_2, c_2, and K_2 in system (1) are all positive constants, and their ecological meanings are interpreted as follows. r and K represent the intrinsic birth rate and the carrying capacity of the prey population in the absence of disease, respectively. β represents the transmission coefficient. c represents the death rate of the infected prey. c_1 represents the maximum value of the per capita rate of the infected prey due to the predator. c_2 represents the maximum value of the per capita rate of the predator due to the infected prey population. K_1 and K_2 represent the extent to which environment protection to the infected prey and the predator, respectively. Zhou et al. studied the boundedness, stability, and the permanence of system (1). The effect of the transmission coefficient and the predation rate on the dynamics of the system were also investigated.

However, an important aspect which should be kept in mind while formulating an epidemiological system is the fact that it is often necessary to incorporate time delays into the system in order to reflect the dynamical behaviors of the system depending on the past history of the system, and epidemiological systems with delay have been studied extensively [4, 5, 15–17]. Zhang et al. [5] formulated a delayed predator-prey epidemiological system with disease spreading in predator. Hu and Li [15] considered a delayed predator-prey system with disease in prey, and they studied Hopf bifurcation and the stability of the periodic solutions induced by the time delay. Motivated by the work above, in the present paper, we incorporate the feedback delay of the predator into system (1) and get the following delayed system:

$$\begin{aligned} \frac{dS}{dt} &= rS\left(1 - \frac{S+I}{K}\right) - \beta SI, \\ \frac{dI}{dt} &= \beta SI - cI - \frac{c_1 I y}{I + K_1}, \\ \frac{dy}{dt} &= \left(a_2 - \frac{c_2 y(t-\tau)}{I(t-\tau) + K_2}\right) y, \end{aligned} \tag{2}$$

where τ is the negative feedback delay of the predator. The main purpose of this paper is to consider the effect of the delay on the dynamics of system (2). We will study the local existence of Hopf bifurcation and the properties of periodic solutions. In addition, in order to delay the onset of Hopf bifurcation, we will incorporate the state feedback and parameter perturbation into system (2).

The initial conditions for system (2) take the following form $S(\theta) = \phi_1(\theta)$, $I(\theta) = \phi_2(\theta)$, $y(\theta) = \phi_3(\theta)$, $\theta \in [-\tau, 0]$, where $(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\tau, 0], R_+^3)$ and $\phi_1(0) > 0$, $\phi_2(0) > 0$, and $\phi_3(0) > 0$.

This paper is organized as follows. In Section 2, we will study the stability of the positive equilibrium and the existence of local Hopf bifurcation of system (2). In Section 3, the state feedback and parameter perturbation are incorporated into system (2) to control the Hopf bifurcation. The direction and the stability of the bifurcated periodic solutions are also determined for the controlled system. Some numerical simulations are given to support the theoretical prediction in Section 4.

2. Stability of Positive Equilibrium and Hopf Bifurcation

According to [10], we can know that if the condition (H_1): $c_1 a_2 K_2 < c_2 K_1(\beta K - c)$ holds, then system (2) has a unique positive equilibrium $E_*(S_*, I_*, y_*)$, where

$$\begin{aligned} I_* &= \frac{-\left[c_2 K_1\left(\beta + \beta^2 K/r\right) + c_1 a_2 - c_2(\beta K - c)\right] + \sqrt{\Delta_1}}{2c_2\left(\beta + \beta^2 K/r\right)}, \\ y_* &= \frac{a_2\left(I_* + K_2\right)}{c_2}, \\ S_* &= \frac{1}{\beta}\left(c + \frac{c_1 y_*}{I_* + K_1}\right), \end{aligned} \tag{3}$$

with

$$\begin{aligned} \Delta_1 = {} & \left[c_2 K_1\left(\beta + \frac{\beta^2 K}{r}\right) + c_1 a_2 - c_2(\beta K - c)\right]^2 \\ & - 4c_2\left(\beta + \frac{\beta^2 K}{r}\right)\left(c_1 a_2 K_2 - c_2 K_1(\beta K - c)\right). \end{aligned} \tag{4}$$

The variational matrix at E_* takes the form

$$J(E_*) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & b_{32} e^{-\lambda\tau} & b_{33} e^{-\lambda\tau} \end{pmatrix}, \tag{5}$$

where

$$\begin{aligned} & a_{11} = -\frac{r}{K} S_*, \qquad a_{12} = -S_*\left(\beta + \frac{r}{K}\right), \\ & a_{21} = \beta I_*, \qquad a_{22} = \frac{c_1 I_* y_*}{\left(I_* + K_1\right)^2}, \qquad a_{23} = -\frac{c_1 I_*}{I_* + K_1}, \\ & b_{32} = \frac{c_2 y_*^2}{\left(I_* + K_2\right)^2}, \qquad b_{33} = -\frac{c_2 y_*}{I_* + K_2}. \end{aligned} \tag{6}$$

The characteristic equation corresponding to $J(E_*)$ will be

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + \left(B_2\lambda^2 + B_1\lambda + B_0\right)e^{-\lambda\tau}, \tag{7}$$

where

$$\begin{aligned} & A_2 = -\left(a_{11} + a_{22}\right), \qquad A_1 = a_{11}a_{22} - a_{12}a_{21}, \\ & B_2 = -b_{33}, \qquad B_1 = \left(a_{11} + a_{22}\right)b_{33} - a_{23}b_{32}, \\ & B_0 = a_{11}a_{23}b_{32} + \left(a_{12}a_{21} - a_{11}a_{22}\right)b_{33}. \end{aligned} \tag{8}$$

For $\tau = 0$, characteristic equation (7) reduces to

$$\lambda^3 + \left(A_2 + B_2\right)\lambda^2 + \left(A_1 + B_1\right)\lambda + B_0 = 0. \tag{9}$$

Obviously, if the condition (H_2): $A_2 + B_2 > 0$ and $(A_2 + B_2)(A_1 + B_1) > B_0$ holds, then the positive equilibrium $E_*(S_*, I_*, y_*)$ is locally asymptotically stable in the absence of delay.

For $\tau > 0$, substituting $\lambda = i\omega(\omega > 0)$ into (7) and separating the real and imaginary parts, one can get

$$\begin{aligned} B_1\omega\sin\tau\omega + \left(B_0 - B_2\omega^2\right)\cos\tau\omega &= A_2\omega^2, \\ B_1\omega\cos\tau\omega - \left(B_0 - B_2\omega^2\right)\sin\tau\omega &= \omega^3 - A_1\omega, \end{aligned} \tag{10}$$

which leads to

$$\omega^6 + p_0\omega^4 + q_0\omega^2 + r_0 = 0, \tag{11}$$

where

$$\begin{aligned} & p_0 = A_2^2 - B_2^2 - 2A_1, \\ & q_0 = A_1^2 - B_1^2 + 2B_0B_2, \qquad r_0 = -B_0^2. \end{aligned} \tag{12}$$

Let $z = \omega^2$. Equation (11) can be written as

$$h(z) := z^3 + p_0 z^2 + q_0 z + r_0 = 0. \tag{13}$$

Obviously, $r_0 = -B_0^2 \le 0$. Discussion about the roots of (13) is similar to that in [15], so we have the following lemma.

