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Research Article

Positive Definiteness of High-Order Subdifferential and High-Order Optimality Conditions in Vector Optimization Problems

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We obtain a new Taylor's formula in terms of the $k + 1$ order subdifferential of a $C^{k,1}$ function from R^n to R^m. As its applications in optimization problems, we build $k + 1$ order sufficient optimality conditions of this kind of functions and $k + 1$ order necessary conditions for strongly C-quasiconvex functions.

1. Introduction

For a function from R^n to R, Luc [1] studied the $k + 1$ order subdifferential of it, established a Taylor-type formula in terms of such $k + 1$ order subdifferential, and applied such Taylor-type formula to consider two-order optimality conditions in vector optimization and characterizations of quasiconvex functions. In vector optimization, notions of Pareto solution, weak Pareto solution, sharp minima and weak sharp minima are very important; see [2–14] and the references therein. Some authors have attained many necessary or sufficient optimality conditions in optimization problems. In particular, Zheng and Yang provided some results on sharp minima, and weak sharp minima for high-order smooth vector optimization problems in Banach spaces. By the tools of nonsmooth analysis, many optimality conditions were obtained; for examples, one can see [6, 7, 15, 16] and the references therein. Such optimality conditions play a key role in many issues of mathematical programming such as sensitivity analysis and error bounds.

Motivated by Luc [1] and Zheng and Yang [17], in this paper, we consider the $k + 1$ order subdifferential and optimality conditions of a $C^{k,1}$ vector-valued function from R^n to R^m. We will first prove a new Taylor's formula in the terms of $k+1$ order subdifferential for $C^{k,1}$ functions from R^n to R^m, which is analogous to that for real-valued functions in [1]. Then, under the positive definiteness assumption of $k + 1$ order subdifferential, we will use this formula to derive $k + 1$ order optimality conditions of weak Pareto and Pareto solutions in the terms of $k + 1$ order subdifferential for a $C^{k,1}$ function from R^n to R^m. Finally, we will define a kind of strongly C-quasiconvex functions and prove a necessary condition in the terms of $(k + 1)$th order subdifferential for such kind of functions. Our results extend the corresponding results in [1] for $C^{k,1}$ functions from R^n to R to that for $C^{k,1}$ vector-valued functions from R^n to R^m and in [17] for functions in smooth setting to that in nonsmooth setting, respectively.

The outline of the paper is as follows. In the next section, we give some notions and preliminary results in vector optimization problems. In Section 3, we build our Taylor's formula in the terms of $k + 1$ order subdifferential for a $C^{k,1}$ function from R^n to R^m. In Section 4, as applications in optimization problems, we establish some optimality conditions in terms of $(k + 1)$th order subdifferential. In Section 5, we give a necessary condition in the terms of

$(k+1)$th order subdifferential for a strongly C-quasiconvex vector-valued function.

2. Preliminaries

Let X, Y be Banach spaces, Y^* the dual space of Y, $C \subset Y$ a closed convex cone with $\operatorname{int}(C) \neq \emptyset$, and C^+ the dual cone of C; that is,

$$C^+ = \left\{ y^* \in Y^* : 0 \leq \left\langle y^*, c \right\rangle \ \forall c \in C \right\}. \tag{1}$$

For $y_1, y_2 \in Y$, we define $y_1 <_C y_2$ and $y_1 \leq_C y_2$ if $y_2 - y_1 \in \operatorname{int}(C)$ and $y_2 - y_1 \in C$, respectively. Let A be a subset of Y and $a \in A$. Recall that (i) a is a weak Pareto point of A if there exists no point $y \in A$ such that $y <_C a$; (ii) a is a Pareto point of A if there exists no point $y \in A \setminus \{a\}$ such that $y \leq_C a$; (iii) a is an ideal point of A if $a \leq_C y$ for all $y \in A$. Let $\mathrm{WE}(A, C)$, $E(A, C)$, and $I(A, C)$ denote the sets of all weak Pareto, Pareto, and ideal points of A, respectively. It is easy to verify that

$$\begin{gathered} a \in \mathrm{WE}(A, C) \Longleftrightarrow (a - \operatorname{int}(C)) \cap A = \emptyset, \\ a \in E(A, C) \Longleftrightarrow (a - C) \cap A = \{a\}, \\ I(A, C) \subset E(A, C) \subset \mathrm{WE}(A, C). \end{gathered} \tag{2}$$

Let $X^n := \{(x_1, \ldots, x_n) : x_i \in X, i = 1, \ldots, n\}$ be equipped with the norm $\|(x_1, \ldots, x_n)\| = \sum_{i=1}^n \|x_i\|$.

