Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 675373, 6 pages
http://dx.doi.org/10.1155/2013/675373

Research Article

Computing Hypercrossed Complex Pairings in Digital Images

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Received 3 October 2013; Accepted 9 November 2013

Academic Editor: Abdon Atangana

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We consider an additive group structure in digital images and introduce the commutator in digital images. Then we calculate the hypercrossed complex pairings which generates a normal subgroup in dimension 2 and in dimension 3 by using 8-*adjacency* and 26-*adjacency*.

1. Introduction

In this paper we denote the set of integers by $\mathbb{Z}$. Then $\mathbb{Z}^n$ represents the set of lattice points in Euclidean n-dimensional spaces. A finite subset of $\mathbb{Z}^n$ with an adjacency relation is called a digital image.

Definition 1 (see [1, 2]). Consider the following.

(1) Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p-q|=1$.

(2) Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.

(3) Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ by exactly one coordinate.

(4) Two points p and q in $\mathbb{Z}^2$ are 26-adjacent if they are distinct and differ by at most 1 in each coordinate.

(5) Two points p and q in $\mathbb{Z}^2$ are 18-adjacent if they are 26-adjacent and differ in at most two coordinates.

(6) Two points p and q in $\mathbb{Z}^2$ are 6-adjacent if they are 18-adjacent and differ by exactly one coordinate.

Definition 2. Let $\mathbf{G}$ be a subset of a digital image. A simplicial group $\mathbf{G}$ in digital images consists of a sequence of groups $\mathbf{G}$ and collections of group homomorphisms $d_i : G_n \to G_{n-1}$ and $s_i : G_n \to G_{n+1}$, $0 \leq i \leq n$, that satisfies the following axioms:

$$\begin{gathered} d_i d_j = d_{j-1} d_i, \quad i<j, \\ d_i s_j = s_{j-1} d_i, \quad i<j, \\ d_j s_j = d_{j+1} s_j = id, \quad i=j \text{ or } i=j+1, \\ d_i s_j = s_j d_{i-1}, \quad i>j+1, \\ s_i s_j = s_{j+1} s_i, \quad i \leq j. \end{gathered} \tag{1}$$

Definition 3. Given a simplicial group $\mathbf{G}$ with κ-*adjacency*, the Moore complex $(\mathbf{NG}, \partial)$ of $\mathbf{G}$ is the chain complex defined by

$$NG_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i, \tag{2}$$

with $\partial : NG_n \to NG_{n-1}$ induced from d_n by restriction.

The nth homology group of the Moore complex of $\mathbf{G}$ is

$$H_n(\mathbf{NG}, \partial) = \frac{\bigcap_{i=0}^{n} \operatorname{Ker} d_i}{d_{n+1}\left(\bigcap_{i=0}^{n} \operatorname{Ker} d_i\right)}. \tag{3}$$

2. Hypercrossed Complex Pairings in Digital Images

First of all we adapt ideas from Carrasco and Cegarra [3–5] to get the construction in digital images. We define a set $P(n)$

consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \varnothing$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_l, \ldots, i_1)$ and $\beta = (j_m, \ldots, j_1) \in S(n)$.

Consider the following diagram:

$$\begin{array}{ccc} NG_{n-\#\alpha} \times NG_{n-\#\beta} & \xrightarrow{F_{\alpha,\beta}} & NG_n \\ \Big\downarrow{\scriptstyle s_\alpha \times s_\beta} & & \Big\downarrow{\scriptstyle p} \\ G_n \times G_n & \xrightarrow{\mu} & G_n \end{array} \tag{4}$$

$$\{F_{\alpha,\beta}: NG_{n-\#\alpha} \times NG_{n-\#\beta} \longrightarrow NG_n: (\alpha,\beta) \in P(n), n \geq 0\}$$

where

$$\begin{gathered} s_\alpha = s_{i_l} \cdots s_{i_1} : NG_{n-\#\alpha} \longrightarrow G_n, \\ s_\beta = s_{j_m} \cdots s_{j_1} : NG_{n-\#\beta} \longrightarrow G_n, \end{gathered} \tag{5}$$

and define $p : G_n \to NG_n$ and $p(x) = p_{n-1} \cdots p_0(x)$ as $p_j(z) = z - s_i d_i z$ and $j = 0, \ldots, n-1$. Since a digital image has the additive group structure, define the commutator as

$$[x, y] = xy - yx. \tag{6}$$

Thus

$$\begin{aligned} \mu : G_n \times G_n &\longrightarrow G_n, \\ F_{\alpha,\beta}(x_\alpha, y_\beta) &= p\mu(s_\alpha \times s_\beta)(x_\alpha, y_\beta) \\ &= p[s_\alpha x_\alpha, s_\beta y_\beta]. \end{aligned} \tag{7}$$