Lemma 1. *For the polynomial equation* (13)*, since* $r_0 \leq 0$*, one has the following results:*

(i) *if* $r_0 < 0$*, then* (13) *has at least one positive root;*

(ii) *if* $r_0 = 0$ *and* $p_0^2 - 3q_0 \leq 0$*, then* (13) *has no positive roots;*

(iii) *if* $r_0 = 0$ *and* $p_0^2 - 3q_0 > 0$*, then* (13) *has positive roots if and only if* $z_1^* = ((-p_0 + \sqrt{p_0^2 - 3q_0})/3) > 0$ *and* $h(z_1^*) \leq 0$.

Suppose that the coefficients in $h(z)$ satisfy the following condition (H_3): $r_0 < 0$ or $r_0 = 0$, $p_0^2 - 3q_0 > 0$, $z_1^* > 0$, and $h(z_1^*) < 0$.

If the condition (H_3) holds, then (13) has at least one positive root. Without loss of generality, we assume that (13) has three positive roots that are denoted as z_1, z_2, and z_3. Then, (11) has tree positive roots $\omega_k = \sqrt{z_k}$, $k = 1, 2, 3$, and for every fixed ω_k, the corresponding critical value of time delay $\tau_k^{(j)}$ is

$$\tau_k^{(j)} = \frac{1}{\omega_k} \arccos \frac{(A_2B_2 + B_1)\,\omega_k^4 - (A_1B_1 + A_2B_0)\,\omega_k^2}{B_2^2\omega_k^4 + (B_1^2 - 2B_0B_2)\,\omega_k^2 + B_0^2} + \frac{2j\pi}{\omega_k}, \quad k = 1, 2, 3; \quad j = 0, 1, 2, \ldots. \tag{14}$$

Let

$$\tau_0 = \tau_k^{(0)} = \min\left\{\tau_k^{(0)}\right\}, \quad \omega_0 = \omega_{k_0}, \quad k = 1, 2, 3. \tag{15}$$

Next, we give the transversality condition by the following Lemma.

Lemma 2. *Suppose that* $z_0 = \omega_0^2$ *and* $h'(z_0) \neq 0$*. Then* $d\,\mathrm{Re}\,\lambda(\tau_0)/d\tau \neq 0$.

Proof. Taking the derivative of λ with respect to τ in (7), we obtain

$$\begin{aligned}&\left(3\lambda^2 + 2A_2\lambda + A_1\right)\frac{d\lambda}{d\tau} + (2B_2\lambda + B_1)\,e^{-\lambda\tau}\frac{d\lambda}{d\tau}\\&\quad - \left(B_2\lambda^2 + B_1\lambda + B_0\right)e^{-\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) = 0,\end{aligned} \tag{16}$$

which yields

$$\begin{aligned}\frac{d\lambda}{d\tau}^{-1} &= -\frac{3\lambda^2 + 2A_2\lambda + A_1}{\lambda(\lambda^3 + A_2\lambda^2 + A_1\lambda)}\\&\quad + \frac{2B_2\lambda + B_1}{\lambda(B_2\lambda^2 + B_1\lambda + B_0)} - \frac{d\tau}{d\lambda}.\end{aligned} \tag{17}$$

Hence, a direct calculation shows that

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda(\tau_0)}{d\tau}\right)^{-1} &= \frac{3\omega_0^4 + 2\left(A_2^2 - 2A_1\right)\omega_0^2 + A_1^2}{\omega_0^6 + (A_2^2 - 2A_1)\,\omega_0^4 + A_1^2\omega_0^2}\\&\quad - \frac{2B_2^2\omega_0^2 + B_1^2 - 2B_0B_2}{B_2^2\omega_0^4 + (B_1^2 - 2B_0B_2)\,\omega_0^2 + B_0^2}.\end{aligned} \tag{18}$$

From (11), we have

$$\begin{aligned}&\omega_0^6 + \left(A_2^2 - 2A_1\right)\omega_0^4 + A_1^2\omega_0^2\\&\quad = B_2^2\omega_0^4 + \left(B_1^2 - 2B_0B_2\right)\omega_0^2 + B_0^2.\end{aligned} \tag{19}$$

Thus,

$$\begin{aligned}&\mathrm{Re}\left(\frac{d\lambda(\tau_0)}{d\tau}\right)^{-1}\\&= \frac{3\omega_0^4 + 2\left(A_2^2 - B_2^2 - 2A_1\right)\omega_0^2 + A_1^2 - B_1^2 + 2B_0B_2}{\omega_0^6 + (A_2^2 - 2A_1)\,\omega_0^4 + A_1^2\omega_0^2}\\&= \frac{h'(z_0)}{\omega_0^6 + (A_2^2 - 2A_1)\,\omega_0^4 + A_1^2\omega_0^2}.\end{aligned} \tag{20}$$

Obviously, if $h'(z_0) \neq 0$, then $\mathrm{Re}(d\lambda(\tau_0)/d\tau)^{-1} \neq 0$. In addition, $\mathrm{sgn}(d\,\mathrm{Re}\,\lambda(\tau_0)/d\tau) = \mathrm{sgn}\,\mathrm{Re}(d\lambda(\tau_0)/d\tau)$.

Thus, the proof is completed. □

By Lemmas 1 and 2 and Corollary 2.4 in [18], we have the following theorem.

Theorem 3. *For system* (2)*, if the conditions* (H_1)*–*(H_3) *hold, then*

(i) *the positive equilibrium* $E_*(S_*, I_*, y_*)$ *is asymptotically stable for* $\tau \in [0, \tau_0)$*;*

(ii) *the positive equilibrium* $E_*(S_*, I_*, y_*)$ *is unstable when* $\tau > \tau_0$*;*

(iii) *if* $h'(z_0) \neq 0$*, then system* (2) *undergoes a Hopf bifurcation at* $E_*(S_*, I_*, y_*)$ *when* $\tau = \tau_0$*. That is, system* (2) *has a branch of periodic solutions bifurcating from the zero solution near* $\tau = \tau_0$.