Let $\Phi : X^n \to Y$ be n-linear and symmetric mapping [17]; that is, for any $s, t \in \mathbb{R}$ and $x_1, z_1, x_2, \ldots, x_n \in X$,

$$\begin{aligned} &\Phi(sx_1 + tz_1, x_2, \ldots, x_n) \\ &\quad = s\Phi(x_1, x_2, \ldots, x_n) + t\Phi(z_1, x_2, \ldots, x_n), \\ &\quad \Phi(x_1, \ldots, x_n) = \Phi\left(x_{i_1}, \ldots, x_{i_n}\right), \end{aligned} \tag{3}$$

where $(i_1, \ldots, i_n)$ is an arbitrary permutation of $(1, \ldots, n)$. Let $f : X \to Y$ be a mapping. It is known that its derivative $f^{(n)}(x)$ is n-linear, symmetric, and continuous mapping if f is n-time smooth.

Let f be a function from R^n to R^m and $C \subset R^m$ be a closed convex cone. Consider the following vector optimization problem

$$C - \min_{x \in R^n} f(x). \tag{4}$$

A vector $\overline{x} \in X$ is said to be a local weak Pareto (resp., Pareto and ideal) solution of (4) if there exists $\delta > 0$ such that $f(\overline{x})$ is a weak Pareto (resp., Pareto and ideal) point of $f(B(\overline{x}, \delta))$, where $B(\overline{x}, \delta)$ denotes the open ball with center $\overline{x}$ and radius δ. We say that $\overline{x}$ is a sharp Pareto solution of (4) of order r if there exist $\eta, \delta \in (0, +\infty)$ such that

$$\eta \|x - \overline{x}\| \leq \left[f(x) - f(\overline{x}) \right]_+^r, \quad \forall x \in B(\overline{x}, \delta), \tag{5}$$

where $[f(x) - f(\overline{x})]_+ := d(f(x) - f(\overline{x}), -C)$.

We denote by $C^{k,1}, k > 0$, the class of k-time differentiable mappings from R^n to R^m whose kth order derivatives are locally Lipschitz mappings and by $C^{0,1}$ the class of locally Lipschitz functions from R^n to R^m. By Rademacher's theorem (see [18]), for any $f \in C^{k,1}$, $f(x) = (f_1(x), \ldots, f_m(x))$, its kth order derivative $D^k f(x)$ is a function differentiable almost everywhere. The $(k+1)$th order subdifferential of f at $x \in R^n$ is defined as "generalized Jacobian" of $D^{k+1} f$ at x in Clarke's sense [18] as follows:

$$\begin{aligned} \partial^{k+1} f(x) := \operatorname{co} \Big\{ &\lim D^{k+1} f(x_i) : x_i \longrightarrow x, \\ &D^{k+1} f(x_i) \text{ exists at } x_i \Big\}. \end{aligned} \tag{6}$$

It is worth mentioning that each element in $\partial^{k+1} f(x)$ is a $k+1$ linear and symmetric mapping from $(R^n)^{k+1}$ to R^m. For more details about $\partial^{k+1} f(x)$, we refer the reader to [18].

It is similar to the proof of Lemma 2.1 in [1], and one can verify the following chain rule.

Lemma 1. *Let x, u in R^n, g be a function from R to R^n defined by $g(t) = x + tu$ for every $t \in R$, and let f be a $C^{k,1}$ function from R^n to R^m. Then,*

$$\partial^{k+1} (f \circ g)(t) \subseteq \partial^{k+1} f(x + tu)(u, \ldots, u). \tag{7}$$

3. A New Taylor's Formula in Form of High-Order Subdifferential

By Lemma 1, we have the following Taylor-type formula for a $C^{k,1}$ vector-valued function from R^n to R^m which will be useful in the sequel.