The normal subgroup NG_n of G_n is generated by the elements of the form

$$F_{\alpha,\beta}(x_\alpha, y_\beta), \tag{8}$$

where $x_\alpha \in NG_{n-\#\alpha}$ and $y_\beta \in NG_{n-\#\beta}$.

Theorem 4. *2-dimensional normal subgroup N_2 with 8-adjacency is generated by the elements of the form*

$$[s_0 x_1 - s_1 x_1, s_1 y_1]. \tag{9}$$

Proof. Let $\alpha = (1)$ and $\beta = (0)$ for $n = 2$. For x_1 and $y_1 \in NG_1 = \operatorname{Ker} d_0$,

$$\begin{aligned} F_{(0),(1)}(x_1, y_1) &= p_1 p_0 [s_0 x_1, s_1 y_1] \\ &= p_1 \{[s_0 x_1, s_1 y_1] - s_0 d_0 [s_0 x_1, s_1 y_1]\} \\ &= [s_0 x_1 - s_1 x_1, s_1 y_1]. \end{aligned} \tag{10}$$

Thus $F_{(0),(1)}(x_1, y_1) = [s_0 x_1 - s_1 x_1, s_1 y_1]$ and this is the element generating N_2 normal subgroups. □

Proposition 5. *3-dimensional normal subgroup N_3 with 26-adjacency is generated by the elements of the following forms:*

(i) $[s_1 s_0 x_1 - s_0 s_1 x_1, s_2 y_2]$,
(ii) $[s_2 s_0 x_1 - s_2 s_1 x_1, s_1 y_2 - s_2 y_2]$,
(iii) $[s_0 x_2 - s_1 x_2 + s_2 x_2, s_1 s_1 y_1]$,
(iv) $[s_1 x_2 - s_2 x_2, s_2 y_2]$,
(v) $[s_0 x_2, s_2 y_2]$,
(vi) $[s_0 x_2 - s_1 x_2, s_1 y_2] + [s_2 x_2, s_2 y_2]$.

Proof. For $n = 3$ the possible pairings are the following:

(i) $F_{(1,0)(2)}$,
(ii) $F_{(2,0)(1)}$,
(iii) $F_{(0)(2,1)}$,
(iv) $F_{(1)(2)}$,
(v) $F_{(0)(2)}$,
(vi) $F_{(0)(1)}$.

For all $x_1 \in NG_1$ and $y_2 \in NG_2$ the corresponding generators of N_3 are the following with $F_{\alpha,\beta} : NG_1 \times NG_2 \to NG_3$ and for $n = 3$, $p(x) = p_2 p_1 p_0(x)$:

(i)

$$\begin{aligned} F_{(1,0)(2)}(x_1, y_2) &= p[s_1 s_0 x_1, s_2 y_2] \\ &= p_2 p_1 p_0 [s_1 s_0 x_1, s_2 y_2] \\ &= [s_1 s_0 x_1 - s_2 s_1 x_1, s_2 y_2], \end{aligned} \tag{11}$$

(ii)

$$\begin{aligned} F_{(2,0)(1)}(x_1, y_2) &= p[s_2 s_0 x_1, s_1 y_2] \\ &= p_2 p_1 p_0 [s_2 s_0 x_1, s_1 y_2] \\ &= [s_2 s_0 x_1 - s_2 s_1 x_1, s_1 y_2 - s_2 y_2]. \end{aligned} \tag{12}$$

For all $x_2 \in NG_2$ and $y_1 \in NG_1$ and considering the map $F_{\alpha,\beta} : NG_2 \times NG_1 \to NG_3$, the corresponding generator of N_3 is