3. Hopf Bifurcation Control

In this section, we will incorporate the state feedback and parameter perturbation into system (2) in order to delay the onset of Hopf bifurcation in the system or make the bifurcation disappear. Then we get the following system with controller:

$$\begin{aligned}\frac{dS}{dt} &= \alpha\left[rS\left(1 - \frac{S + I}{K}\right) - \beta SI\right] + \gamma S,\\\frac{dI}{dt} &= \alpha\left[\beta SI - cI - \frac{c_1 I y}{I + K_1}\right],\\\frac{dy}{dt} &= \alpha\left[\left(a_2 - \frac{c_2 y(t-\tau)}{I(t-\tau) + K_2}\right)y\right] + \gamma_2 y,\end{aligned} \tag{21}$$

where α, γ, and γ_2 are parameters, which can control the system to relocate the onset of an inherent bifurcation.

Similar as in Section 2, we can easily get that if the following condition (H_4) holds (H_4): $c_1 r K_2(a_2\alpha + \gamma_2) + c c_2 r\alpha K_1 < c_2 \beta K K_1(r\alpha + \gamma)$, then, system (21) has a unique positive equilibrium $E^*(S^*, I^*, y^*)$ where

$$I^* = \frac{-[B_1 - B_2] + \sqrt{\Delta_2}}{2\alpha c_2(\beta + \beta^2 K/r)},$$
$$y^* = \frac{(a_2\alpha + \gamma_2)(I^* + K_2)}{\alpha c_2} \tag{22}$$
$$S^* = \frac{1}{\beta}\left(c + \frac{c_1 y^*}{I^* + K_1}\right),$$

with

$$B_1 = \alpha c_2 K_1\left(\beta + \frac{\beta^2 K}{r}\right) + c_1(a_2\alpha + \gamma_2),$$
$$B_2 = \alpha c_2(\beta K - c) + \frac{c_2 K\beta\gamma}{r},$$
$$\Delta_2 = \left[\alpha c_2 K_1\left(\beta + \frac{\beta^2 K}{r}\right) + c_1(a_2\alpha + \gamma_2) - \alpha c_2(\beta K - c) - \frac{c_2 K\beta\gamma}{r}\right]^2 - 4\alpha c_2\left(\beta + \frac{\beta^2 K}{r}\right)\left(c_1 K_2(a_2\alpha + \gamma_2) - \alpha c_2 K_1(\beta K - c) - \frac{c_2}{r}KK_1\beta\gamma\right). \tag{23}$$

Using Taylor expansion to expand the right-hand side of system (21) at the positive equilibrium $E^*(S^*, I^*, y^*)$, we have

$$\frac{dS}{dt} = a'_{11}S + a'_{12}I + f_1,$$
$$\frac{dI}{dt} = a'_{21}S + a'_{22}I + a'_{23} + f_2, \tag{24}$$
$$\frac{dy}{dt} = b'_{32}I(t-\tau) + b'_{33}y(t-\tau) + f_3,$$

where

$$a'_{11} = -\frac{\alpha r S^*}{K}, \qquad a'_{12} = -\alpha S^*\left(\beta + \frac{r}{K}\right),$$
$$a'_{21} = \alpha\beta I^*, \qquad a'_{22} = \frac{\alpha c_1 I^* y^*}{(I^* + K_1)^2}, \qquad a_{23} = -\frac{\alpha c_1 I^*}{I^* + K_1},$$
$$b'_{32} = \frac{\alpha c_2 y^{*2}}{(I^* + K_2)^2}, \qquad b'_{33} = -\frac{\alpha c_2 y^*}{I^* + K_2},$$
$$f_1 = a_{13}S^2 + a_{14}SI,$$
$$f_2 = a_{24}SI + a_{25}I^2 + a_{26}Iy + a_{27}I^2 y + a_{28}I^3 + \cdots,$$
$$f_3 = c_{31}yy(t-\tau) + c_{32}yI(t-\tau) + c_{33}I^2(t-\tau) + c_{34}I(t-\tau)y(t-\tau) + c_{35}I^3(t-\tau) + c_{36}I^2(t-\tau)y + c_{37}I^2(t-\tau)y(t-\tau) + c_{38}yI(t-\tau)y(t-\tau) + \cdots, \tag{25}$$

with

$$a_{13} = -\frac{\alpha r}{K}, \qquad a_{14} = -\frac{\alpha r}{K} - \alpha\beta,$$
$$a_{24} = \alpha\beta, \qquad a_{25} = \frac{c_1 K_1 \alpha y^*}{(I^* + K_1)^3},$$
$$a_{26} = -\frac{c_1 K_1 \alpha}{(I^* + K_1)^2}, \qquad a_{27} = \frac{2c_1 K_1 \alpha}{(I^* + K_1)^3},$$
$$a_{28} = -\frac{c_1 K_1 \alpha y^*}{(I^* + K_1)^4}, \qquad c_{31} = -\frac{\alpha c_2}{I^* + K_2},$$
$$c_{32} = \frac{\alpha c_2 y^*}{(I^* + K_2)^2}, \qquad c_{33} = \frac{\alpha c_2 y^{*2}}{(I^* + K_2)^3}, \tag{26}$$
$$c_{34} = \frac{\alpha c_2 y^*}{(I^* + K_2)^2}, \qquad c_{35} = \frac{\alpha c_2 y^{*2}}{(I^* + K_2)^4},$$
$$c_{36} = -\frac{\alpha c_2 y^*}{(I^* + K_2)^3}, \qquad c_{37} = -\frac{\alpha c_2 y^*}{(I^* + K_2)^3},$$
$$c_{38} = \frac{\alpha c_2}{(I^* + K_2)^2}.$$

The linear system of (24) is

$$\frac{dS}{dt} = a'_{11}S + a'_{12}I,$$
$$\frac{dI}{dt} = a'_{21}S + a'_{22}I + a'_{23}, \tag{27}$$
$$\frac{dy}{dt} = b'_{32}I(t-\tau) + b'_{33}y(t-\tau).$$

The characteristic equation of system (27) is

$$\lambda^3 + A'_2\lambda^2 + A'_1\lambda + \left(B'_2\lambda^2 + B'_1\lambda + B'_0\right)e^{-\lambda\tau}, \tag{28}$$

where

$$A'_2 = -\left(a'_{11} + a'_{22}\right), \qquad A'_1 = a'_{11}a'_{22} - a'_{12}a'_{21},$$
$$B'_2 = -b'_{33}, \qquad B'_1 = \left(a'_{11} + a'_{22}\right)b'_{33} - a'_{23}b'_{32}, \tag{29}$$
$$B'_0 = a'_{11}a'_{23}b'_{32} + \left(a'_{12}a'_{21} - a'_{11}a'_{22}\right)b'_{33}.$$