Theorem 2. *Let x, u, and f be as in Lemma 1. Then, there exists $A \in \operatorname{clco} \partial^{k+1} f(x, x+u)$ such that*

$$f(x+u) - f(x) = \sum_{i=1}^{k} \frac{1}{i!} D^i f(x)\left(u^i\right) + \frac{1}{(k+1)!} A\left(u^{k+1}\right), \tag{8}$$

where u^i denotes $(u, \ldots, u) \in (R^n)^i$ and

$$\operatorname{clco} \partial^{k+1} f(x, x+u) := \operatorname{clco}\left(\bigcup_{t \in (0,1)} \partial^{k+1} f(x + tu) \right). \tag{9}$$

Proof. Let $\alpha \in R^m$ be a vector satisfying

$$f(x+u) - f(x) = \sum_{i=1}^{k} \frac{1}{i!} D^i f(x)\left(u^i\right) + \frac{1}{(k+1)!} \alpha. \tag{10}$$

We only need to show that there exists $A \in \operatorname{clco} \partial^{k+1} f(x, x+u)$ such that

$$\alpha = A\left(u^{k+1}\right). \tag{11}$$

Let g be as Lemma 1. Set $\varphi(t) := (f \circ g)(t)$ and

$$\begin{aligned} h(t) := {} & \varphi(1) - \varphi(t) \\ & - \sum_{i=1}^{k} \frac{1}{i!} D^i \varphi(t) (1-t)^i - \frac{1}{(k+1)!} (1-t)^{k+1} \alpha. \end{aligned} \tag{12}$$

Let $y \in R^m$ be arbitrarily given. Since the function $\langle y, h(\cdot)\rangle$ is locally Lipschitz and $h(0) = h(1)$, applying Lebourg mean value theorem [18, Theorem 2.3.7 and Theorem 2.3.9], there exists $t_0 \in (0,1)$ such that

$$0 \in \partial \langle y, h(t_0)\rangle \subset \langle y, \partial h(t_0)\rangle. \tag{13}$$

Noting that $\varphi(\cdot)$ and each $D^i\varphi(\cdot)(1-\cdot)^i$ $(1 \le i \le k-1)$ have derivatives which are continuous, it follows that they are strictly H-differentiable. We have

$$\begin{aligned}
\partial h(t_0) = & -\partial\varphi(t_0) - \sum_{i=1}^{k-1}\frac{1}{i!}\partial\left(D^i\varphi(\cdot)(1-\cdot)^i\right)(t_0) \\
& - \frac{1}{k!}\partial\left(D^k\varphi(\cdot)(1-\cdot)^k\right)(t_0) \\
& - \frac{1}{(k+1)!}\partial\left((1-\cdot)^{k+1}\alpha\right)(t_0) \\
= & -\varphi'(t_0) \\
& - \sum_{i=1}^{k-1}\frac{1}{i!}\left(D^{i+1}\varphi(t_0)(1-t_0)^i - iD^i\varphi(t_0)(1-t_0)^{i-1}\right) \\
& - \frac{1}{k!}\partial\left(D^k\varphi(\cdot)(1-\cdot)^k\right)(t_0) - \frac{1}{k!}(1-t_0)^k\alpha \\
= & -\frac{1}{(k-1)!}D^k\varphi(t_0)(1-t_0)^{k-1} \\
& - \frac{1}{k!}\partial\left(D^k\varphi(\cdot)(1-\cdot)^k\right)(t_0) - \frac{1}{k!}(1-t_0)^k\alpha.
\end{aligned} \tag{14}$$

Here, the first equation holds by Propositions 7.4.3(b), and 7.3.5 in [19] and the second holds by Proposition 7.3.9 in [19]. By the chain rule [19, Theorem 7.4.5(a)], we also have

$$\begin{aligned}
\partial\left(D^k\varphi(\cdot)(1-\cdot)^k\right)(t_0) \subset & \partial^{k+1}\varphi(t_0)(1-t_0)^k \\
& - kD^k\varphi(t_0)(1-t_0)^k.
\end{aligned} \tag{15}$$

Hence, we have

$$\partial h(t_0) \subset -\frac{1}{k!}\partial^{k+1}\varphi(t_0)(1-t_0)^k + \frac{1}{k!}(1-t_0)^k\alpha. \tag{16}$$

From (13) and (16), we have

$$0 \in \left\langle y, -\frac{1}{k!}\partial^{k+1}\varphi(t_0)(1-t_0)^k + \frac{1}{k!}(1-t_0)^k\alpha\right\rangle. \tag{17}$$