(iii)

$$\begin{aligned} F_{(0)(2,1)}(x_2, y_1) &= p[s_0 x_2, s_2 s_1 y_1] \\ &= p_2 p_1 p_0 [s_0 x_2, s_2 s_1 y_1] \\ &= [s_0 x_2 - s_1 x_2 + s_2 x_2, s_2 s_1 y_1]. \end{aligned} \tag{13}$$

For all $x_2, y_2 \in NG_2$ and for $F_{\alpha,\beta} : NG_2 \times NG_2 \to NG_3$ the corresponding generators of N_3 are

(iv)

$$\begin{aligned} F_{(1)(2)}(x_2, y_2) &= p[s_1 x_2, s_2 y_2] \\ &= p_2 p_1 p_0 [s_1 x_2, s_2 y_2] \\ &= [s_1 x_2 - s_2 x_2, s_2 y_2], \end{aligned} \tag{14}$$

(v)

$$\begin{aligned} F_{(0)(2)}(x_2, y_2) &= p[s_0 x_2, s_2 y_2] \\ &= p_2 p_1 p_0 [s_0 x_2, s_2 y_2] \\ &= [s_0 x_2, s_2 y_2], \end{aligned} \tag{15}$$

(vi)

$$\begin{aligned} F_{(0)(1)}(x_2, y_2) &= p[s_0x_2, s_1y_2] \\ &= p_2p_1p_0[s_0x_2, s_1y_2] \\ &= [s_0x_2 - s_1x_2, s_1y_2] + [s_2x_2, s_2y_2]. \end{aligned} \quad (16)$$

□

Theorem 6. *Let NG_2 be a 2-dimensional Moore complex of a simplicial group* **G**. *Then $\partial_2(NG_2) = [\operatorname{Ker} d_0, \operatorname{Ker} d_1]$ where ∂_2 is induced from d_2 by restriction.*

Proof. For $n = 2$, assume that $\alpha = (0)$, $\beta = (1)$, and $x_1, y_1 \in NG_1 = \operatorname{Ker} d_0$. Now calculate $d_nF_{\alpha,\beta}$.

Since $F_{(0),(1)}(x_1, y_1) = [s_0x_1 - s_1x_1, s_1y_1]$, from Proposition 5

$$\begin{aligned} d_2F_{(0),(1)}(x_1, y_1) &= \left[d_2s_0x_1 - \underbrace{d_2s_1}_{id}x_1, \underbrace{d_2s_1}_{id}y_1\right] \\ &= [s_0d_1x_1 - x_1, y_1]. \end{aligned} \quad (17)$$

At first we investigate whether $s_0d_1x_1 - x_1$ is in $\operatorname{Ker} d_0$ or not.

$$\underbrace{d_0s_0}_{id}d_1x_1 - \underbrace{d_0x_1}_{=0} = d_1x_1; \quad (18)$$

therefore $s_0d_1x_1 - x_1 \notin \operatorname{Ker} d_0$.

Secondly we examine whether $s_0d_1x_1 - x_1$ is in $\operatorname{Ker} d_1$ or not.

Since

$$\underbrace{d_1s_0}_{id}d_1x_1 - \underbrace{d_1x_1}_{=0} = d_1x_1 - d_1x_1 = 0, \quad s_0d_1x_1 - x_1 \in \operatorname{Ker} d_1. \quad (19)$$

From the assumption $y_1 \in \operatorname{Ker} d_0$, we get

$$F_{(0),(1)}(x_1, y_1) \in [\operatorname{Ker} d_1, \operatorname{Ker} d_0]. \quad (20)$$

□

Theorem 7. *Let NG_3 be a 3-dimensional Moore complex of a simplicial group* **G** *with 26-adjacency. Then*

$$\begin{aligned} \partial_3(NG_3) \subseteq{} & [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ &+ [\operatorname{Ker} d_1, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_2] \\ &+ [\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0] \\ &+ [\operatorname{Ker} d_1, \operatorname{Ker} d_0] \\ &+ [\operatorname{Ker} d_0 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ &+ [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1], \end{aligned} \quad (21)$$

where ∂_3 is induced from d_3 by restriction.

Proof. For $n = 3$ investigate $d_nF_{\alpha,\beta}$ where $x_1 \in NG_1$ and $y_2 \in NG_2 = \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1$.