Obviously, the characteristic equation of system (27) is similar to (7). As the analysis method is similar to Section 2, we omit the linear stability and Hopf bifurcation analysis of system (21). By the similar computation as in Section 2, we can get that the critical value of time delay $\tau_k^{(j)}$ for system (21) is

$$\tau_k'^{(j)} = \frac{1}{\omega_k'}\arccos\frac{g_1(\omega_k')}{g_2(\omega_k')} + \frac{2j\pi}{\omega_k'}, \quad k = 1,2,3; \quad j = 0,1,2,\ldots, \tag{30}$$

with

$$\begin{aligned} g_1(\omega_k') &= (A_2'B_2' + B_1')\omega_k'^4 - (A_1'B_1' + A_2'B_0')\omega_k'^2, \\ g_2(\omega_k') &= B_2'^2\omega_k'^4 + (B_1'^2 - 2B_0'B_2')\omega_k'^2 + B_0'^2, \end{aligned} \tag{31}$$

where ω_k' is a positive root of the following equation:

$$\omega'^6 + p_0'\omega'^4 + q_0'\omega'^2 + r_0' = 0, \tag{32}$$

with

$$\begin{aligned} p_0' &= A_2'^2 - B_2'^2 - 2A_1', \\ q_0' &= A_1'^2 - B_1'^2 + 2B_0'B_2', \qquad r_0' = -B_0'^2. \end{aligned} \tag{33}$$

Let

$$\tau_0' = \tau_k'^{(0)} = \min\left\{\tau_k'^{(0)}\right\}, \quad \omega_0' = \omega_{k_0}', \quad k = 1,2,3. \tag{34}$$

In the following, we will use the normal form method and center manifold theorem introduced by Hassard et al. [19] to determine the property of the bifurcated periodic solutions of the controlled system (21) at τ_0'.

Let $\tau = \tau_0' + \mu, \mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value of the controlled system (21). Rescaling the time $t \to t/\tau$, then system (21) can be written as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{35}$$

where $u(t) = (S(t), I(t), y(t))^T \in C([-1,0], R_+^3)$ and L_μ and F are given respectively by

$$\begin{aligned} L_\mu\phi &= (\tau_0' + \mu)(A'\phi(0) + B'\phi(-1)), \\ F(\mu,\phi) &= (\tau_0' + \mu)(F_1, F_2, F_3)^T, \end{aligned} \tag{36}$$

with

$$A' = \begin{pmatrix} a_{11}' & a_{12}' & 0 \\ a_{21}' & a_{22}' & a_{23}' \\ 0 & 0 & 0 \end{pmatrix}, \qquad B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{32}' & b_{33}' \end{pmatrix},$$

$$\begin{aligned} F_1 &= a_{13}\phi_1^2(0) + a_{14}\phi_1(0)\phi_2(0), \\ F_2 &= a_{24}\phi_1(0)\phi_2(0) + a_{25}\phi_2^2(0) + a_{26}\phi_2(0)\phi_3(0) \\ &\quad + a_{27}\phi_2^2(0)\phi_3(0) + a_{28}\phi_2^3(0) + \cdots, \\ F_3 &= c_{31}\phi_3(0)\phi_3(-1) + c_{32}\phi_2(-1)\phi_3(0) \\ &\quad + c_{33}\phi_2^2(-1) + c_{34}\phi_2(-1)\phi_3(-1) \\ &\quad + c_{35}\phi_2^3(-1) + c_{36}\phi_2^2(-1)\phi_3(0) \\ &\quad + c_{37}\phi_2^2(-1)\phi_3(-1) \\ &\quad + c_{38}\phi_2(-1)\phi_3(0)\phi_3(-1) + \cdots. \end{aligned} \tag{37}$$

Thus, by the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta,\mu) : [-1,0] \to R_+^3$ whose elements are of bounded variation such that

$$L_\mu\phi = \int_{-1}^0 d\eta(\theta,\mu)\phi(\theta), \quad \phi \in C([-1,0], R_+^3). \tag{38}$$

In fact, we choose

$$\eta(\theta,\mu) = (\tau_0 + \mu)A'\delta(\theta) - (\tau_0 + \mu)B'\delta(\theta + 1). \tag{39}$$

For $\phi \in C^1[-1,0]$, we define

$$\begin{aligned} A(\mu)\phi &= \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^0 d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases} \\ R(\mu)\phi &= \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases} \end{aligned} \tag{40}$$

Then system (21) can be transformed into the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{41}$$

The adjoint operator A^* of $A(0)$ is defined by

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^0 d\eta^T(\xi,0)\varphi(-s), & s = 0, \end{cases} \tag{42}$$

and the bilinear inner product:

$$\langle \varphi(s), \phi(\theta)\rangle = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\varphi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \tag{43}$$

with $\eta(\theta)=\eta(\theta,0)$.

By the previous discussions, we know that $\pm i\tau_0'\omega_0'$ are eigenvalues of $A(0)$ and $A^*(0)$. We assume that $q(\theta) = (1,q_2,q_3)^T e^{i\tau_0'\omega_0'\theta}$ is the eigenvector of $A(0)$ belonging to the eigenvalue $+i\tau_0'\omega_0'$, and $q^*(s) = D(1,q_2^*,q_3^*)e^{i\tau_0'\omega_0'}$ is the eigenvector of $A^*(0)$ belonging to $-i\tau_0'\omega_0'$. Then we have

$$A(0)q(\theta) = i\tau_0'\omega_0' q(\theta), \qquad A^*(0)q^*(\theta) = -i\tau_0'\omega_0' q^*(\theta). \tag{44}$$

By a simple computation, we can get

$$\begin{aligned} q_2 &= \frac{i\omega_0' - a_{11}'}{a_{12}'}, \qquad & q_3 &= \frac{b_{32}'\left(i\omega_0' - a_{11}'\right)}{a_{12}'\left(i\omega_0' e^{i\tau_0'\omega_0'} - b_{33}'\right)}, \\ q_2^* &= -\frac{i\omega_0' + a_{11}'}{a_{21}'}, \qquad & q_3^* &= \frac{a_{23}'\left(i\omega_0' + a_{11}'\right)}{a_{21}'\left(i\omega_0' + b_{33}' e^{i\tau_0'\omega_0'}\right)}. \end{aligned} \tag{45}$$

Then from (43), we can obtain

$$\overline{D} = \left[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + \left(b_{32}'q_2 + b_{33}'q_3\right)\overline{q}_3^*\tau_0' e^{-i\tau_0'\omega_0'\theta}\right]^{-1}, \tag{46}$$

such that $\langle q^*, q\rangle = 1$, $\langle q^*, \overline{q}\rangle = 0$.