Together with Lemma 1, it follows that

$$0 \in \left\langle y, -\frac{1}{k!}(1-t_0)^k\partial^{k+1}f(x+t_0u)\left(u^{k+1}\right) + \frac{1}{k!}(1-t_0)^k\alpha\right\rangle; \tag{18}$$

that is,

$$\begin{aligned}
\langle y, \alpha\rangle \in & \left\langle y, \partial^{k+1}f(x+t_0u)\left(u^{k+1}\right)\right\rangle \\
& \subset \left\langle y, \operatorname{clco}\partial^{k+1}f(x, x+u)\left(u^{k+1}\right)\right\rangle.
\end{aligned} \tag{19}$$

Since y is arbitrary in R^n and $\operatorname{clco}\partial^{k+1}f(x, x+u)(u^{k+1})$ is convex and compact, by the separation theorem, we can easily show that $\alpha \in \operatorname{clco}\partial^{k+1}f(x, x+u)(u^{k+1})$. Hence, we can take $A \in \operatorname{clco}\partial^{k+1}f(x, x+u)$ such that $\alpha = A(u^{k+1})$. The proof is completed. □

Corollary 3. *Let f be as in Theorem 2 and $a \in R^n$. Then, for every $x \in R^n$, there exist $A_x \in \partial^{k+1}f(a)$ and a $(k+1)$-linear mapping $r(x)$ from $(R^n)^{k+1}$ to R^m such that*

$$\begin{aligned}
& \lim_{x\to a}\|r(x)\| = 0, \\
& f(x) = f(a) + \sum_{i=1}^{k}\frac{1}{i!}D^if(a)(x-a)^i \\
& \qquad + \frac{1}{(k+1)!}A_x(x-a)^{k+1} + r(x)(x-a)^{k+1}.
\end{aligned} \tag{20}$$

Proof. By Theorem 2, for a given $x \in R^n$, there exists $B_x \in \operatorname{clco}\partial^{k+1}f(a, a+x)$ such that

$$f(x) = f(a) + \sum_{i=1}^{k}\frac{1}{i!}D^if(a)(x-a)^i + \frac{1}{(k+1)!}B_x(x-a)^{k+1}. \tag{21}$$

Let $A_x \in \partial^{k+1}f(a)$ be an element minimizing the distance from B_x to the convex and compact set $\partial^{k+1}f(a)$. Set

$$r(x) := \frac{B_x - A_x}{(k+1)!}. \tag{22}$$

Then, from (21), we obtain the formula of the corollary. Moreover, since the mapping $\partial^{k+1}f$ is upper continuous, nonempty, convex, and compact valued (see [18]), for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $y \in a + \delta B_{R^n}$ (where B_{R^n} denotes the closed unit ball of R^n),

$$\partial^{k+1}f(y) \subset \partial^{k+1}f(a) + (k+1)!\varepsilon B_{L((R^n)^{k+1}, R^m)}, \tag{23}$$

where $B_{L((R^n)^{k+1}, R^m)}$ denotes the closed unit ball of the space $L((R^n)^{k+1}, R^m)$ of all bounded linear operators from $(R^n)^{k+1}$ to R^m. If $x \in a + \delta B_{R^n}$, then

$$\operatorname{clco}\partial^{k+1}f(a, a+x) \subset \partial^{k+1}f(a) + (k+1)!\varepsilon B_{L((R^n)^{k+1}, R^m)}. \tag{24}$$

With this we obtain $\|r(x)\| \le \varepsilon$. The proof is completed. □

4. The Positive Definiteness of High-Order Subdifferential and Optimality Conditions

Recall [17] that n-linear symmetric mapping $\Phi : X^n \to Y$ is said to be positively definite (resp., positively semidefinite) with respect to the ordering cone C if

$$0 <_C \Phi(x^n) \ (\text{resp., } 0 \le_C \Phi(x^n)), \quad \forall x \in X \setminus \{0\}, \tag{25}$$

where x^n denotes $(x,\ldots,x)$. If n is odd and the ordering cone C is pointed (i.e., $C\cap(-C)=\{0\}$), then Φ is positively semidefinite if and only if $\Phi=0$; see [17].