From Proposition 5 we have $F_{(1,0)(2)}(x_1, y_2) = [s_1s_0x_1 - s_2s_0x_1, s_2y_2]$. Then applying d_3 to $F_{(1,0)(2)}(x_1, y_2)$, we get the following:

$$\begin{aligned} d_3F_{(1,0)(2)}(x_1, y_2) &= \left[d_3s_1s_0x_1 - \underbrace{d_3s_2}_{id}s_0x_1, \underbrace{d_3s_2}_{id}y_2\right] \\ &= [s_1d_2s_0x_1 - s_0x_1, y_2] \\ &= [s_1s_0d_1x_1 - s_0x_1, y_2]. \end{aligned} \quad (22)$$

Firstly, examine whether $s_1s_0d_1x_1 - s_0x_1$ is in $\operatorname{Ker} d_0$ or not:

$$\begin{aligned} d_0(s_1s_0d_1x_1 - s_0x_1) &= d_0s_1s_0d_1x_1 - d_0s_0x_1 \\ &= s_0\underbrace{d_0s_0}_{id}d_1x_1 - \underbrace{d_0s_0}_{id}x_1 \\ &= s_0d_1x_1 - x_1. \end{aligned} \quad (23)$$

So $s_1s_0d_1x_1 - s_0x_1 \notin \operatorname{Ker} d_0$.

Secondly we investigate whether $s_1s_0d_1x_1 - s_0x_1$ is in $\operatorname{Ker} d_1$ or not:

$$\begin{aligned} d_1(s_1s_0d_1x_1 - s_0x_1) &= d_1s_1s_0d_1x_1 - d_1s_0x_1 \\ &= \underbrace{d_1s_1}_{id}s_0d_1x_1 - \underbrace{d_1s_0}_{id}x_1 \\ &= s_0d_1x_1 - x_1. \end{aligned} \quad (24)$$

Therefore $s_1s_0d_1x_1 - s_0x_1 \notin \operatorname{Ker} d_1$.

Finally we check whether $s_1s_0d_1x_1 - s_0x_1$ is in $\operatorname{Ker} d_2$ or not.

Since

$$\begin{aligned} d_2(s_1s_0d_1x_1 - s_0x_1) &= \underbrace{d_2s_1}_{id}s_0d_1x_1 - d_2s_0x_1 \\ &= s_0d_1x_1 - s_0d_1x_1 = 0, \end{aligned} \quad (25)$$

therefore $s_1s_0d_1x_1 - s_0x_1 \in \operatorname{Ker} d_2$.

We get

$$F_{(0,1),(2)}(x_1, y_1) \in [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1], \quad (26)$$

since $y_2 \in \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1$.

If

$$F_{(2,0)(1)}(x_1, y_2) = [s_2s_0x_1 - s_2s_1x_1, s_1y_2 - s_2y_2], \quad (27)$$

then

$$\begin{aligned} & d_3F_{(2,0)(1)}(x_1, y_2) \\ &= d_3([s_2s_0x_1 - s_2s_1x_1, s_1y_2 - s_2y_2]) \\ &= \left[\underbrace{d_3s_2}_{id}s_0x_1 - \underbrace{d_3s_2}_{id}s_1x_1, d_3s_1y_2 - \underbrace{d_3s_2}_{id}y_2\right] \\ &= [s_0x_1 - s_1x_1, s_1d_2y_2 - y_2]. \end{aligned} \quad (28)$$

At first we check whether $s_0x_1 - s_1x_1$ is in $\operatorname{Ker} d_0$, $\operatorname{Ker} d_1$, and $\operatorname{Ker} d_2$ or not.

$$d_0\left(s_0x_1 - s_1x_1\right) = \underbrace{d_0s_0}_{id}x_1 - d_0s_1x_1 = x_1 - d_0s_1x_1. \tag{29}$$

Thus $s_0x_1 - s_1x_1 \notin \operatorname{Ker} d_0$.