Next, we can get the coefficients determining the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions by the algorithms given in [19]:

$$\begin{aligned} g_{20} = 2\tau_0'\overline{D}\Big[& a_{13} + a_{14}q^{(2)}(0) \\ &+ \overline{q}_2^*\Big(a_{24}q^{(2)}(0) + a_{25}\big(q^{(2)}(0)\big)^2 + a_{26}q^{(2)}(0)q^{(3)}(0)\Big) \\ &+ \overline{q}_3^*\Big(c_{31}q^{(3)}(0)q^{(3)}(-1) + c_{32}q^{(2)}(-1)q^{(3)}(0) + c_{33}\big(q^{(2)}(-1)\big)^2 + c_{34}q^{(2)}(-1)q^{(3)}(-1)\Big)\Big], \end{aligned}$$

$$\begin{aligned} g_{11} = \tau_0'\overline{D}\Big[& 2a_{13} + a_{14}\left(\overline{q}^{(1)}(0) + \overline{q}^{(2)}(0)\right) \\ &+ \overline{q}_2^*\Big(2a_{25}q^{(2)}(0)\overline{q}^{(2)}(0) + a_{26}\left(\overline{q}^{(3)}(0)q^{(2)}(0) + q^{(3)}(0)\overline{q}^{(2)}(0)\right)\Big) \\ &+ \overline{q}_3^*\Big(c_{31}\left(q^{(3)}(0)\overline{q}^{(3)}(-1) + \overline{q}^{(3)}(0)q^{(3)}(-1)\right) \\ &+ c_{32}\left(\overline{q}^{(2)}(-1)q^{(3)}(0) + q^{(2)}(-1)\overline{q}^{(3)}(0)\right) \\ &+ 2c_{33}q^{(2)}(-1)\overline{q}^{(2)}(-1) \\ &+ c_{34}\left(\overline{q}^{(2)}(-1)q^{(3)}(-1) + q^{(2)}(-1)\overline{q}^{(3)}(-1)\right)\Big)\Big], \end{aligned}$$

$$\begin{aligned} g_{02} = 2\tau_0'\overline{D}\Big[& a_{13} + a_{14}\overline{q}^{(2)}(0) \\ &+ \overline{q}_2^*\Big(a_{24}\overline{q}^{(2)}(0) + a_{25}\big(\overline{q}^{(2)}(0)\big)^2 + a_{26}\overline{q}^{(2)}(0)\overline{q}^{(3)}(0)\Big) \\ &+ \overline{q}_3^*\Big(c_{31}\overline{q}^{(3)}(0)\overline{q}^{(3)}(-1) + c_{32}\overline{q}^{(2)}(-1)\overline{q}^{(3)}(0) + c_{33}\big(\overline{q}^{(2)}(-1)\big)^2 + c_{34}\overline{q}^{(2)}(-1)\overline{q}^{(3)}(-1)\Big)\Big], \end{aligned}$$

$$\begin{aligned} g_{21} = 2\tau_0'\overline{D}\Big[& a_{13}\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) \\ &+ a_{14}\Big(W_{11}^{(1)}(0)q^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}^{(2)}(0) + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big) \\ &+ \overline{q}_2^*\Big(a_{24}\Big(W_{11}^{(1)}(0)q^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}^{(2)}(0) + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big) \\ &+ a_{25}\left(2W_{11}^{(2)}(0)q^{(2)}(0) + W_{20}^{(2)}(0)\overline{q}^{(2)}(0)\right) \\ &+ a_{26}\Big(W_{11}^{(2)}(0)q^{(3)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\overline{q}^{(3)}(0) + W_{11}^{(3)}(0)q^{(2)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\overline{q}^{(2)}(0)\Big) \\ &+ a_{27}\Big(\big(q^{(2)}(0)\big)^2\overline{q}^{(3)}(0) + 2q^{(2)}(0)\overline{q}^{(2)}(0)q^{(3)}(0)\Big) \\ &+ 3a_{28}\big(q^{(2)}(0)\big)^2\overline{q}^{(2)}(0)\Big) \\ &+ \overline{q}_3^*\Big(c_{31}\Big(W_{11}^{(3)}(0)q^{(3)}(-1) + \frac{1}{2}W_{20}^{(3)}(0)\overline{q}^{(3)}(-1) \end{aligned}$$

$$
\begin{aligned}
&+ W_{11}^{(3)}(-1)\, q^{(3)}(0) \\
&+ \frac{1}{2} W_{20}^{(3)}(-1)\, \overline{q}^{(3)}(0) \Big) \\
&+ c_{32} \Big(W_{11}^{(2)}(-1)\, q^{(3)}(0) \\
&\quad + \frac{1}{2} W_{20}^{(2)}(-1)\, \overline{q}^{(3)}(0) \\
&\quad + W_{11}^{(3)}(0)\, q^{(2)}(-1) \\
&\quad + \frac{1}{2} W_{20}^{(3)}(0)\, \overline{q}^{(2)}(-1) \Big) \\
&+ c_{33} \big(2W_{11}^{(2)}(-1)\, q^{(2)}(-1) \\
&\quad + W_{20}^{(2)}(-1)\, \overline{q}^{(2)}(-1) \big) \\
&+ c_{34} \Big(W_{11}^{(2)}(-1)\, q^{(3)}(-1) \\
&\quad + \frac{1}{2} W_{20}^{(2)}(-1)\, \overline{q}^{(3)}(-1) \\
&\quad + W_{11}^{(3)}(-1)\, q^{(2)}(-1) \\
&\quad + \frac{1}{2} W_{20}^{(3)}(-1)\, \overline{q}^{(2)}(-1) \Big) \\
&+ 3c_{35} \left(q^{(2)}(-1) \right)^2 \overline{q}^{(2)}(-1) \\
&+ c_{36} \Big(\left(q^{(2)}(-1) \right)^2 \overline{q}^{(3)}(0) \\
&\quad + 2q^{(2)}(-1)\, \overline{q}^{(2)}(-1)\, q^{(3)}(0) \Big) \\
&+ c_{37} \Big(\left(q^{(2)}(-1) \right)^2 \overline{q}^{(3)}(-1) \\
&\quad + 2q^{(2)}(-1)\, \overline{q}^{(2)}(-1)\, q^{(3)}(-1) \Big) \\
&+ c_{38} \Big(q^{(2)}(-1)\, q^{(3)}(0)\, \overline{q}^{(3)}(-1) \\
&\quad + q^{(2)}(-1)\, \overline{q}^{(3)}(0)\, q^{(3)}(-1) \\
&\quad + \overline{q}^{(2)}(-1)\, q^{(3)}(0)\, q^{(3)}(-1) \Big) \Big) \Big],
\end{aligned}
\tag{47}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{i g_{20} q(0)}{\tau_0' \omega_0'} e^{i\tau_0'\omega_0'\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau_0'\omega_0'} e^{-i\tau_0'\omega_0'\theta} \\
&\quad + E_1 e^{2i\tau_0'\omega_0'\theta}, \\
W_{11}(\theta) &= -\frac{i g_{11} q(0)}{\tau_0'\omega_0'} e^{i\tau_0'\omega_0'\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\tau_0'\omega_0'} e^{-i\tau_0'\omega_0'\theta} \\
&\quad + E_2,
\end{aligned}
\tag{48}
$$