By the separation theorem, it is easy to verify that a n-linear symmetric mapping Φ is positively semidefinite with respect to the ordering cone C if and only if the composite $c^*\circ\Phi$ is positively semidefinite for any $c^*\in C^+$. Recall that a mapping $f:X\to Y$ is C-convex if

$$f\left(tx_1+(1-t)x_2\right)\leq_C tf\left(x_1\right)+(1-t)f\left(x_2\right),\quad \forall x_1,x_2\in X,\quad \forall t\in[0,1]. \tag{26}$$

Noting that f is C-convex if and only if $c^*\circ f$ is convex for all $c^*\in C^+$, one can see that a twice differentiable function f is C-convex if and only if $f''(x)$ is positively semidefinite for all $x\in X$.

Inspired by the notion of positive definiteness, we introduce positive definiteness of the $(k+1)$th order subdifferential for $C^{k,1}$ functions.

Definition 4. Let f be a $C^{k,1}$ function from R^n to R^m and C a closed convex cone of R^m. We say that the $(k+1)$th order subdifferential mapping $\partial^{k+1}f$ is positively definite at $\overline{x}\in R^n$ with respect to the ordering cone C if each $A\in\partial^{k+1}f(\overline{x})$ is positively definite with respect to C.

Proposition 5. *Let f be a $C^{k,1}$ function from R^n to R^m, and let C be a closed convex cone of R^m. Suppose that the subdifferential mapping $\partial^{k+1}f$ is positively definite at $\overline{x}\in R^n$ with respect to C. Then, there exists $\eta>0$ such that*

$$A\left(x^{k+1}\right)+\eta B_{R^m}\subset C,\quad \forall A\in\partial^{k+1}f\left(\overline{x}\right),\ x\in S_{R^n}, \tag{27}$$

where $S_{R^n}:=\{x\in R^n:\|x\|=1\}$.

Proof. From [17, Proposition 3.4], for any $A\in\partial^{k+1}f(\overline{x})$, there exists $\eta_A>0$ such that

$$A\left(x^{k+1}\right)+\eta_A B_{R^m}\subset C,\quad \forall x\in S_{R^n}. \tag{28}$$

If the conclusion is not true, then, for every natural number i, there exist $A_i\in\partial^{k+1}f(\overline{x})$, $x_i\in S_{R^n}$ and $b_i\in B_{R^n}$ such that

$$A_i\left(x_i^{k+1}\right)+\frac{1}{i}b_i\notin C. \tag{29}$$

Since $\partial^{k+1}f(\overline{x})$ and S_{R^n} are compact, we can assume that $A_i\to A_0\in\partial^{k+1}f(\overline{x})$, $x_i\to x_0\in S_{R^n}$ (passing to a subsequence if necessary). Then,

$$A_0\left(x_0^{k+1}\right)+\left(A_i\left(x_i^{k+1}\right)-A_0\left(x_0^{k+1}\right)\right)+\frac{1}{i}b_i\notin C \tag{30}$$

for all i. But from (28), for large enough i, we have

$$A_0\left(x_0^{k+1}\right)+\left(A_i\left(x_i^{k+1}\right)-A_0\left(x_0^{k+1}\right)\right)+\frac{1}{i}b_i\in A_0\left(x_0^{k+1}\right)+\eta_{A_0}B_{R^m}\subset C, \tag{31}$$

which is a contradiction with (29). The proof is completed. □

Under the positive definiteness assumption, we will provide a $(k+1)$th order sufficient condition for $\overline{x}$ to be a sharp local Pareto solution of (4) for a $C^{k,1}$ function f.

Theorem 6. *Let f be a $C^{k,1}$ function from R^n to R^m, C a closed convex cone of R^m, and $\overline{x}\in R^n$. Suppose that there exists $c^*\in C^+$ with $\|c^*\|=1$ such that $\sum_{i=1}^{k}(1/i!)c^*\circ D^i f(\overline{x})=0$, and that $\partial^{(k+1)}f$ is positively definite at $\overline{x}$ with respect to the ordering cone C. Then, $\overline{x}$ is a local Pareto solution of* (4)*, and there exist $\eta,\delta\in(0,\ +\infty)$ such that*

$$\eta\|x-\overline{x}\|\leq\left[f(x)-f(\overline{x})\right]_+^{1/(k+1)},\quad \forall x\in B(\overline{x},\delta). \tag{32}$$