Next, since

$$\begin{gathered} d_1\left(s_0x_1 - s_1x_1\right) = \underbrace{d_1s_0}_{id}x_1 - \underbrace{d_1s_1}_{id}x_1 = x_1 - x_1 = 0, \\ s_0x_1 - s_1x_1 \in \operatorname{Ker} d_1, \end{gathered} \tag{30}$$

and, finally,

$$d_2\left(s_0x_1 - s_1x_1\right) = d_2s_0x_1 - \underbrace{d_2s_1}_{id}x_1 = s_0d_1x_1 - x_1 \notin \operatorname{Ker} d_2. \tag{31}$$

Now examine whether $s_1d_2y_2 - y_2$ is in $\operatorname{Ker} d_0$, $\operatorname{Ker} d_1$, and $\operatorname{Ker} d_2$ or not:

$$\begin{aligned} d_0\left(s_1d_2y_2 - y_2\right) &= d_0s_1d_2y_2 - d_0y_2 \\ &= s_0d_0d_2y_2 - d_0y_2 \\ &= s_0d_1\underbrace{d_0y_2}_{=0} - \underbrace{d_0y_2}_{=0} = 0. \end{aligned} \tag{32}$$

Therefore $s_1d_2y_2 - y_2 \in \operatorname{Ker} d_0$. We have the following:

$$\begin{gathered} d_1\left(s_1d_2y_2 - y_2\right) = \underbrace{d_1s_1}_{id}d_2y_2 - \underbrace{d_1y_2}_{=0} = d_2y_2 \notin \operatorname{Ker} d_1; \\ d_2\left(s_1d_2y_2 - y_2\right) = \underbrace{d_2s_1}_{id}d_2y_2 - d_2y_2 = d_2y_2 - d_2y_2 = 0 \\ \Longrightarrow s_1d_2y_2 - y_2 \in \operatorname{Ker} d_2. \end{gathered} \tag{33}$$

So $F_{(2,0)(1)}(x_1, y_2) = [\operatorname{Ker} d_1, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_2]$.

For all $x_2 \in NG_2$ and $y_1 \in NG_1$ if

$$F_{(0)(2,1)}\left(x_2, y_1\right) = \left[s_0x_2 - s_1x_2 + s_2x_2,\ s_1s_1y_1\right], \tag{34}$$

then

$$\begin{aligned} &d_3F_{(0)(2,1)}\left(x_2, y_1\right) \\ &\quad= d_3\left(\left[s_0x_2 - s_1x_2 + s_2x_2, s_2s_1y_1\right]\right) \\ &\quad= \left[d_3s_0x_2 - d_3s_1x_2 + \underbrace{d_3s_2}_{id}x_2, \underbrace{d_3s_2}_{id}s_1y_1\right] \\ &\quad= \left[s_0d_2x_2 - s_1d_2x_2 + x_2, s_1y_1\right]. \end{aligned} \tag{35}$$

Firstly investigate whether $s_0d_2x_2 - s_1d_2x_2 + x_2$ is in $\operatorname{Ker} d_0$, $\operatorname{Ker} d_1$, and $\operatorname{Ker} d_2$ or not:

$$\begin{aligned} d_0&\left(s_0d_2x_2 - s_1d_2x_2 + x_2\right) \\ &= \underbrace{d_0s_0}_{id}d_2x_2 - d_0s_1d_2x_2 + \underbrace{d_0x_2}_{=0} \\ &= d_2x_2 - s_0d_1\underbrace{d_0x_2}_{=0} = d_2x_2. \end{aligned} \tag{36}$$

Thereby $s_0d_2x_2 - s_1d_2x_2 + x_2 \notin \operatorname{Ker} d_0$. We have

$$\begin{aligned} d_1&\left(s_0d_2x_2 - s_1d_2x_2 + x_2\right) \\ &= \underbrace{d_1s_0}_{id}d_2x_2 - \underbrace{d_1s_1}_{id}d_2x_2 + \underbrace{d_1x_2}_{=0} \\ &= d_2x_2 - d_2x_2 = 0. \end{aligned} \tag{37}$$

For this reason $s_0d_2x_2 - s_1d_2x_2 + x_2 \in \operatorname{Ker} d_1$. We also have

$$\begin{aligned} d_2&\left(s_0d_2x_2 - s_1d_2x_2 + x_2\right) \\ &= d_2s_0d_2x_2 - \underbrace{d_2s_1}_{id}d_2x_2 + d_2x_2 \\ &= s_0d_1d_2x_2 - d_2x_2 + d_2x_2 \\ &= s_0d_1\underbrace{d_1x_2}_{=0} = 0. \end{aligned} \tag{38}$$

Hence $s_0d_2x_2 - s_1d_2x_2 + x_2 \in \operatorname{Ker} d_2$.