where E_1 and E_2 can be computed as the following equations, respectively,

$$
\begin{aligned}
E_1 &= 2 \begin{pmatrix} 2i\omega_0' - a_{11}' & -a_{12}' & 0 \\ -a_{21}' & 2i\omega_0' - a_{22}' & -a_{23}' \\ 0 & -b_{32}' e^{-2i\tau_0'\omega_0'} & 2i\omega_0' - b_{33}' e^{-2i\tau_0'\omega_0'} \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} G_{11} \\ G_{21} \\ G_{31} \end{pmatrix}, \\
E_2 &= -\begin{pmatrix} a_{11}' & a_{12}' & 0 \\ a_{21}' & a_{22}' & a_{23}' \\ 0 & b_{32}' & b_{33}' \end{pmatrix}^{-1} \times \begin{pmatrix} H_{11} \\ H_{21} \\ H_{31} \end{pmatrix},
\end{aligned}
\tag{49}
$$

with

$$
\begin{aligned}
G_{11} &= a_{13} + a_{14} q^{(2)}(0), \\
G_{21} &= a_{24} q^{(2)}(0) + a_{25} \left(q^{(2)}(0) \right)^2 + a_{26} q^{(2)}(0)\, q^{(3)}(0), \\
G_{31} &= c_{31} q^{(3)}(0)\, q^{(3)}(-1) + c_{32} q^{(2)}(-1)\, q^{(3)}(0) \\
&\quad + c_{33} \left(q^{(2)}(-1) \right)^2 + c_{34} q^{(2)}(-1)\, q^{(3)}(-1), \\
H_{11} &= 2a_{13} + a_{14} \left(\overline{q}^{(1)}(0) + \overline{q}^{(2)}(0) \right), \\
H_{21} &= 2a_{25} q^{(2)}(0)\, \overline{q}^{(2)}(0) \\
&\quad + a_{26} \left(\overline{q}^{(3)}(0)\, q^{(2)}(0) + q^{(3)}(0)\, \overline{q}^{(2)}(0) \right), \\
H_{31} &= c_{31} \left(q^{(3)}(0)\, \overline{q}^{(3)}(-1) + \overline{q}^{(3)}(0)\, q^{(3)}(-1) \right) \\
&\quad + c_{32} \left(\overline{q}^{(2)}(-1)\, q^{(3)}(0) + q^{(2)}(-1)\, \overline{q}^{(3)}(0) \right) \\
&\quad + 2c_{33} q^{(2)}(-1)\, \overline{q}^{(2)}(-1) \\
&\quad + c_{34} \left(\overline{q}^{(2)}(-1)\, q^{(3)}(-1) + q^{(2)}(-1)\, \overline{q}^{(3)}(-1) \right).
\end{aligned}
\tag{50}
$$

Therefore, we can calculate the following values:

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\tau_0'\omega_0'} \left(g_{11} g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\
\delta &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0')\}}, \\
\sigma &= 2\operatorname{Re}\{C_1(0)\}, \\
T &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\left\{\lambda'\left(\tau_0'\right)\right\}}{\tau_0'\omega_0'}.
\end{aligned}
\tag{51}
$$

Based on the previous discussion, we can obtain the following results.

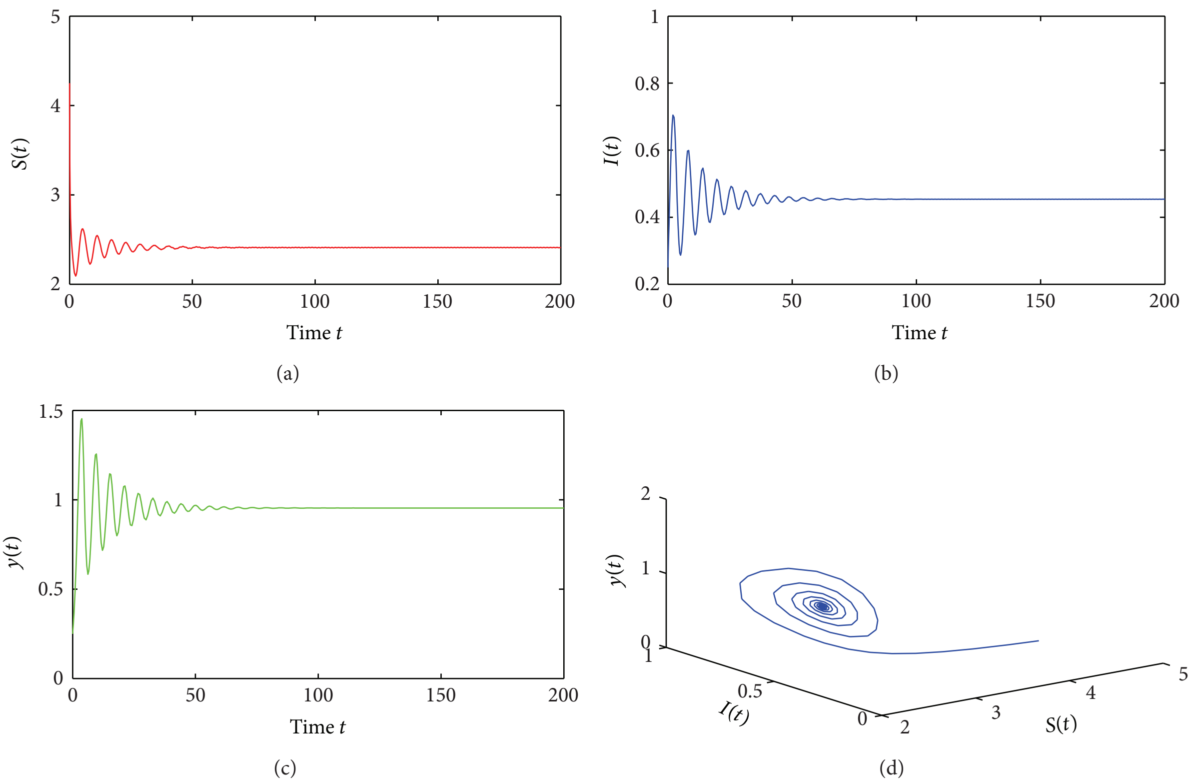

Figure 1: E_* is locally asymptotically stable for $\tau = 1.05 < \tau_0 = 1.0877$.

Theorem 4. *For system* (21), *when* $\tau = \tau_0'$, *the direction of the Hopf bifurcation and stability of periodic solutions are determined by the formulas* (51), *and the following results hold.*

The Hopf bifurcation is supercritical (subcritical) if $\delta > 0$ ($\delta < 0$); the bifurcating periodic solutions are stable (unstable) if $\sigma < 0$ ($\sigma > 0$); the period of the bifurcating periodic solution increases (decreases) if $T > 0$ ($T < 0$).