Proof. Since $\partial^{(k+1)}f(\overline{x})$ is positively definite with respect to C, by Proposition 5, there exists $\eta>0$ such that

$$\frac{1}{(k+1)!}\partial^{(k+1)}f\left(\overline{x}\right)\left(h^{k+1}\right)+2\eta^{k+1}B_{R^m}\subset C,\quad \forall h\in S_{R^n}. \tag{33}$$

Noting that $c^*\in C^+$ and $\|c^*\|=1$, we have that

$$\frac{1}{(k+1)!}\left\langle c^*,Ah^{k+1}\right\rangle\geq 2\eta^{k+1}\|h\|^{k+1},\quad \forall A\in\partial^{(k+1)}f\left(\overline{x}\right),\quad h\in R^n. \tag{34}$$

Let $\phi(x):=\langle c^*,f(x)\rangle$ for all $x\in X$. Since f is a $C^{k,1}$ function, so is ϕ. Noting that $D^{(i)}\phi(\overline{x})=c^*\circ D^{(i)}f(\overline{x})$ with Corollary 3, there exist $A_x\in\partial^{k+1}f(\overline{x})$ and $(k+1)$-linear mapping $r(x)$ with $\lim_{x\to\overline{x}}\|r(x)\|=0$ such that

$$\phi(x)=\phi(\overline{x})+\sum_{i=1}^{k}\frac{1}{i!}D^{(i)}\phi(\overline{x})\left((x-\overline{x})^i\right)+\frac{1}{(k+1)!}\times\left\langle c^*,A_x(x-\overline{x})^{k+1}\right\rangle+\left\langle c^*,r(x)(x-\overline{x})^{k+1}\right\rangle. \tag{35}$$

It follows that there exists $\delta>0$ such that

$$\phi(x)-\phi(\overline{x})-\sum_{i=1}^{k}\frac{1}{i!}D^{(i)}\phi(\overline{x})\left((x-\overline{x})^i\right)-\frac{1}{(k+1)!}\left\langle c^*,A_x(x-\overline{x})^{k+1}\right\rangle\geq-\eta^{k+1}\|x-\overline{x}\|^{k+1}, \tag{36}$$

for all $x\in B(\overline{x},\delta)$. Since $\sum_{i=1}^{k}(1/i!)c^*\circ D^i f(\overline{x})=0$, it follows from (34) and (36) that

$$\eta^{k+1}\|x-\overline{x}\|^{k+1}\leq\phi(x)-\phi(\overline{x}),\quad \forall x\in B(\overline{x},\delta). \tag{37}$$

On the other hand, for any $c\in C$, one has

$$\phi(x)-\phi(\overline{x})=\left\langle c^*,f(x)-f(\overline{x})\right\rangle\leq\left\langle c^*,f(x)-f(\overline{x})+c\right\rangle\leq\left\|f(x)-f(\overline{x})+c\right\|. \tag{38}$$

This implies that (32) holds. It remains to show that $\overline{x}$ is a local Pareto solution of (4). Let $x \in B(\overline{x}, \delta)$ such that $f(x) \leq_C f(\overline{x})$. Then, $\|f(x) - f(\overline{x})\|_+ = 0$. It follows from (32) that $x = \overline{x}$, and hence $f(x) = f(\overline{x})$. This shows that $\overline{x}$ is a local Pareto solution of (4). □

In Theorem 6, if f is a C-convex $C^{k,1}$ function, then $\overline{x}$ is a global Pareto solution of (4).

Theorem 7. *Let f be a C-convex $C^{k,1}$ function from R^n to R^m, C a closed convex cone of R^m, and $\overline{x} \in R^n$. Suppose that there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that $\sum_{i=1}^{k}(1/i!)c^* \circ D^i f(\overline{x}) = 0$ and that $\partial^{k+1} f(\overline{x})$ is positively definite. Then, $\overline{x}$ is a global Pareto solution of (4), and there exists $\eta_0 \in (0, +\infty)$ such that*

$$\eta_0 \|x - \overline{x}\| \leq \max\left\{\left[f(x) - f(\overline{x})\right]_+^{1/(k+1)}, \left[f(x) - f(\overline{x})\right]_+\right\}, \quad \forall x \in R^n. \tag{39}$$