Later on we research whether $s_2s_1y_1$ is in $\operatorname{Ker} d_0$, $\operatorname{Ker} d_1$, and $\operatorname{Ker} d_2$ or not.

Since $d_0(s_1y_1) = s_0\underbrace{d_0y_1}_{=0} = 0$, $s_1y_1 \in \operatorname{Ker} d_0$.

Since $d_1(s_1y_1) = \underbrace{d_1s_1}_{id}y_1 = y_1$, $s_1y_1 \notin \operatorname{Ker} d_1$.

Since $d_2(s_1y_1) = \underbrace{d_2s_1}_{id}y_1 = y_1$, $s_1y_1 \notin \operatorname{Ker} d_2$.

Thus $d_3F_{(0)(2,1)}(x_2, y_1) \in [\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0]$.

For all $x_2, y_2 \in NG_2$ since $F_{(0)(2)}(x_2, y_2) = [s_0x_2, s_2y_2]$,

$$\begin{aligned} d_3F_{(0)(2)}\left(x_2, y_2\right) &= d_3\left(\left[s_0x_2, s_2y_2\right]\right) \\ &= \left[d_3s_0x_2, \underbrace{d_3s_2}_{id}y_2\right] \\ &= \left[s_0d_2x_2, y_2\right]. \end{aligned} \tag{39}$$

By using properties of the commutator we have

$$\begin{aligned} &\left[s_0d_2x_2 - s_1d_2x_2 + x_2, y_2\right] \\ &\quad= \left[s_0d_2x_2 + \left(x_2 - s_1d_2x_2\ \right), y_2\right] \\ &\quad= \left[s_0d_2x_2, y_2\right] + \left[x_2 - s_1d_2x_2, y_2\right], \\ &\left[s_0d_2x_2 - s_1d_2x_2 + x_2, y_2\right] + \left[y_2, x_2 - s_1d_2x_2\ \right] \\ &\quad= \left[s_0d_2x_2, y_2\right] \\ &\quad= d_3F_{(0)(2)}\left(x_2, y_2\right). \end{aligned} \tag{40}$$

Thus

$$\begin{aligned} &d_3F_{(0)(2)}\left(x_2, y_2\right) \\ &\quad\in \left[\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_1 \cap \operatorname{Ker} d_0\right] \\ &\qquad+ \left[\operatorname{Ker} d_0 \cap \operatorname{Ker} d_1, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_2\right]. \end{aligned} \tag{41}$$

If $F_{(1)(2)}(x_2, y_2) = [s_1x_2 - s_2x_2, s_2y_2]$, then

$$\begin{aligned} d_3F_{(1)(2)}(x_2, y_2) &= d_3([s_1x_2 - s_2x_2, s_2y_2]) \\ &= \left[d_3s_1x_2 - \underbrace{d_3s_2}_{id}x_2, \underbrace{d_3s_2}_{id}y_2\right] \\ &= [s_1d_2x_2 - x_2, y_2]. \end{aligned} \tag{42}$$

Firstly we check whether $s_1d_2x_2 - x_2$ is in $\operatorname{Ker} d_0$, $\operatorname{Ker} d_1$, and $\operatorname{Ker} d_2$ or not:

$$\begin{aligned} d_0(s_1d_2x_2 - x_2) &= d_0s_1d_2x_2 - \underbrace{d_0x_2}_{=0} \\ &= s_0d_0d_2x_2 \\ &= s_0d_1\underbrace{d_0x_2}_{=0} \\ &= 0. \end{aligned} \tag{43}$$

Therefore $s_1d_2x_2 - x_2 \in \operatorname{Ker} d_0$. Since

$$\begin{aligned} d_1(s_1d_2x_2 - x_2) &= \underbrace{d_1s_1}_{id}d_2x_2 - \underbrace{d_1x_2}_{=0} \\ &= d_2x_2, \end{aligned} \tag{44}$$

$s_1d_2x_2 - x_2 \notin \operatorname{Ker} d_1$. We have

$$\begin{aligned} d_2(s_1d_2x_2 - x_2) &= \underbrace{d_2s_1}_{id}d_2x_2 - d_2x_2 \\ &= d_2x_2 - d_2x_2 \\ &= 0. \end{aligned} \tag{45}$$

Hence $s_1d_2x_2 - x_2 \in \operatorname{Ker} d_2$.