4. Numerical Simulation Examples

In this section, we give some numerical simulations to illustrate our theoretical analysis in Sections 2 and 3. As an example, we consider the following particular case of system (2):

$$\begin{aligned}\frac{dS}{dt} &= 5S\left(1-\frac{S+I}{3}\right)-0.5SI,\\ \frac{dI}{dt} &= 0.5SI-0.3I-\frac{Iy}{I+0.6},\\ \frac{dy}{dt} &= \left(1-\frac{y(t-\tau)}{I(t-\tau)+0.5}\right)y.\end{aligned} \tag{52}$$

By a simple computation, we have $c_1a_2K_2 = 0.5$, $c_2K_1(\beta K - c) = 0.72$. Obviously, $c_1a_2K_2 < c_2K_1(\beta K - c)$. Namely, the condition (H_1) holds, and we can get that system (52) has an unique positive equilibrium $E_*(3.4102, 0.4537, 0.9537)$. Then, we obtain $A_2 + B_2 = 4.6273 > 0$, $(A_2 + B_2)(A_1 + B_1) = 13.3390 > B_0 = 2.1102$. Thus, the condition (H_2) holds. Further, we have $\omega_0 = 0.7284$, $\tau_0 = 1.0877$, and $h'(z_0) = 15.7784 > 0$. That is, the transversality condition is satisfied. By Theorem 3, we can get that the positive equilibrium $E_*(3.4102, 0.4537, 0.9537)$ is locally asymptotically stable for $\tau \in [0, 1.0877)$, which can be seen from Figure 1, and $E_*(3.4102, 0.4537, 0.9537)$ is unstable when $\tau > \tau_0$. This property can be illustrated by Figure 2.

Next, we choose $\alpha = 0.5$, $\gamma = 0.3$, and $\gamma_2 = 0.2$ to control the Hopf bifurcation, and we get a particular case of system (21):

$$\begin{aligned}\frac{dS}{dt} &= 2.5S\left(1-\frac{S+I}{3}\right)-0.25SI+0.3S,\\ \frac{dI}{dt} &= 0.25SI-0.15I-\frac{0.5Iy}{I+0.6},\\ \frac{dy}{dt} &= 0.5\left(1-\frac{y(t-\tau)}{I(t-\tau)+0.5}\right)y+0.2y.\end{aligned} \tag{53}$$

Then, we can easily get the unique positive equilibrium of system (53) $E^*(3.0619, 0.2291, 1.0206)$. From the analysis in Section 3, we get $\omega_0' = 0.2603$ and $\tau_0' = 1.8034$. By choosing $\tau = 1.65$ and $\tau = 1.92$, the dynamical behavior of the controlled system (53) is illustrated in Figures 3 and 4. From the two figures we can see that, when $\tau = 1.65 < \tau_0' = 1.9034$, the positive equilibrium E^* is asymptotically stable (see Figure 3). However, once the time delay passes through the critical value τ_0', the system loses stability and a Hopf bifurcation occurs (see Figure 4).

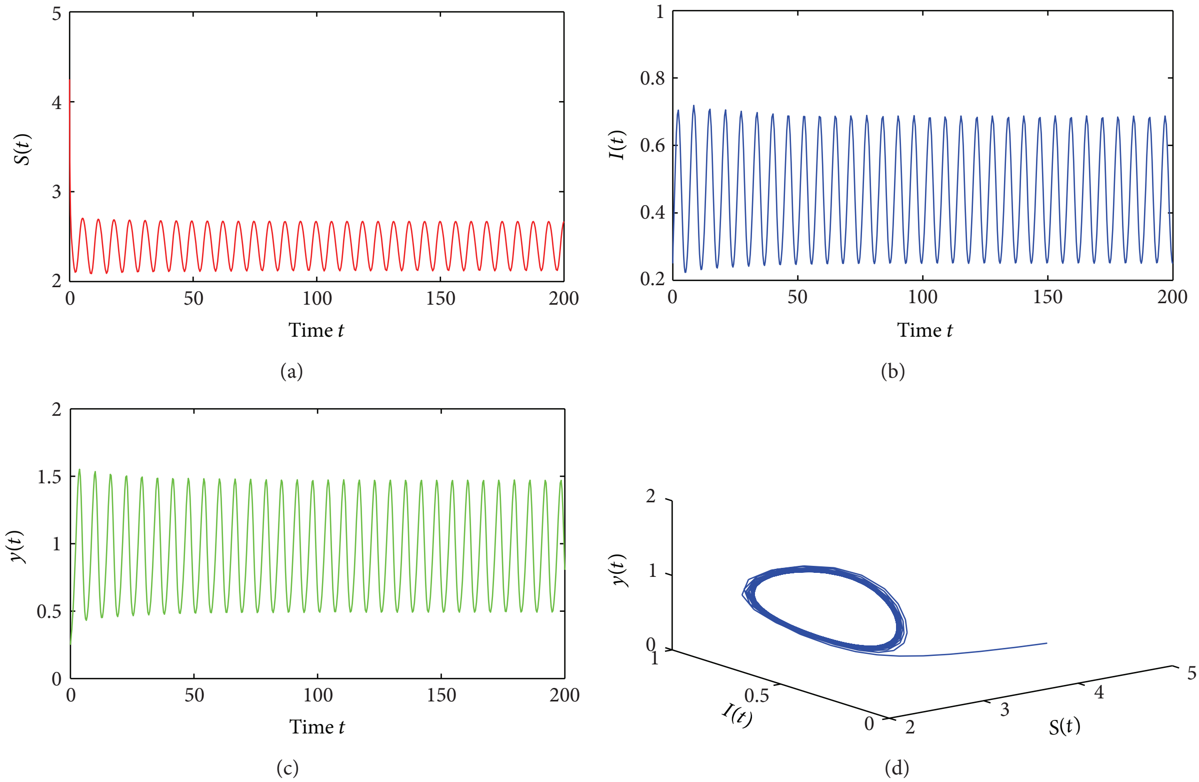

FIGURE 2: E_* is unstable for $\tau = 1.12 > \tau_0 = 1.0877$.