Proof. Similar to the proof of Theorem 6, one can show that (39) implies that $\overline{x}$ is a global Pareto solution of (4). It remains to show that (39) holds. By Theorem 6, there exist $\eta, \delta \in (0, +\infty)$ such that (32) holds. Since f is C-convex, it is easy to verify that $x \mapsto [f(x) - f(\overline{x})]_+$ is a convex function. Let $x \in R^n \setminus B(\overline{x}, \delta)$. Then,

$$\begin{aligned}\eta^{k+1}\delta^{k+1} &\leq \left[f\left(\overline{x} + \delta\frac{x - \overline{x}}{\|x - \overline{x}\|}\right) - f(\overline{x})\right]_+ \\ &\leq \left(1 - \frac{\delta}{\|x - \overline{x}\|}\right)\left[f(\overline{x}) - f(\overline{x})\right]_+ \\ &\quad + \frac{\delta}{\|x - \overline{x}\|}\left[f(x) - f(\overline{x})\right]_+ \\ &= \frac{\delta}{\|x - \overline{x}\|}\left[f(x) - f(\overline{x})\right]_+.\end{aligned} \tag{40}$$

Hence, $\eta^{k+1}\delta^k\|x - \overline{x}\| \leq [f(x) - f(\overline{x})]_+$. Letting $\eta_0 := \min\{\eta, \eta^{k+1}\delta^k\}$, it follows from (32) that (39) holds. The proof is completed. □

With $\sum_{i=1}^{k}(1/i!)c^* \circ D^i f(\overline{x}) = 0$ in Theorem 7 replaced by a stronger assumption, we have the following sufficient condition for sharp ideal solutions of (4).

Theorem 8. *Let f be a $C^{k,1}$ function from R^n to R^m, C a closed convex cone of R^m, and $\overline{x} \in R^n$. Suppose that $\sum_{i=1}^{k}(1/i!)D^i f(\overline{x}) = 0$ and that $\partial^{k+1} f$ is positively definite at $\overline{x}$ with respect to the ordering cone C. Then, there exist $\eta, \delta \in (0, +\infty)$ such that*

$$f(\overline{x}) \leq_C f(x), \quad \forall x \in B(\overline{x}, \delta), \tag{41}$$

$$\eta\|x - \overline{x}\| \leq \left[f(x) - f(\overline{x})\right]_+^{1/(k+1)}, \quad \forall x \in B(\overline{x}, \delta). \tag{42}$$

Proof. By Theorem 6, we need only to show that there exists $\delta > 0$ such that (41) holds. Since $\partial^{k+1} f(\overline{x})$ is positively definite, there exists $\eta > 0$ such that

$$\frac{1}{(k+1)!}\partial^{k+1} f(\overline{x})\left(h^{k+1}\right) + 2\eta B_{R^m} \subset C, \quad \forall h \in S_{R^m}. \tag{43}$$

It follows that

$$\frac{1}{(k+1)!}\partial^{k+1} f(\overline{x})\left(x^{k+1}\right) + \eta\|x\|^{k+1} B_{R^m} \subset C, \quad \forall x \in R^m. \tag{44}$$

On the other hand, since $\sum_{i=1}^{k}(1/i!)D^i f(\overline{x}) = 0$, with Corollary 3, we can assume that for any $x \in R^n$ close to $\overline{x}$, there exists n-linear symmetric and continuous mapping $r(x)$ from $(R^n)^{k+1}$ to R^m such that $\lim_{x \to \overline{x}} r(x) = 0$ and

$$f(x) - f(\overline{x}) \in \frac{1}{(k+1)!}\partial^{(k+1)} f(\overline{x})\left((x - \overline{x})^{k+1}\right) + r(x)(x - \overline{x})^{k+1}. \tag{45}$$

Hence, there exists $\delta > 0$ such that

$$f(x) - f(\overline{x}) \in \frac{1}{(k+1)!}\partial^{k+1} f(\overline{x})\left((x - \overline{x})^{k+1}\right) + \eta\|x - \overline{x}\|^{k+1} B_{R^m}, \quad \forall x \in B(\overline{x}, \delta). \tag{46}$$