Because of the case $y_2 \in \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1$,

$$d_3F_{(1)(2)}(x_2, y_2) \in [\operatorname{Ker} d_0 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1]. \tag{46}$$

If $F_{(0)(1)}(x_2, y_2) = [s_0x_2 - s_1x_2, s_1y_2] + [s_2x_2, s_2y_2]$, then

$$\begin{aligned} &d_3F_{(0)(1)}(x_2, y_2) \\ &= [d_3s_0x_2 - d_3s_1x_2, s_1y_2] + \left[\underbrace{d_3s_2}_{id}x_2, \underbrace{d_3s_2}_{id}y_2\right] \\ &= [s_0d_2x_2 - s_1d_2x_2, s_1d_2y_2] + [x_2, y_2]. \end{aligned} \tag{47}$$

Consider the following commutator:

$$[s_0d_2x_2 - s_1d_2x_2 + x_2, s_1d_2y_2 - y_2], \tag{48}$$

and code the terms of this commutator such as

$$\begin{aligned} &a = s_0d_2x_2, \qquad b = s_1d_2y_2, \\ &c = s_1d_2x_2, \qquad d = x_2, \qquad e = y_2, \end{aligned} \tag{49}$$

in order to simplify the algebraic operations. Thus, by using the properties and definition of the commutator we obtain the following:

$$\begin{aligned} [a - c + d, b - e] &= [a - c, b] + [d, e], \\ (a - c + d)(b - e) &- (b - e)(a - c + d) \\ &= ab - cb + db - ae + ce - de \\ &\quad - \{ba - bc + bd - ea + ec - ed\}. \end{aligned} \tag{50}$$

Consider the following cases:

$$\begin{aligned} ab - cb - ba + bc &= (a - c)b - b(a - c) \\ &= [a - c, b], \\ ce - de - ec + ed &= (c - d)e - e(c - d) \\ &= [c - d, e]. \end{aligned} \tag{51}$$

And from the remaining terms we get

$$\begin{aligned} db - bd - [d, e] &= db - bd - de + ed \\ &= d(b - e) - (b - e)d \\ &= [d, b - e]. \end{aligned} \tag{52}$$

Consequently for $n = 3$ we have

$$\begin{aligned} \partial_3(NG_3) \subseteq\ & [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ & + [\operatorname{Ker} d_1, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_2] \\ & + [\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0] \\ & + [\operatorname{Ker} d_1, \operatorname{Ker} d_0] \\ & + [\operatorname{Ker} d_0 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ & + [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1]. \end{aligned} \tag{53}$$

□

Corollary 8. *Let NG_3 be a 3-dimensional Moore complex of a simplicial group* **G** *with 26-adjacency. Then*

$$\begin{aligned} \partial_3(NG_3) \subseteq\ & [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ & + [\operatorname{Ker} d_1, Kerd_0 \cap \operatorname{Ker} d_2] \\ & + [\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0] \\ & + [\operatorname{Ker} d_1, \operatorname{Ker} d_0] \\ & + [\operatorname{Ker} d_0 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ & + [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1]. \end{aligned} \tag{54}$$

Proof. Otherwise inclusion for the previous theorem is obtained from [4, 5]. Therefore

$$\begin{aligned}\partial_3(NG_3) = {} & [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ & + [\operatorname{Ker} d_1, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_2] \\ & + [\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0] \\ & + [\operatorname{Ker} d_1, \operatorname{Ker} d_0] \\ & + [\operatorname{Ker} d_0 \cap \operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1] \\ & + [\operatorname{Ker} d_2, \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1].\end{aligned} \tag{55}$$

□

3. Conclusion

In this paper for dimension 2 and dimension 3, we obtained the Moore complex of simplicial groups generated by hypercrossed complex pairings in digital images.

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