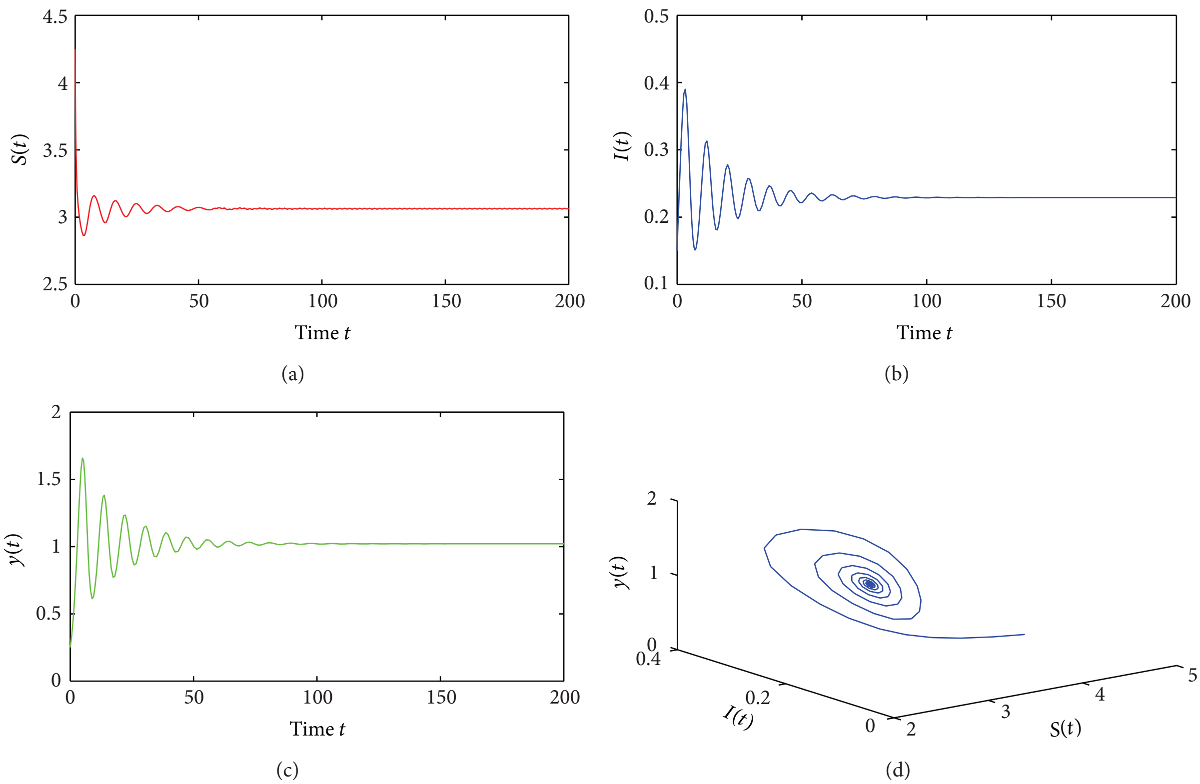

FIGURE 3: E^* is locally asymptotically stable for $\tau = 1.65 < \tau_0' = 1.8034$.

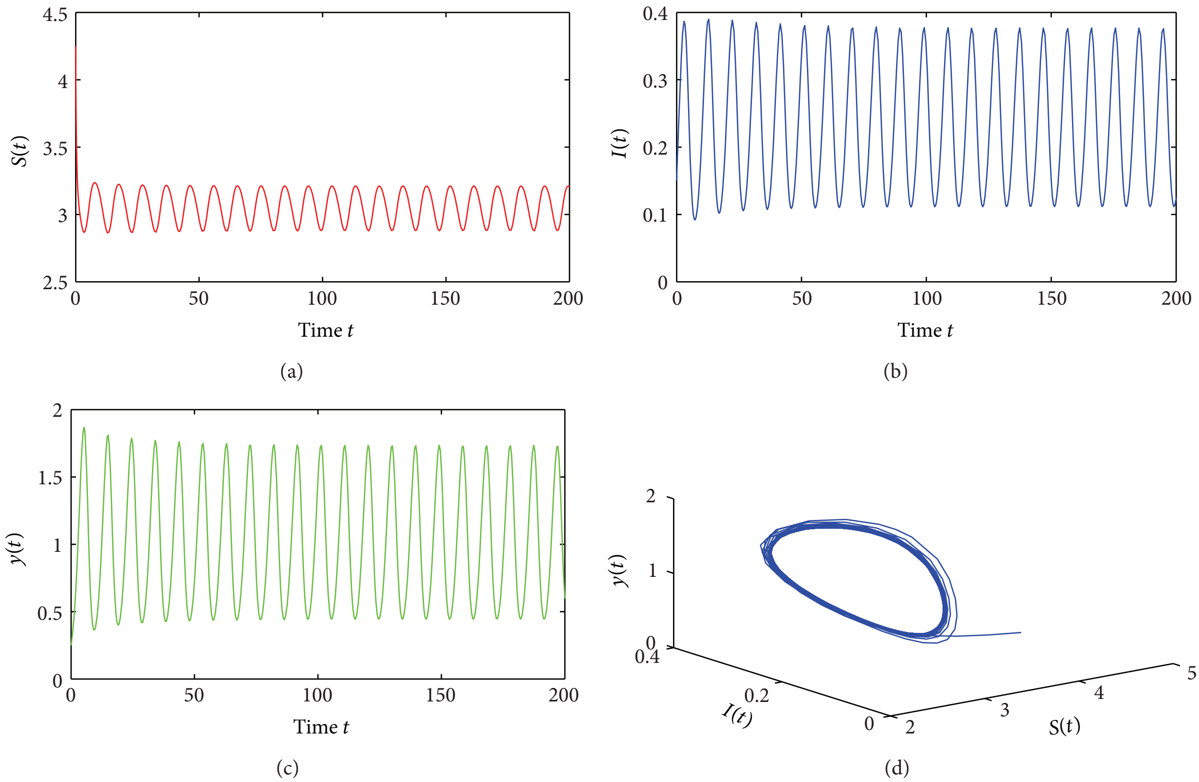

FIGURE 4: E^* is unstable for $\tau = 1.92 > \tau_0' = 1.8034$.

Comparing Figures 3 and 4 with Figures 1 and 2, it shows that the onset of Hopf bifurcation is delayed when controller has been incorporated into the system, and the critical value of the delay increases from $\tau_0 = 1.0877$ to $\tau_0' = 1.8034$.

In addition, from (51), we get $\mu_2 = 271.5206 > 0$, $\beta_2 = -33.2968 < 0$, and $T_2 = 113.2090 > 0$. Thus, from Theorem 4, we know that the Hopf bifurcation is supercritical, the bifurcated periodic solutions are stable, and the period of the bifurcated periodic solutions increases. Since the bifurcated periodic solutions are stable, then the species in system (53) can coexist under some conditions in an oscillatory mode from the viewpoint of biology.

5. Conclusions

A delayed predator-prey system with prey infection and the modified Leslie-Gower scheme is investigated. Regarding the negative feedback delay of the predator as a parameter, the local stability of the positive equilibrium and the existence of Hopf bifurcation are analyzed. The results show that, when the delay crosses a critical value, the system will lose its stability and a Hopf bifurcation occurs. To delay the onset of the Hopf bifurcation, we incorporate the state feedback and parameter perturbation into the system, and simulation results show the effectiveness of the controller. In addition, the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions for the controlled system are also determined by the normal form theory and the center manifold argument.

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