This and (44) imply that (41) holds. The proof is completed. □

5. $(k+1)$th Order Necessary Conditions for Strongly C-Quasiconvex Functions

We recall that a function f from R^n to R is quasiconvex if, for every $x, y \in R^n$ and for every $\lambda \in (0, 1)$, one has $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$. Inspired by this, we introduce the notion of strong C-quasiconvexity for functions from R^n to R^m. A function f from R^n to R^m is said to be strongly C-quasiconvex if, for every $x, y \in R^n$ and for every $\lambda \in (0, 1)$, one has

$$f(\lambda x + (1 - \lambda)y) \in \{f(x), f(y)\} - C. \tag{47}$$

Using the generalized Hessian (see [20]), Luc [1] gave a second-order criterion for quasiconvex functions. We will give a $(k+1)$th order necessary codition for a function to be strongly C-quasiconvex.

Theorem 9. *Let f be a strongly C-quasiconvex function from R^n to R^m, k an odd number and $C \subset R^m$ the closed pointed ordering cone. Then, for any $x, u \in R^n$ with $D^i f(x)(u^i) = 0$ $(i = 1, \ldots, k)$, there exists $A_0 \in \partial^{k+1} f(x)$ such that $A_0(u^{k+1}) \in C$.*

Proof. Suppose that the conclusion is not true. Then, there exist some $x, u \in R^n$, with $D^i f(x)(u^i) = 0$ $(i = 1, \ldots, k)$ such that $\partial^{k+1} f(x)(u^{k+1}) \subset R^m \setminus C$. Since $R^m \setminus C$ is open and $\partial^{k+1} f(x)(u^{k+1})$ is compact, there exists $\varepsilon > 0$ such that

$$\partial^{k+1} f(x)\left(u^{k+1}\right) + \varepsilon B_{L((R^n)^{k+1}, R^m)}\left(u^{k+1}\right) \subset R^m \setminus C. \tag{48}$$

Since $\partial^{k+1} f(\cdot)$ is upper continuous, for the previous ε, there exists $t_0 > 0$ such that

$$\partial^{k+1} f(x + tu) \subset \partial^{k+1} f(x) + \varepsilon B_{L((R^n)^{k+1}, R^m)}, \tag{49}$$

for any $t \in [-t_0, t_0]$. Noting that $\partial^{k+1} f(x)$ is closed convex, from (48) and (49), we have

$$\begin{aligned} &\operatorname{clco} \partial^{k+1} f\left(x - tu, x + tu\right)\left(u^{k+1}\right) \\ &\quad \subset \partial^{k+1} f\left(x\right)\left(u^{k+1}\right) + \varepsilon B_{L((R^n)^{k+1}, R^m)}\left(u^{k+1}\right) \subset R^m \setminus C. \end{aligned} \tag{50}$$

From Theorem 2, for any $t \in [-t_0, t_0]$, we can take $A_t \in \operatorname{clco} \partial^{k+1} f(x, x + tu)$ such that

$$\begin{aligned} f\left(x+tu\right) - f\left(x\right) &= \sum_{i=1}^{k} \frac{1}{i!} t^i D^i f\left(x\right)\left(u^i\right) + \frac{1}{(k+1)!} t^{k+1} A_t\left(u^{k+1}\right) \\ &= \frac{1}{(k+1)!} t^{k+1} A_t\left(u^{k+1}\right) \in R^m \setminus C. \end{aligned} \tag{51}$$

Noting that $k+1$ is even, we have $f(x+tu) - f(x) \in R^m \setminus C$ and $f(x-tu) - f(x) \in R^m \setminus C$, for all $t \in [-t_0, t_0]$.

On the other hand, since f is C-quasiconvex and $f(x) = f((1/2)(x+tu) + (1/2)(x-tu))$, one has $f(x+tu) - f(x) \in C$ or $f(x-tu) - f(x) \in C$. This is a contradiction. □

If $m = 1$ and $C = R_+$, then $C^+ = R_+$. We have the following.

Corollary 10 (see [1]). *Let f be a quasiconvex function from R^n to R and k an odd number. Then, for any $x, u \in R^n$ with $D^i f(x)(u^i) = 0$ $(i = 1, \ldots, k)$, one has $D_+^{k+1} f(x; u) \geq 0$, where $D_+^{k+1} f(x; u) := \max\{A(u, \ldots, u) : A \in \partial^{k+1} f(x)\}$.*

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